Transactions of the Nigerian Association of Mathematical Physics
Volume 8, (January, 2019), **p189 –194**
© Trans. of NAMP

COEFFICIENT ESTIMATES FOR A SUBCLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES.

[1]E. Ukeje and[2] K. Nwachukwu-Nwokeafor

[1]Dept of Mathematics, Michael Okpara University of Agriculture, Umudike.
[2]Dept. of Computer Engineering, Michael Okpara University of Agriculture, Umudike.

Abstract

The work redefined a general subclass of bi-univalent functions using Ruscheweyh derivative, establish connections with already existing class and obtained their coefficient estimates using Faber polynomial expansion .There is an improvement on the bounds following introduction of $\omega -$ fixed-point.

***Keywords*:** Bi-univalent function, Faberpolynomial, Ruscheweyh derivative, subordination,

1. Introduction

Let $A(w) \subset A$ be the class of analytic functions of the form

$$f(z) = z - \omega + \sum_{k=n}^{\infty} a_k (z-\omega)^k \tag{1.1}$$

in the open unit disk $E = \{z : |z| < 1\}$ which are univalent , normalized with $f(\omega) = 0$ and $f'(\omega) = 1$ where ω is a fixed point , [1].

We denote by S the class of all analytic functions in A. For $f \in A(\omega)$, the Ruscheweyh derivative is

$$D^n f(z) = (z-\omega) + \sum_{k=n}^{\infty} \frac{n!(k-1)!}{(k+n-1)!} a_k (z-\omega)^k \tag{1.2}$$

where $n \in$ □ see [1,2] .

Let p be the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z$ that are analytic in E and satisfy the condition $\mathrm{Re}(p(z)) > 0$ in E. By the CaratheodoryLemma [2] It is well known that every function $f \in S$ has an inverse f^{-1}, which is defined by $f^{-1}(f(z)) = z$, $z \in E$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f) : r_0(f) \geq \frac{1}{4}$) .

Suppose $\omega = 0$ in (1.1) the inverse functions $g = f^{-1}$ is given by $g(w) = f^{-1}(w)$

$$= w - a_2 w^2 + (2a_2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots$$
$$= w + \sum_{n=2}^{\infty} A_n w^n \tag{1.3}$$

A function $f \in A$ is said to be bi-univalent in E if both $f \in S$ and $g \in f^{-1} \in S$. Let $\sum$ be the class of bi-univalent functions in E given by (1.1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [3] where it was proved that $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's result to

$|a_2| \leq \sqrt{2}$ and later Netanyahu [5] proved that $|a_2| \leq 4/3$.

Brannan and Taha [6] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor –Maclaurin coefficient $|a_2|$ and $|a_3|$. For a brief history and interesting examples of the functions in the class $\sum$ see [6].

Correspondence Author: Ukeje E., Email: Ukeje.emelike@mouao.edu.ng, Tel: +2348035636237

Other works include Frasin and Aouf [7] , Ali etal [8] Hamidietal [9] and others .
Recently, SerapBulut [10] ,SibelYalcin [11] provided interesting geometric properties of the class Σ .

For $f(z)$ and $F(z)$ analytic in E, we say that f subordinate to F, written $f \prec F$,
if there exists a Schwarz function $u(z)=\sum_{k=1}^{\infty} c_k z^k$ with $|u(z)|<1$ in E, such that $f(z)=F(u(z))$, see [12].

A function $D^n f \in \Sigma$ is said to be $\Lambda_\Sigma(n,\mu,\lambda,\varphi)$, $\lambda \geq 1$ and $\mu \geq 0$, if the following subordination hold

$$(1-\lambda)\left(\frac{D^n f(z)}{z-\omega}\right)^{\mu} + \lambda\left(D^n f(z)\right)'\left(\frac{D^n f(z)}{z-\omega}\right)^{\mu-1} \prec \varphi(z) \tag{1.4}$$

and

$$(1-\lambda)\left(\frac{D^n g(w)}{w-\omega}\right)^{\mu} + \lambda\left(D^n g(w)\right)'\left(\frac{D^n g(w)}{w-\omega}\right)^{\mu-1} \prec \varphi(w) \tag{1.5}$$

where $g(w) = f^{-1}(w)$

Here, this work will use Faber polynomial expansions to establish bounds for the general coefficients $|a_n|$ of a redefined bi-univalent functions with ω fixed point $\Lambda_\Sigma(n,\mu,\lambda,\varphi)$.

2. Main Results

Using the Faber polynomial expansion of functions $f \in A$ of (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as [13] .

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...) w^n$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1]!(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4$$
$$+ \frac{(n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5}[a_5 + (-n+2)a_3^2]$$

$$+ \frac{(-n!)}{(-2n+5)!(n-6)!} a_2^{n-6}[a_6 + (-2n+5)a_3 a_4]$$
$$+ \sum_{j\geq 7} a_2^{n-j} V_j \tag{1.6}$$

Such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$ [14].In particular , the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2} K_1^{-2} = -a_2$$
$$\frac{1}{3} K_2^{-3} = 2a_2^2 - a_3$$
$$\frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4) \tag{1.7}$$

In general , for any $P \in \square$, an expansion of K_n^p is as [13]

$$K_n^p = pa_n + \frac{p(p-1)}{2} E_n^2 + \frac{p!}{(p-3)!3!} E_n^3 + ... + \frac{p!}{(p-n)!n!} E_n^n \tag{1.8}$$

where $E_n^p = E_n^p(a_2, a_3, ...)$ and by [14]

$$E_n^m(a_1,a_2,...,a_n)=\sum_{m=1}^{\infty}\frac{m!(a_1)^{\mu_1}...(a_n)^{\mu_n}}{\mu_1!...\mu_n!} \tag{1.9}$$

while $a_1=1$, and the sum is taken over all nonnegative integers $\mu_1,...,\mu_n$ satisfying

$$\mu_1+\mu_2+...+\mu_n=m$$
$$\mu_1+2\mu_2+...+n\mu_n=n \tag{2.0}$$

Evidently , $E_n^n(a_1,a_2,...,a_n)=a_1^n$ [15] .

Theorem 2.1

For $\lambda\geq 1$ and $\mu\geq 0$, let $D^n f\in\Lambda_{\Sigma}(n,\mu,\lambda,\varphi)$. If $a_k=0;\ 2\leq k\leq n-1$, then

$$|a_n|\leq\frac{2}{\psi_{n,k}^{n-1}(r+d)^{n-1}[\mu+(n-1)\lambda]} \tag{2.1}$$

where $\psi_{nk}=\dfrac{n!(k-1)!}{(n+k-1)!}$, $|z|=r<1$ and $|\omega|=d$

Proof: Let functions f given by (1.1) , we have

$$(1-\lambda)\left(\frac{D^n f(z)}{z-\omega}\right)^{\mu}+\lambda\left(D^n f(z)\right)'\left(\frac{D^n f(z)}{z-\omega}\right)^{\mu-1}=1+\sum_{n=2}^{\infty}F_{n-1}(a_2,a_3,...,a_n)(z-\omega)^{n-1} \tag{2.2}$$

and

$$(1-\lambda)\left(\frac{D^n g(w)}{w-\omega}\right)^{\mu}+\lambda\left(D^n g(w)\right)'\left(\frac{D^n g(w)}{w-\omega}\right)^{\mu-1}=1+\sum_{n=2}^{\infty}F_{n-1}(b_2,b_3,...,b_n)(w-\omega)^{n-1} \tag{2.3}$$

where

$$F_1=\psi_{2,k}(r+d)(\mu+\lambda)a_2$$
$$F_2=\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{\frac{\mu-1}{2}a_2^2+a_3\right\} \tag{2.4}$$
$$F_3=\psi_{4,k}^3\left(r+d\right)^3\left(\mu+3\lambda\right)\left\{\frac{\psi\left(\mu-1\right)\left(\mu-2\right)}{3!}a_2^3+\left(\mu-1\right)a_4a_2+a_4\right\}$$

In general,see[11 ,12].Consequently,the inequality (1.4) and (1.5) imply the existence of two positive real part functions

$u(z)=1+\sum_{n=1}^{\infty}c_n(z-\omega)^n$ and $v(w)=1+\sum_{n=1}^{\infty}t_n(w-\omega)^n$

where $\mathrm{Re}\,u(z)>0$ and $\mathrm{Re}\,v(\omega)>0$ in p so that

$$(1-\lambda)\left(\frac{D^n f(z)}{z-\omega}\right)^{\mu}+\lambda\left(D^n f(z)\right)'\left(\frac{D^n f(z)}{z-\omega}\right)^{\mu-1}=\varphi\left(u(z)\right) \tag{2.5}$$

and

$$(1-\lambda)\left(\frac{D^n g(w)}{w-\omega}\right)^{\mu}+\lambda\left(D^n g(w)\right)'\left(\frac{D^n g(w)}{w-\omega}\right)^{\mu-1}=\varphi\left(v(w)\right) \tag{2.6}$$

where $\varphi\left(u(z)\right)=1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\psi_{n,k}^k\varphi_k E_n^k(c_1,c_2,...,c_n)(z-\omega)^n$ (2.7)

and $\varphi\left(v(w)\right)=1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\psi_{n,k}^k\varphi_k E_n^k(d_1,d_2,...,d_n)(w-\omega)^n$ (2.8)

Comparing the corresponding coefficients of (2.5) and (2.7) yields

$$\psi_{n,k}^{n-1}(r+d)^{n-1}\left(\mu+(n-1)\lambda\right)a_n=1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\varphi_k E_{n-1}^k(c_1,c_2,...,c_{n-1})(z-\omega)^n \tag{2.9}$$

similarly, from (2.6) and (2.8) will give

$$\psi_{n,k}^{n-1}(r+d)^{n-1}\left(\mu+(n-1)\lambda\right)b_n=1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\varphi_k E_{n-1}^k(t_1,t_2,...,t_{n-1})(w-\omega)^n \tag{2.10}$$

Note that for $a_k = 0$: $2 \le k \le n-1$ we have $b_n = -a_n$ and so

$$\psi_{n,k}^{n-1}(r+d)^{n-1}\left(\mu+(n-1)\lambda\right)a_n = \varphi_1 c_{n-1}$$

$$-\psi_{n,k}^{n-1}(r+d)^{n-1}\left(\mu+(n-1)\lambda\right)b_n = \varphi_1 t_{n-1}$$

Now taking the absolute values of either of the above two equations and using the facts that $|\varphi_1| \le 2$, $|c_{n-1}| \le 1$ and $|t_{n-1}| \le 1$ will give

$$|a_n| \le \frac{|\varphi_1 c_{n-1}|}{\psi_{n,k}^{n-1}(r+d)^{n-1}\left(\mu+(n-1)\lambda\right)} = \frac{|\varphi_1 t_{n-1}|}{\psi_{n,k}^{n-1}(r+d)^{n-1}\left(\mu+(n-1)\lambda\right)} \qquad (2.11)$$

$$\le \frac{2}{\psi_{n,k}^{n-1}(r+d)^{n-1}\left(\mu+(n-1)\lambda\right)}$$

Theorem 2.2 . Let $D^n f \in \Lambda_\Sigma(n,\mu,\lambda,\varphi)$, $\lambda \ge 1$ and $\mu \ge 0$. Then

i $\cdot |a_2| \le \min\left\{\frac{2}{\psi_{2,k}(r+d)\left(\mu+\lambda\right)}, \sqrt{\frac{8}{\psi_{3,k}^2(r+d)^2\left(\mu+2\lambda\right)(\mu+1)}}\right\}$

ii $\cdot |a_3| \le \min\left\{\begin{array}{l}\frac{4}{\psi_{2,k}^2(r+d)^2(\mu+\lambda)^2} + \frac{2}{\psi_{3,k}^2(r+d)^2(\mu+2\lambda)}, \\ \frac{8}{\psi_{3,k}^2(r+d)^2(\mu+2\lambda)(\mu+1)} + \frac{2}{\psi_{3,k}^2(r+d)^2(\mu+2\lambda)}\end{array}\right\}$ (2.12)

Proof.
Replacing n by 2 and 3 in (2.7) and (2.8) ,respectively , we find that

$$\psi_{2,k}(r+d)(\mu+\lambda)a_2 = \varphi_1 c_1 \qquad (2.13)$$

$$\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{\frac{\mu-1}{2}a_2^2 + a_3\right\} = \varphi_1 c_2 + \varphi_2 c_1^2 \qquad (2.14)$$

while the inverse of (2.13) and (2.14) respectively are :

$$-\psi_{2,k}(r+d)(\mu+\lambda)a_2 = \varphi_1 d_1 \qquad (2.15)$$

$$\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{\frac{\mu-1}{2}a_2^2 + 2a_2^2 - a_3\right\} = \varphi_1 d_2 + \varphi_2 d_1^2$$

$$\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{\frac{\mu+3}{2}a_2^2 - a_3\right\} = \varphi_1 d_2 + \varphi_2 d_1^2 \qquad (2.16)$$

From (2.13) or (2.15) , one gets

$$|a_2| \le \frac{|\varphi_1 c_1|}{\psi_{2,k}(r+d)(\mu+\lambda)} = \frac{|\varphi_1 d_1|}{\psi_{2,k}(r+d)(\mu+\lambda)} \le \frac{2}{\psi_{2,k}(r+d)(\mu+\lambda)} \qquad (2.17)$$

Adding2.11 to 2.16 gives

$$\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{a_2^2\left(\frac{\mu-1}{2} + \frac{\mu+3}{2}\right)\right\} = \varphi_1(c_2+d_2) + \varphi_2(c_1^2+d_1^2)$$

ie $\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{a_2^2\left(\mu+1\right)\right\} = \varphi_1(c_2+d_2) + \varphi_2(c_1^2+d_1^2)$ (2.18)

or equivalently

$$|a_2| \le \sqrt{\frac{8}{\psi_{3,k}^2(r+d)^2(\mu+2\lambda)(\mu+1)}} \qquad (2.19)$$

Subtracting (2.16) from (2.14), we have

$$\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{\left(\frac{\mu-1}{2} - \frac{\mu+3}{2}\right)a_2^2 + 2a_3\right\} = \varphi_1(c_2-d_2) + \varphi_2(c_1^2+d_1^2)$$

or $2\psi_{3,k}^2(r+d)^2(\mu+2\lambda)\left\{-a_2^2 + a_3\right\} = \varphi_1(c_2-d_2) + \varphi_2(c_1^2+d_1^2)$ (2.20)

i.e. $|a_3| = \left|a_2^2\right|^2 + \frac{|\varphi_1(c_2-d_2)|}{2\psi_{3,k}^2(r+d)^2(\mu+2\lambda)} \le |a_2|^2 + \frac{2}{\psi_{3,k}^2(r+d)^2(\mu+2\lambda)}$ (2.21)

Using the value of a_2^2 from(2.17) and into (2.20)will yield respectively

$$|a_3| \le \frac{4}{\psi_{2,k}^2 (r+d)^2 (\mu+\lambda)^2} + \frac{2}{\psi_{3,k}^2 (r+d)^2 (\mu+2\lambda)}$$

and $|a_3| \le \frac{8}{\psi_{3,k}^2 (r+d)^2 (\mu+2\lambda)(\mu+1)} + \frac{2}{\psi_{3,k}^2 (r+d)^2 (\mu+2\lambda)}$

The corollary below establishes a connection between this work and [15]

Corollary 2.3 Let $D^n f \in \Lambda_\Sigma(n,\mu,\lambda,\varphi), n \ge 2, \mu \ge 0$ and $\lambda = 1, k = 1$ then

$$|a_2| = \frac{2}{(r+d)(\mu+1)}$$

and $|a_3| \le \frac{4}{(r+d)^2 (\mu+1)^2} + \frac{2}{(r+d)^2 (\mu+2)}$.

Corollary 2.4Let $D^n f \in \Lambda_\Sigma(n,\mu,\lambda,\varphi), n \ge 2, \lambda \ge 1$ and $\mu = 1, k = 1$ then

$$|a_2| = \frac{2}{(r+d)(\lambda+1)}$$

$$|a_3| \le \frac{4}{(r+d)^2 (\lambda+1)^2} + \frac{2}{(r+d)^2 (1+2\lambda)}$$

CONCLUSION

Linear property of Ruscheweyh derivative was establish in this work following some of the results shown

ACKNOWLEDGEMENT.

I write to express my sincere gratitude to my Supervisor ,Prof. A.T. Oladipo of Ladoke Akintola University of Technology Ogbomoso who provided some of the invaluable materials used in this work. I thank him immensely for his continuous support.

References

[1] Emelike Ukeje, Generalization of Coefficient Inequalities and Convolution properties of a class of Analytic Functions with Respect to a differential operator. Journal of the Nigerian Association of Mathematical Physics, volume 34, (March, 2016), pp5-12 J. of NAMP.

[2] P.L. Duren, univalent functions, vol. 259 of Grundlehren der Mathematischen, springer, New York, NY, USA,1983.

[3] M. Lewin 'On a coefficient problem for bi-univalent functions' Proceedings of the American Mathematical Society, vol. 18 , 63-68 , 1967.

[4] D. A. Brannan and J. S. Clunie (Edc) Aspects of contemporary complex analysis (Proceedings of the NATO advanced study institute held at the University Durham : July 1 , 20,1979) Academic Press , New York and London ,1980.

[5] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. Arch Rational Mech. Anal 32 (1969) , 100-112

[6] D. A. Brannan and T.S. Taha , on some classes of bi-univalent functions in Mathematical Analysis and its applications (S. M. Mazhar, A. Hamoui and N.S. Faour, Editors) (Kuwait: February 18-21, 1985), KFAS Proceedings series , Vol, 3, Pergamon Press (Elsvier Science Limited), Oxford, 1988 , pp, 53-60: see also Studia Univ. Babes-Bolyai Math . 31(2) (1986), 70-77.

[7] H. M. Srivastava, A. K. Mishra and P. Gochihayat Certain subclasses of analytic and bi-univalent functions, Appl. Math Lett. 23(2010) 1188-1192.

[8] B. A. Frasin and M. K. Aouf , New subclasses of bi-univalent functions , Appl. Math Lett. 24 (2011)

[9] R. M. Ali , S. K. Lee , V. Ravichandran and S. Supramaniam, 'coefficient estimate for bi-univalent Ma-Mindastarlike and convex functions ' Applied Mathematics Letters , vol.25 no. 3 , pp 344-351, 2012.

[10] S. G. Hamidi, S. A. Halim and J. M. Jahangiri, 'Faber polynomial coefficient estimates for meromorphic bi-starlike functions 'International Journal of Mathematics and Mathematical Sciences, Vol. 2013, Article 1D 498159, 4 pages, 2013.

[11] SerapBulut, Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions defined by Salagean differential operator MATEMATNYKµ BECHNK, 67, 3(2015) 185-193 September 2015.

[12] Sahsene Altinkaya and Sibel Yalcin, Faber polynomial coefficient bounds for a subclass of bi-univalent functions stud. Univ. Babes-Bolyai Math. 61(2016), No 1, 37-44.

[13] H. Airault and A. Bouali "Differential calculus on the Faber polynomials "Bulletin des Sciences Mathematiques, Vol.130 , no 3 , pp. 179-222 , 2006.

[14] H. Airault and J. Ren, "An algebra of differential operators and generating functions on the set of univalent" Bulletin des Sciences Mathematiques , vol. 126 , no. 5 , pp. 345-367 , 2002.

[15] P. G. Todorov, On the Faber polynomials of the univalent functions of class Σ , J. Math. Anal. Appl. 162 (1) (1991), 268-278 .

[16] H. Airault, "Remarks on Faber polynomials" international Mathematical Forum., Journal for Theory and Applications, Vol.3, No.9-12,pp.449-456,2008.