Transactions of the Nigerian Association of Mathematical Physics
***Volume 8, (January, 2019*), p185 –188**
© Trans. of NAMP

CONTINUOUS NUMERICAL INTERPOLANT FOR THE SOLUTION OF WAVE EQUATIONS

Sunday Babuba

Federal University Dutse, Department of Mathematics, Ibrahim Aliyu Bye-Pass, P.M.B 7156, Dutse, Jigawa State - Nigeria.

Abstract

A new Continuous interpolant method based on polynomial approximation is here proposed for solving wave equation subject to some initial and boundary conditions. The method results from discretization of the wave equation which leads to the production of a system of algebraic equations. By solving the system of algebraic equations by employing the continuous interpolant scheme we obtain the problem approximate solutions.

***Keywords*:** Polynomials, interpolation, collocation, wave equation, lines, Continuous interpolant

1.0 Introduction

There is a growing interest in the recent literatures concerning continuous numerical methods for solving ODEs. In science and engineering, this interest is extended to the development of continuous numerical techniques for solving wave equation subject to initial and boundary conditions. Their advantages over discrete ones are now well known, including their connection to large families. In (1)we saw the presentation of an extension of this continuous method for solving ODEs to solve PDEs in two dimensions as a conjecture. Hitherto, efforts have been on top gear to derive continuous numerical interpolant for solving wave equation. When this is achieved then a generalized scheme that can solve all the branches of PDEs- parabolic, hyperbolic and elliptic equations is possible. In this paper therefore, we develop a new continuous numerical interpolant which is based on interpolation and collocation at some points along the coordinates.

2.0 Solution Method

To set up the solution method we select an integer N such that $N > 0$. Then subdivide the interval $0 \le x \le X$ into N equal subintervals with mesh points along the space coordinate given by $x_i = ih,\ i = \frac{1}{\beta}\left(\frac{1}{\beta}\right)N$, where $Nh = X$. Similarly, reverse the roles of x and t and select another integer M such that $M > 0$. Also, subdivide the interval $0 \le t \le T$ into M equal subintervals with mesh points along time axis given by $t_j = jk$, $j = \frac{1}{\alpha}\left(\frac{1}{\alpha}\right)M$ where $Mk = T$ and h, k are the mesh sizes along space and time axes respectively. Here, we seek for the approximate solution $\overline{U}(x,t)$ to $\overline{U}_{p-1}(x,t)$ of the form

$$\overline{U}(x,t) \approx \overline{U}_{p-1}(x,t) = \sum_{r=0}^{p-1} a_r \left[Q_r(x,t)\right],\ x \in \left[x_i, x_{i+h}\right],\ t \in \left[t_j, t_{j+k}\right] \tag{2.0}$$

Over $h > 0, k > 0$ mesh sizes, such that

$0 = x_0 < \ldots < x_1 < \ldots x_N, 0 = t_0 < \ldots < t_1 < \ldots t_M$. Let p be the sum of interpolation points along space and time coordinates. Hence, $\rho = g + b$ where g is the number of interpolation points along the space axis and b the number of interpolation points along time coordinate. The basis function $Q_r(x,t),\ r = 0,1,\ldots, p-1$ is the Taylor's polynomials which is known, a_r are the constants to be determined. There will be flexibility in the choice of the basis function as may be

Correspondence Author: Sunday B., Email: sundaydzupu@yahoo.com, Tel: +2348039282881

desired for specific application. For this work, we consider the Taylor's polynomial $Q_r(x,t) = x^r t^r$. The interpolation values $\overline{U}_{i,j},\ldots,\overline{U}_{i+h-1,j}$ are assumed to have been determined from previous steps, while the method seeks to obtain $\overline{U}_{i+h,j}$ as in (1). Applying the above interpolation conditions on eqn. (2.0) we obtain,

$$a_0Q_0(x_{i+h},t_{j+k}) + a_1Q_1(x_{i+h},t_{j+k}) + \ldots a_{p-2}Q_{p-1}(x_{i+h},t_{j+k}) = \overline{U}(x_{i+h},t_{j+k}) \quad (2.1)$$

We let $h = -\frac{1}{\beta}\left(\frac{1}{\beta}\right)\left[g - \left(\frac{2\beta-1}{\beta}\right)\right]$ arbitrarily and $k = 0$, then by Crammer's rule, eqn. (2.1) becomes

$$\left.\begin{array}{l} W\underline{a} = \underline{F}, \quad \underline{F} = \left(\overline{U}_{V,j}, \overline{U}_{v+\frac{1}{\beta},j}, \ldots, U_{z,j}\right)^T \\ \underline{a} = (a_0,\ldots,a_{p-1})^T \end{array}\right\} \quad (2.2)$$

and

$$W = \begin{bmatrix} Q_0(x_v,t_j), & Q_1(x_v,t_j), & \ldots, & Q_{p-1}(x_v,t_j) \\ Q_0\left(x_{v+\frac{1}{\beta}},t_j\right), & Q_1\left(x_{v+\frac{1}{\beta}},t_j\right), & \ldots, & Q_{p-1}\left(x_{v+\frac{1}{\beta}},t_j\right) \\ \ldots, & \ldots, & \ldots, & \\ Q_0(x_z,t_j), & Q_1(x_z,t_j), & \ldots, & Q_{p-1}(x_z,t_j) \end{bmatrix}$$

Where $z = i + g - \left(\frac{2\beta-1}{\beta}\right)$, $v = i - \frac{1}{\beta}$ and W^{-1} exists. Hence, by equation (2.2) we obtain

$$\underline{a} = \varpi\underline{F}, \quad \varpi = W^{-1} \quad (2.3)$$

The vector $\underline{a} = (a_0,\ldots,a_{p-1})^T$ is now determined in terms of known parameters in $\varpi\underline{F}$. If ϖ_{r+1} is the $(r+1)^{th}$ row of ϖ then

$$a_r = \varpi_{r+1}\underline{F} \quad (2.4)$$

Eqn. (2.4) determines the values of a_r. Let us take first and second derivatives of eqn. (2.0) with respect to x,

$$\overline{U}'(x,t) = \sum_{r=0}^{p-1} a_r\left[Q_r'(x,t)\right]$$

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1} a_r\left[Q_r''(x,t)\right] \quad (2.5)$$

Substituting eqn. (2.4) into eqn. (2.5), we obtain

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1}\left[\varpi_{r+1}\underline{F}\left(Q_r''(x,t)\right)\right] \quad (2.6)$$

We reverse the roles of x and t in eqn. (2.1) and we arbitrarily set $k = 0\left(\frac{1}{\alpha}\right)\left[b - \left(\frac{\alpha-1}{\alpha}\right)\right]$ and $k = 0$, then againby Cramer's rule eqn. (2.1) becomes.

$$\left.\begin{array}{l} Y\underline{a} = \underline{E}, \quad \underline{E} = \left(\overline{U}_{i,\eta-\frac{1}{\alpha}}, \overline{U}_{i,\eta}, \ldots, U_{i,\gamma}\right)^T \\ \underline{a} = (a_0,\ldots,a_{p-1})^T \end{array}\right\} \quad (2.7)$$

and

$$Y = \begin{bmatrix} Q_0\left(x_i,t_{\eta-\frac{1}{\alpha}}\right), & Q_1\left(x_i,t_{\eta-\frac{1}{\alpha}}\right), & \ldots, & Q_{p-1}\left(x_i,t_{\eta-\frac{1}{\alpha}}\right) \\ Q_0(x_i,t_\eta), & Q_1(x_i,t_\eta), & \ldots, & Q_{p-1}(x_i,t_\eta) \\ \ldots, & \ldots, & \ldots, & \\ Q_0(x_i,t_\gamma), & Q_1(x_i,t_\lambda), & \ldots, & Q_{p-1}(x_i,t_\gamma) \end{bmatrix}$$

Where $\eta = j + \frac{1}{\alpha}$, $\gamma = j + b - \left(\frac{\alpha-1}{\alpha}\right)$, and Y^{-1} exists (1-17). Hence from equation (2.7) we obtain

$$\underline{a} = L\underline{E}, \quad L = Y^{-1} \quad (2.8)$$

Continuous Numerical Interpolant for… Sunday Trans. Of NAMP

The vector $\underline{a} = (a_0,...,a_{p-1})^T$ is now determined in terms of known parameters in $L\underline{E}$. If L_{r+1} is the $(r+1)^{th}$ row of L then

$$a_r = L_{r+1}\underline{E} \quad (2.9)$$

Also, eqn. (2.9) determines the values of a_r. Taking the first and second derivatives of eqn. (2.0) with respect to t, we obtain

$$\bar{U}'(x,t) = \sum_{r=0}^{p-1} a_r \left[Q_r'(x,t) \right]$$

$$\bar{U}''(x,t) = \sum_{r=0}^{p-1} a_r \left[Q_r''(x,t) \right] \quad (2.10)$$

Substituting eqn. (2.9) in eqn. (2.10) we have

$$\bar{U}''(x,t) = \sum_{r=0}^{p-1} \left[L_{r+1}\underline{E}\left(Q_r''(x,t) \right) \right] \quad (2.11)$$

But by eqn. (1.0) it is obvious that eqn. (2.11) is equal to eqn. (2.6), therefore,

$$\sum_{r=0}^{p-1} \left[L_{r+1}\underline{E}\left(Q_r''(x,t) \right) \right] - \sum_{r=0}^{p-1} \left[\varpi_{r+1}\underline{E}\left(Q_r''(x,t) \right) \right] = 0 \quad (2.12)$$

Collocating eqn. (2.12) at $x = x_i$ and $t = t_j$ we obtain a new continuous numerical interpolant that solves eqn. (2.0) explicitly.

3.0 Numerical Examples

In this section we give some numerical examples to compute approximate solutions for equation (2.0) by the method discussed in this paper. This is in order to test the numerical accuracy of the new method. To achieve this, we truncate the Taylor's polynomial after second degree and use it as the basis function for the computations. The resultant interpolant is used to solve the following two test problems.

Example 1

Use the scheme to approximate the solution to the wave equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0, \quad 0 < x < 1 \quad 0 < t, \; U(0,t) = U(1,t) = 0, \; t > 0$$

$$U(x,0) = \sin \pi x, \quad 0 \le x \le 1, \quad \frac{\partial U}{\partial x}(x,0) = 0, \quad 0 \le x \le 1$$

Table 1: Result of action of Eqn. (2.12) on example 1

x	Exact solution $U(x,t)$	Schmidt method $U(x,t)$	New Method $U(x,t)$	Errors	
				New Method	Schmidt method
0	0	0	0	0	0
0.1	0.305212482	0.305992120	0.305235901	2.3419 X E-5	7.7963840 X E- 4
0.2	0.580548640	0.582031600	0.580593187	4.4547 X E-5	1.4829604 X E -3
0.3	0.799056652	0. 801097772	0.799117966	6.1314 X E-5	2.0411200 X E- 3
0.4	0.939347432	0.941746912	0.939419511	7.2079 X E-5	2.3994802 X E -3
0.5	0.987688340	0.990211303	0.987764129	7.5789 X E-5	2.5229632 X E -3
0.6	0.939347432	0.941746912	0.939419511	7.2079 X E-5	2.3994802 X E -3
0.7	0.799056652	0. 801097772	0.799117966	6.1314 X E-5	2.0411200 X E- 3
0.8	0.580548640	0.582031600	0.580593187	4.4547 X E-5	2.0411200 X E- 3
0.9	0.305212482	0.305992120	0.305235901	2.3419 X E-5	7.7963840 X E- 4
1	0	0	0	0	0

Example 2

Use the scheme to approximate the solution to the wave equation

$$\frac{\partial^2 U}{\partial t^2} - 4\frac{\partial^2 U}{\partial x^2} = 0 \qquad 0 < x < 1, \quad 0 < t, \; U(0,t) = U(1,t) = 0, \qquad t > 0$$

$$U(x,0) = \sin \pi x, \qquad 0 \le x \le 1, \; \frac{\partial U}{\partial x}(x,0) = 0, \qquad 0 \le x \le 1$$

Table2: Result of action of Eqn. (2.12) on example2

x	Exact Solution $U(x,t)$	Schmidt method $U(x,t)$	New method $U(x,t)$	Errors	
				New Method	Schmidt Method
0	0	0	0	0	0
0.1	0.305212482	0.304983829	0.305235901	2.3419 X E-5	2.2865 X E -4
0.2	0.58054864	0.580113718	0.580593187	4.4547 X E-5	4.3492 X E -4
0.3	0.799056652	0.798458034	0.799117966	6.1314X E-5	5.9862 X E -4
0.4	0.939347432	0.9386437114	0.939419511	7.2079 X E-5	7.0372 X E– 4
0.5	0.987688340	0.986948407	0.987764129	7.5789 X E-5	7.3993 X E– 4
0.6	0.939347432	0.305992120	0.939419511	7.2079 X E-5	7.0372 X E– 4
0.7	0.799056652	0.798458034	0.799117966	6.1314 X E-5	5.9862 X E -4
0.8	0.58054864	0.580113718	0.580593187	4.4547 X E-5	4.3492 X E -4
0.9	0.305212482	0.304983829	0.305235901	2.3419 X E-5	2.2865 X E -4
1	0	0	0	0	0

References

[1] Odekunle, M. R. (2008). Solution of partial differential equation using collocation Nigeria interpolation method. A conjecture. Paper presented at the annual conference of the Mathematical Society, July, at University of Lagos, Lagos.

[2] Adam, A. & David, R. (2002): One dimensional heat equation. http://www.ng/online.redwoods.cc.ca.us/instruct/darnold/deproj/sp02/.../paper.pdf

[3] Awoyemi, D. O. (2002): An Algorithmic collocation approach for direct solution of special fourth – order initial value problems of ordinary differential equations. *Journal of the Nigerian Association of Mathematical Physics,* **vol 6,** pp 271 – 284.

[4] Awoyemi, D. O. (2003): A p – stable linear multistep method for solving general third order Ordinary differential equations. *Int. J. Computer Math*. **80** (**8**), 987 - 993.

[5] Bao, W., Jaksch, P. &Markowich, P.A. (2003): Numerical solution of the Gross – Pitaevskii equation for Bose – Einstein condensation. *J. Compt. Phys*. **187**(1), 318- 342.

[6] Benner, P. & Mena, H. (2004): BDF methods for large scale differential Riccati equations in proc. of mathematical theory of network and systems. *MTNS.* Edited by Moore, B. D., Motmans, B., Willems, J., Dooren, P.V. &Blondel, V.

[7] Bensoussan, A,. Da Prato, G., Delfour, M. &Mitter, S. (2007): Representation and control of infinite dimensional systems. 2nd edition. Birkhauser: Boston, MA.

[8] Motmans, B., Willems, J., Dooren, P. V. &Blondel, V.

[9] Biazar, J. &Ebrahimi, H. (2005): An approximation to the solution of hyperbolic equation by a domain decomposition method and comparison with characteristics method. *Appl. Math. andComput.* **163**, 633 - 648.

[10] Brown, P. L. T. (1979): A transient heat conduction problem. *AICHEJournal*, **16**, 207 - 215.

[11] Chawla, M. M. &Katti, C. P. (1979): Finite difference methods for two – point boundary value problems involving high – order differential equations. *BIT*. **19**, 27-39.

[12] Cook, R. D. (1974): *Concepts and Application of Finite Element Analysis:* NY: Wiley Eastern Limited.

[13] Crandall, S. H. (1955): An optimum implicit recurrence formula for the heat conduction equation. *JACM*.**13**, 318 - 327.

[14] Crane, R. L. & Klopfenstein, R. W. (1965): A predictor – corrector algorithm with increased range of absolute stability. *JACM*. **12**, 227-237.

[15] Crank, J. & Nicolson, P. (1947): A practical method for numerical evaluation of solutions of partial differential equations of heat conduction type. *Proc. Camb. Phil.Soc.* **6**, 32 - 50.

[16] Dahlquist, G. &Bjorck, A. (1974): Numerical methods. NY: Prentice Hall.

[17] Dehghan, M. (2003): Numerical solution of a parabolic equation with non – local boundary specification. *Appl. Math. Comput*. **145**, 185 – 194.

[18] Dieci, L. (1992): Numerical analysis. *SIAM Journal.* **29**(**3**), 781-815.

[19] Douglas, J., (1961): *A Survey of Numerical Methods for Parabolic Differential Equations in advances in computer II*. Academic press.

[20] D' Yakonov, Ye. G. (1963): On the application of disintegrating difference operators. *Z. Vycist. Mat. I. Mat. Fiz.* **3**, 385 - 395.

[21] Eyaya, B. E. (2010): Computation of the matrix exponential with application to linear parabolic PDEs. http://www.dip.sun.ac.za/~eyaya/PGD-Essay-Template-2009_10.pdf

[22] Fox, L. (1962): Numerical Solution of Ordinary and Partial Differential Equation. New York: Pergamon.

[23] Penzl, T. (2000): Matrix analysis. *SIAM J.* **21**, 1401- 1418.

[24] Pierre, J. (2008): Numerical solution of the dirichlet problem for elliptic parabolic equations. SIAM J. *Soc. Indust. Appl. Math.* **6**(3), 458 – 466.

[25] Richard, L. B. & Albert, C. (1981): Numerical analysis. *Berlin:*Prindle, Weber and Schmidt, inc.

[26] Richard, L., Burden, J. & Douglas, F. (2001): *Numerical analysis*. Seventh ed., Berlin: Thomson Learning Academic Resource Center.

[27] Saumaya, B., Neela, N. & Amiya, Y. Y. (2012): Semi discrete Galerkin method for equations of Motion arising in Kelvin – Voitght model of visco-elastic fluid flow. *Journal of Pure and Applied Science,* **3** (2 & 3), 321- 343.

[28] Yildiz, B. &Subasi, M. (2001): On the optimal control problem for linear Schrodinger equation. *Appl. Math.and Comput.* **121**, 373-381.

[29] Zheyin, H. R. &Qiang, X. (2012): An approximation of incompressible miscible displacement in porous media by mixed finite elements and symmetric finite volume element method of characteristics. *Applied Mathematics and Computation*, Elsevier, **143**, 654 - 672.