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ANTIPLANE ELASTODYNAMIC ANALYSIS OF A CRACKED ORTHOTROPIC STRIP USING THE WIENER-HOPF TECHNIQUE

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Abstract

The antiplane strain problem of a semi-infinite crack in an orthotropic strip is investigated. The problem is formulated using Fourier transform and solved by the Wiener-Hopf technique. A closed form solution for the displacement is obtained from which the stress intensity factor is calculated. The stress intensity factor is seen to possess the square root singularity; a character which deformation fields display in the analysis of a small region around the crack front.

1. Introduction

Contributions to analytic solutions of crack problems in orthotropic strips has been carried out by many researchers. Different methods ranging from complex variable methods and integral transform has been employed to derive the stress intensity factor. This is a fundamental quantity which governs the linear elastic stress field near the crack tip. It is used to predict the failure of a cracked plate. The stress intensity factor depends on both the geometrical configuration and the loading conditions of the body.

Notably, the problem of the influence of crack propagation velocity on the stress intensity factor has been described by [1]. Solutions for this elastodynamic have been sought under various assumptions, in particular various problems relating the dynamics of cracks located in a strip has been investigated by [2]and approximate solution obtained via Fredholm integral equations of the second kind. The problem of a single crack moving in elastic strip under antiplane shear stress using integral transform technique was solved by [3]. Similar solutions have been obtained by [4]. The complex variable and Wiener –Hopf techniques has been used by [5,6] insolving the problem of a moving semi-infinite crack in an infinite sheet. A transient crack problem for an infinite strip under antiplane shear was considered by [7]. His result shows that the SIF oscillate with a decreasing amplitude and tends to a steady-state value. In this present paper, the Fourier transform along with the Wiener-Hopf technique are used to study mode-III semi-infinite dynamic crack problem in an orthotropic strip, where the material properties are assumed to vary continuously along the crack direction. The values of the stress and displacement fields are obtained leading to the stress intensity factor.

2. Problem formulation

Consider an infinite elastic orthotropic strip occupying the region $-\eta h \le y' \le \eta h$ in a fixed rectangular system (x', y', z') The strip extends from $-\infty$ to $+\infty$ in the x' -direction and contains an infinite crack moving at a constant velocity v in the x' -direction and occupying the region $y' = 0 \quad , -\infty < x' < 0$, The infinite strip has a height $2h$ in the y' -direction. Antiplane shear displacements are applied on the strip faces, while the surface of the crack is free from traction. The geometric configurations and the loading conditions are as depicted in Fig.1.

In the dynamic problem of anti-plane shear, there exists a single non-vanishing component of displacement directed in the Z -direction and independent of Z , i.e.,

$$u = v = 0 \quad , w = w(x', y', t) \tag{1}$$

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Where u , v w are the displacement components in the x' , y' , z -directions, respectively. The corresponding stress field and the equation of motion are

$$\left.\begin{aligned} &\sigma_{x'} = \sigma_{y'} = \sigma_{z'} = \sigma_{x'y'} = 0 \\ &\sigma_{x'z} = C_{44}\frac{\partial w}{\partial x'} \;, \sigma_{y'z} = C_{55}\frac{\partial w}{\partial y'} \\ &C_{44}\frac{\partial^2 w}{\partial x'^2} + C_{55}\frac{\partial^2 w}{\partial y'^2} = \rho\frac{\partial^2 w}{\partial t'^2} \end{aligned}\right\} \qquad (2)$$

where $C_{44} = \mu_{x'z}, C_{55} = \mu_{y'z}$ are the shear moduliof the elastic material in the $x'z$ -and $y'z$ -planes, respectively . Satisfying the equation of motionleads to the classical wave equation in two dimensions for w , viz

$$\frac{\partial^2 w}{\partial x'^2} + \frac{1}{\eta^2}\frac{\partial^2 w}{\partial y'^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t'^2} \qquad (3)$$

Where $\eta = \left(\frac{c_{44}}{c_{55}}\right)^{\frac{1}{2}}$, $c = \left(\frac{C_{44}}{\rho}\right)^{\frac{1}{2}}$

c is the shear speed and ρ is the density of the material.

For a crack moving with constant velocity in the x' -direction, it is convenient to introduce the Galilean transformation

$$x = x' - vt \quad , \; y = \eta y' \quad , \quad z = z' \quad t' = t \qquad (4)$$

where (x, y, z) represents the moving coordinate system. With this transformation, equation (3) reduces to

$$\nabla^2 w(x, y) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \qquad (5)$$

Let the lateral boundaries of the orthotropic strip at $y' = \pm\eta h$ be clamped and displaced by an amount ϑ_0 to produce anti-plane shear motion in the z -direction while the crack moves in the positive x' -direction at a constant speed. The boundary condition for this situation can be stated as

$$\left.\begin{aligned} &w(x', \pm\eta h) = w(x, \pm\eta h) = \pm\vartheta_0 && -\infty < x < \infty \\ &\sigma_{y'z}(x', 0) = \sigma_{yz}(x, 0) = 0 && -\infty < x < 0 \\ &w(x', 0) = w(x, 0) = 0 && 0 < x < \infty \end{aligned}\right\} \qquad (6)$$

Application of the Wiener-Hopf technique requires that the boundary conditions be expressed in a different but equivalent form. From [3],the stress state for the orthotropic strip containing no crack is

$$\sigma_{yz} = \frac{C_{55}}{\eta h}\vartheta_o \;, \; \sigma_{xz} = 0 \qquad (7)$$

Then, applying the negative of σ_{yz} to the crack surface in the strip whose boundaries at $y' = \pm\eta h$ are free from displacement, enables us to express the governing equation and the boundary condition as

The governing equation

$$\left.\begin{aligned} &\frac{\partial^2 w}{\partial x'^2} + \frac{1}{\eta^2}\frac{\partial^2 w}{\partial y'^2} + \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \\ &\textit{The boundary equation} \\ &\frac{\partial w}{\partial y}(x, 0) = -\frac{\vartheta_0}{\eta h} && -\infty < x < 0 \\ &w(x, 0) = 0 && 0 < x < \infty \\ &w(x, \pm\eta h) = 0 && -\infty < x < \infty \end{aligned}\right\} \qquad (8)$$

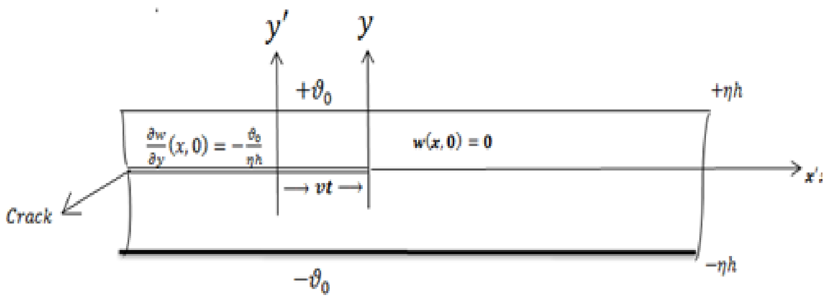

Figure 1. infinite crack moving in a long strip under anti-plane shear.

3. The Wiener- Hopf technique

We introduce Fourier transform in x as :

$$F\left\{f(x,y)\right\} = \bar{F}(\omega,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,y)\exp(i\omega x)\,dx \tag{9}$$

$$F^{\bullet}\left\{\bar{F}(x,y)\right\} = f(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{F}(\omega,y)\exp(-i\omega x)\,d\omega \tag{10}$$

where $\omega = \sigma + i\tau$ is the complex variable in the Fourier transform plane.
Applying eqn. (9) to eqn.(8), we have

$$\frac{\partial^2 \bar{W}(\omega,y)}{\partial y^2} - \omega^2 \bar{W}(\omega,y) = 0 \tag{11}$$

Solving eqn. (11), we have

$$\bar{W}(\omega,y) = A(\omega)\cosh\omega y + B(\omega)\sinh\omega y \tag{12}$$

We determine $A(\omega)$ and $B(\omega)$ by method of Wiener-Hopf.

Consequently, we introduce two unknown functions as follows

$$\frac{\partial w(x,0)}{\partial y} = \tilde{g}(x) \text{ for } 0 < x < \infty \tag{13}$$

$$w(x,0) = \tilde{l}(x) \text{ for} -\infty < x < 0 \tag{14}$$

To determine $A(\omega)$, $B(\omega)$, we transform the boundary conditions.

$$\bar{W}(\omega,\pm\eta h) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x,\pm\eta h)e^{i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 0.e^{i\omega x}dx = 0 \tag{15}$$

But from eqn. (12)

$$0 = \bar{W}(\omega,\pm\eta h) = A(\omega)\cosh\omega\eta h + B(\omega)\sinh\omega\eta h$$

Therefore

$$B(\omega) = -\frac{A(\omega)\cosh\omega\eta h}{\sinh\omega\eta h} = -A(\omega)\coth\omega\eta h \tag{16}$$

$$\bar{W}(\omega,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x,0)e^{i\omega x}dx = N_{-}(\omega) \tag{17}$$

where

$$N_{-}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \tilde{l}(x)e^{i\omega x}dx \tag{18}$$

$$\therefore \bar{W}(\omega,0) = N_{-}(\omega) \tag{19}$$

But from eqn.(13)

$$\bar{W}(\omega,0) = A(\omega)\cosh 0 + B(\omega)\sinh 0 = A(\omega)$$

Therefore from eqn.(19)

$$A(\omega)=N_{-}(\omega) \tag{20}$$

$$\frac{\partial\bar{W}(\omega,0)}{\partial y}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{\partial w(x,0)}{\partial y}e^{i\omega x}dx+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\partial w(x,0)}{\partial y}e^{i\omega x}dx$$

$$=-\frac{\vartheta_0}{\eta h\sqrt{2\pi}}\frac{e^{i\omega x}}{i\omega}\Big|_{x=-\infty}^{x=0}+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\tilde{g}(x)e^{i\omega x}dx=-\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}+M_{+}(\omega) \tag{21}$$

where

$$M_{+}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\tilde{g}(x)e^{i\omega x}dx \tag{22}$$

Also if we differentiate eqn.(13), we have

$$\frac{\partial\bar{W}(\omega,y)}{\partial y}=-\omega A(\omega)\sinh\omega y+\omega B(\omega)\cosh\omega y \tag{23}$$

Therefore

$$\frac{\partial\bar{W}(\omega,0)}{\partial y}=-\omega A(\omega)\sinh 0+\omega B(\omega)\cosh 0=\omega B(\omega) \tag{24}$$

Therefore from eqn.(21)

$$\omega B(\omega)=-\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}+M_{+}(\omega) \tag{25}$$

But from eqn. (17)

$$B(\omega)=-A(\omega)\coth\omega\eta h=-N_{-}(\omega)\coth\omega\eta h \tag{26}$$

Therefore

$$\omega B(\omega)=-\omega N_{-}(\omega)\coth\omega\eta h \tag{27}$$

Thus from eqns.(2) and (27)

$$-\omega N_{-}(\omega)\coth\omega\eta h=-\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}+M_{+}(\omega)$$

Rearranging, we have

$$M_{+}(\omega)+\omega\coth(\omega\eta h)N_{-}(\omega)-\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}=0 \tag{28}$$

This is wiener-Hopf equation with kernel $K(\omega)$ given as

$$K(\omega)=\omega\coth(\omega\eta h) \tag{29}$$

The solution of the problem reduces now to the determination of $M_{+}(\omega)$ and $N_{-}(\omega)$. Since the Wiener-Hopf equation by itself does not determine $M_{+}(\omega)$ or $N_{-}(\omega)$ uniquely, auxiliary conditions must be found from the analytic properties of the Fourier transforms $M_{+}(\omega)$ and $N_{-}(\omega)$ and from the asymptotic behavior of their inverses as $x\to\pm\infty$

From the configuration of the problem it is reasonable to assume that these functions are exponentially bounded at infinity and this ensures the existence of their Fourier transform. In particular, from theorem 1 below

Theorem 1

Let $\bar{F}(\omega)$ be the Fourier transform of $f(x)$ whose

$$f(x)=\begin{cases}g(x) & x>0\\ 0 & x<0\end{cases} \tag{30}$$

So that

$$\bar{F}(\omega)=\int_{-\infty}^{\infty}f(x)e^{i\omega x}dx=\int_{0}^{\infty}\left\{g(x)e^{(-\tau x)}\right\}e^{i\alpha x}dx \tag{31a}$$

With $\omega=\alpha+i\tau$

Suppose there is a τ_{+}

Such $g(x)$ is $O\left(e^{\tau_+ x}\right)$ as $x \to \infty$

Then

(i) $\bar{F}(\omega)$ is a regular function of ω in the half plane $\operatorname{Im}(\omega) = \tau > \tau_+$

And

(ii) $\bar{F}(\omega) \to 0$ as $|\omega| \to \infty$ in $\tau > \tau_+$

Therefore, if $M_+(\omega,0)$ is the Fourier transform of

$$M_+(x,0) = \begin{cases} \tilde{g}(x,0) & x > 0 \\ 0 & x < 0 \end{cases} \tag{31b}$$

We may state as follows that

(iii) $M_+(\omega)$ is regular in the upper half plane $\tau > \tau_+$ and

(iv) $M_+(\omega) \to 0$ as $|\omega| \to \infty$

Similarly

(v) $N_-(\omega)$ is regular in the lower half plane $\tau < \tau_-$ and

(vi) $N_-(\omega) \to 0$ as $|\omega| \to \infty$

5. PRODUCT DECOMPOSITION OF $K(\omega)$

Having a hyperbolic function, enables the kernel to be splitted in the desired product form

$$K(\omega) = K_+(\omega) K_-(\omega) \tag{32}$$

From [8]

$$K(\omega) = K_+(\omega) K_-(\omega) = \pi\omega \coth \pi\omega \tag{33}$$

Where, by means of Gamma function

$$K_+(\omega) = \frac{\sqrt{\pi}\,\Gamma(1 - i\omega)}{\Gamma\left(\frac{1}{2} - i\omega\right)} \tag{34}$$

$$K_-(\omega) = K_+(-\omega) \tag{35}$$

For the problem at hand

$$K(\omega) = \omega \coth \omega\eta h = \frac{1}{\eta h}\pi\left(\frac{\omega\eta h}{\pi}\right)\coth \pi\left(\frac{\omega\eta h}{\pi}\right) \tag{36}$$

Letting $\gamma = \dfrac{\omega\eta h}{\pi}$

$$K(\gamma) = \frac{1}{\eta h}\pi\gamma \coth \pi\gamma = \frac{1}{\eta h}\frac{\sqrt{\pi}\,\Gamma(1 - i\gamma)}{\Gamma\left(\frac{1}{2} - i\gamma\right)}.\frac{\sqrt{\pi}\,\Gamma(1 + i\gamma)}{\Gamma\left(\frac{1}{2} + i\gamma\right)} = K_+(\gamma).K_-(\gamma) \tag{37}$$

Where

$$K_+(\gamma) = \frac{\sqrt{\pi}}{\eta h}\frac{\Gamma(1 - i\gamma)}{\Gamma\left(\frac{1}{2} - i\gamma\right)} \quad and \quad K_-(\gamma) = \sqrt{\pi}\,\frac{\Gamma(1 + i\gamma)}{\Gamma\left(\frac{1}{2} + i\gamma\right)} \tag{38}$$

The equation (28) in view of eqn.(35) can be written as

$$\frac{M_+(\omega)}{K_+(\omega)} - \frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\frac{1}{K_+(\omega)} = -K_-(\omega) N_-(\omega) \tag{39}$$

6. SUM DECOMPOSITION OF $E(\omega)$

The second term in eqn.(39)

$$E(\omega) = -\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\frac{1}{K_+(\omega)} \tag{40}$$

Can be decomposed in the form

$$E(\omega) = E_+(\omega) + E_-(\omega) \tag{41}$$

That is

$$E(\omega) = -\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\frac{1}{K_+(\omega)} = -\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\left[\frac{1}{K_+(\omega)} - \frac{1}{K_+(0)} + \frac{1}{K_+(0)}\right]$$

$$= -\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\left[\frac{1}{K_+(\omega)} - \frac{1}{K_+(0)}\right] - \frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\frac{1}{K_+(0)} \tag{42}$$

$$E(\omega) = E_+(\omega) + E_-(\omega) \tag{43}$$

Where

$$E_+(\omega) = -\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\left[\frac{1}{K_+(\omega)} - \frac{1}{K_+(0)}\right] \tag{44}$$

$$E_-(\omega) = -\frac{\vartheta_0}{i\omega\eta h\sqrt{2\pi}}\frac{1}{K_+(0)} \tag{45}$$

From eqn.(38)

$$K_+(0) = \frac{\sqrt{\pi}}{\eta h}\frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{\eta h}\frac{1}{\sqrt{\pi}} = \frac{1}{\eta h} \tag{46}$$

Therefore

$$\frac{1}{K_+(0)} = \eta h \tag{47}$$

From analytic continuation and the generalized Liouville's theorem, eqn.(39) by virtue of eqn.(40) becomes [9].

$$G(\omega) = \frac{M_+(\omega)}{K_+(\omega)} + E_+(\omega) = -K_-(\omega)N_-(\omega) - E_-(\omega) \tag{48}$$

In the above equation, the first member is analytic in the upper half plane $\text{Im}(\omega) = \tau > \max(\tau_1, \tau_2) < 0$ and the second member in the lower half plane $\text{Im}(\omega) = \tau < 0$. Therefore, the region of analysis overlap and invoking analytic continuation it is concluded that $G(\omega)$ is analytic and entire in the whole ω-plane. The entire function $G(\omega)$ Must vanish by the generalized Liouville's theorem.

$$M_+(\omega) = \frac{\vartheta_o K_+(\omega)}{i\omega\eta h\sqrt{2\pi}}\left[\frac{1}{K_+(\omega)} - \frac{1}{K_+(0)}\right] \tag{49}$$

$$N_-(\omega) = \frac{\vartheta_o}{i\omega\eta h\sqrt{2\pi}}\frac{1}{K_-(\omega)K_+(0)} \tag{50}$$

In linear elastic fracture mechanics(LEFM), the main interest is the near crack-tip stress and displacement field, Which necessitates finding the asymptotic forms of $K_+(\omega)$, $K_-(\omega)$ *and* $K_+(0)$

From the asymptotic behavior of the Gamma function. [10,11].

$$\Gamma(1+z) = \sqrt{2\pi}\, z^{\frac{1}{2}} z^z e^{-z} \tag{51}$$

7. EVALUATION OF $M_+(\omega)$

Recall that

$$M_+(\gamma)=\frac{\vartheta_o K_+(\gamma)}{i\omega\eta h\sqrt{2\pi}}\left[\frac{1}{K_+(\gamma)}-\frac{1}{K_+(0)}\right]$$

$$=\frac{\vartheta_o}{i\omega\eta h\sqrt{2\pi}}\left[1-\eta h K_+(\gamma)\right] \tag{52}$$

But

$$K_+(\gamma)=\frac{\sqrt{\pi}}{\eta h}\frac{\Gamma(1-i\gamma)}{\Gamma\left(\frac{1}{2}-i\gamma\right)} \tag{53}$$

Now

$$\Gamma(1-i\gamma)=\Gamma(1+0-i\gamma)=\Gamma(1+z) \tag{54}$$

where $z=0-i\gamma$

$$\Gamma\left(\frac{1}{2}-i\gamma\right)=\Gamma\left(1-1+\frac{1}{2}-i\gamma\right)=\Gamma\left(1-\frac{1}{2}-i\gamma\right)=\Gamma(1+v) \tag{55}$$

where

$$v=-\frac{1}{2}-i\gamma$$

Now using the Stirling's asymptotic expansion of the gamma function,

$$\underset{\gamma\to\infty}{G(\gamma)}=\frac{\Gamma(1-i\gamma)}{\Gamma\left(\frac{1}{2}-i\gamma\right)}=\frac{\Gamma(1+z)}{\Gamma(1+v)}=\frac{\sqrt{2\pi}\,z^{\frac{1}{2}}z^z e^{-z}}{\sqrt{2\pi}v^{\frac{1}{2}}\,v^v e^{-v}}=\frac{e^{i\frac{3\pi}{4}}\gamma^{\frac{1}{2}}(-i\gamma)^{-i\gamma}e^{i\gamma}}{e^{i\frac{3\pi}{4}}\gamma^{\frac{1}{2}}e^{i\frac{5\pi}{4}}\gamma^{-\frac{1}{2}}(-i\gamma)^{-i\gamma}e^{\frac{1}{2}}e^{i\gamma}}=\frac{e^{-i\frac{5\pi}{4}}\gamma^{\frac{1}{2}}}{e^{\frac{1}{2}}}=e^{-i\frac{5\pi}{4}}\gamma^{\frac{1}{2}} \tag{56}$$

Therefore eqn. (38) becomes

$$K_+(\gamma)=\frac{\sqrt{\pi}}{\eta h}\frac{\Gamma(1-i\gamma)}{\Gamma\left(\frac{1}{2}-i\gamma\right)}=\frac{\sqrt{\pi}}{\eta h}\underset{\gamma\to\infty}{G(\gamma)}=\frac{\sqrt{\pi}}{\eta h}ie^{-i\frac{5\pi}{4}}\gamma^{\frac{1}{2}}$$

$$=\frac{\sqrt{\pi}}{\eta h}ie^{-i\frac{5\pi}{4}}\left(\frac{\omega\eta h}{\pi}\right)^{\frac{1}{2}} \qquad ,\gamma=\frac{\omega\eta h}{\pi} \tag{57}$$

Therefore

$$K_+(\omega)=\left(\frac{1}{\eta h}\right)^{\frac{1}{2}}ie^{-i\frac{5\pi}{4}}\omega^{\frac{1}{2}} \tag{58}$$

Hence substituting eqn.(58) for $M_+(\omega)$

$$\underset{\omega\to+\infty}{M_+(\omega)}=\frac{\vartheta_o K_+(\omega)}{i\omega\eta h\sqrt{2\pi}}\left[\frac{1}{K_+(\omega)}-\frac{1}{K_+(0)}\right]=\frac{\vartheta_o}{i\omega\eta h\sqrt{2\pi}}\left[1-\eta h K_+(\omega)\right]$$

$$=\frac{\vartheta_o}{i\omega\eta h\sqrt{2\pi}}\left[1-\eta h\left(\frac{1}{\eta h}\right)^{\frac{1}{2}}e^{-i\frac{5\pi}{4}}\omega^{\frac{1}{2}}\right]=\frac{\vartheta_o}{i\omega\eta h\sqrt{2\pi}}\left[1-(\eta h)^{\frac{1}{2}}e^{-i\frac{5\pi}{4}}\omega^{\frac{1}{2}}\right]$$

$$=\frac{\vartheta_o}{i\omega\eta h\sqrt{2\pi}}+\frac{\vartheta_o}{\sqrt{2\pi}}\left(\frac{1}{\eta h}\right)^{\frac{1}{2}}\omega^{-\frac{1}{2}}e^{-i\frac{5\pi}{4}} \qquad ,i=e^{i\frac{5\pi}{2}}$$

$$=-\vartheta_0\left(\frac{1}{2\pi\eta h}\right)^{\frac{1}{2}}e^{i\frac{5\pi}{4}}\omega^{-\frac{1}{2}} \tag{59}$$

8. Evaluation of $N_-(\omega)$

Recall that

$$N_-(\omega)=\frac{\vartheta_o}{i\omega\eta h\sqrt{2\pi}}\frac{1}{K_-(\omega)K_+(0)}$$

$$=\frac{\vartheta_o}{i\omega\sqrt{2\pi}}\frac{1}{K_-(\omega)}\quad,\qquad \frac{1}{K_+(0)}=\eta h \tag{60}$$

but

$$K_-(\gamma)=\sqrt{\pi}\frac{\Gamma(1+i\gamma)}{\Gamma\left(\frac{1}{2}+i\gamma\right)}=\sqrt{\pi}\frac{\Gamma(1+z)}{\Gamma(1+v)} \tag{61}$$

where $z=i\gamma,\ v=-\frac{1}{2}+i\gamma$

Therefore

$$\underset{\gamma\to\infty}{G(\gamma)}=\frac{\sqrt{2\pi}z^{\frac{1}{2}}e^{-z}}{\sqrt{2\pi}v^{\frac{1}{2}}v^{v}e^{-v}}=\frac{\sqrt{2\pi}(i\gamma)^{\frac{1}{2}}(i\gamma)^{i\gamma}e^{-i\gamma}}{\sqrt{2\pi}(i\gamma)^{\frac{1}{2}}(i\gamma)^{i\gamma}(i\gamma)^{-\frac{1}{2}}e^{\frac{1}{2}}e^{-i\gamma}}=\frac{1}{e^{\frac{1}{2}}(i)^{-\frac{1}{2}}(\gamma)^{-\frac{1}{2}}}$$

Thus

$$K_-(\gamma)=\frac{\sqrt{\pi}}{(i)^{-\frac{1}{2}}(\gamma)^{-\frac{1}{2}}}$$

$$\therefore\ \frac{1}{K_-(\gamma)}=\frac{(i)^{-\frac{1}{2}}(\gamma)^{-\frac{1}{2}}}{\sqrt{\pi}} \tag{62}$$

We now evaluate

$$N_-(\omega)=\frac{\vartheta_o}{i\omega\sqrt{2\pi}}\frac{1}{K_-(\omega)}=\frac{\vartheta_o}{i\omega\sqrt{2\pi}}\left(\frac{(i)^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}}{\sqrt{\pi}}\right)=\frac{\vartheta_o}{i\omega\sqrt{2\pi}}\left(\frac{(i)^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}}{\sqrt{\pi}}\right) \tag{63}$$

9. STRESS AND DISPLACEMENT FIELDS NEAR THE CRACK TIP

With the derivation of $M_+(\omega)$ and $N_-(\omega)$, we can determine the stress and displacement fields near the crack tip using the inverse Fourier transform.

9.1 Stress field near the crack tip

Recall from eqn.(22)

$$M_+(\omega)=\frac{1}{\sqrt{2\pi}}\int_0^\infty m(x)e^{i\omega x}dx \tag{64}$$

Using the inverse Fourier transform in eqn. (10)

We have

$$m(x)=\frac{1}{\sqrt{2\pi}}\int_0^\infty M_+(\omega)e^{-i\omega x}d\omega=\frac{1}{\sqrt{2\pi}}\int_0^\infty\left(\vartheta_0\left\{\frac{1}{2\pi\eta h}\right\}^{\frac{1}{2}}e^{i\frac{5\pi}{4}}\omega^{-\frac{1}{2}}\right)e^{-i\omega x}d\omega \tag{65}$$

From eqn.(13) i.e.

$$\frac{\partial w(x,0)}{\partial y}=\tilde{g}(x)\ \text{for}\ 0<x<\infty \tag{66}$$

$$\lim_{x\to+0}\frac{\partial w(x,0)}{\partial y}=\tilde{g}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\vartheta_0\left\{\frac{1}{2\pi\eta h}\right\}^{\frac{1}{2}}e^{i\frac{5\pi}{4}}\omega^{-\frac{1}{2}}\right)e^{-i\omega x}d\omega$$

$$=\frac{2\vartheta_0}{\sqrt{2\pi}}\left\{\frac{1}{2\pi\eta h}\right\}^{\frac{1}{2}}e^{i\frac{5\pi}{4}}\int_0^{\infty}\omega^{-\frac{1}{2}}e^{-i\omega x}d\omega \qquad (67)$$

To evaluate $\int_0^{\infty}\omega^{-\frac{1}{2}}e^{-i\omega x}d\omega$

Let $\omega=u^2 \;\therefore\; \omega^{-\frac{1}{2}}=\frac{1}{u},\; d\omega=2udu$

hence

$$\int_0^{\infty}\omega^{-\frac{1}{2}}e^{-i\omega x}d\omega=\int_0^{\infty}\frac{1}{u}e^{-iu^2x}2udu=2\int_0^{\infty}e^{-iu^2x}du=2\int_0^{\infty}e^{-ku^2}du \qquad where \quad k=ix$$

From table of integrals [12]

$$=2\left[\frac{1}{2}\sqrt{\frac{\pi}{k}}\right]=\sqrt{\frac{\pi}{ix}}$$

There eqn. (67) becomes

$$\lim_{x\to+0}\frac{\partial}{\partial y}w(x,0)=\frac{2\vartheta_0}{\sqrt{2\pi}}\left\{\frac{1}{2\pi\eta h}\right\}^{\frac{1}{2}}e^{i\frac{5\pi}{4}}\sqrt{\frac{\pi}{ix}}=\frac{2\vartheta_0}{2}\left\{\frac{1}{\pi\eta h}\right\}^{\frac{1}{2}}e^{i\frac{5\pi}{4}}(i)^{-\frac{1}{2}}x^{-\frac{1}{2}}$$

$$=\vartheta_0\left\{\frac{1}{\pi\eta h}\right\}^{\frac{1}{2}}e^{i\frac{5\pi}{4}}e^{-i\frac{5\pi}{4}}x^{-\frac{1}{2}}, \quad where \qquad (i)^{-\frac{1}{2}}=e^{-i\frac{5\pi}{4}}$$

$$\therefore \quad \lim_{x\to+0}\frac{\partial}{\partial y}w(x,0)=\vartheta_0\left\{\frac{1}{\pi\eta h}\right\}^{\frac{1}{2}}x^{-\frac{1}{2}} \qquad (68)$$

This is the required stress field at the crack tip.

10. STRESS INTENSITY FACTOR

From the linear elastic fracture mechanics, the stress intensity factor K_{111} is given by the formula

$$K_{111}=\lim_{x\to+0}\left[(2\pi x)^{\frac{1}{2}}\sigma_{yz}(x,0)\right] \qquad (69)$$

But from eqn. (2)

$$\sigma_{yz}=\mu\frac{\partial w}{\partial y} \qquad (70)$$

$$K_{111}=\lim_{x\to+0}\left[(2\pi x)^{\frac{1}{2}}\sigma_{yz}(x,0)\right]=\lim_{x\to+0}\left[(2\pi x)^{\frac{1}{2}}\mu\frac{\partial w}{\partial y}(x,0)\right]\therefore \quad K_{111}=\mu\vartheta_0\left(\frac{2}{\eta h}\right)^{\frac{1}{2}} \qquad (71)$$

This is the required stress intensity factor.

11. DISPLACEMENT FIELD NEAR THE CRACK TIP.

From equation (18)

$$N_-(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\tilde{l}(x)e^{i\omega x}dx$$

But from eqn. (63)

$$N_-(\omega)=\frac{\vartheta_o}{\omega\sqrt{2\pi}}\left(\frac{-(i)^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}}{\sqrt{\pi}}\right) \qquad (4.10)$$

From equation (19)

$$N_{-}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} w(x,0)e^{i\omega x}dx$$

Using inverse Fourier transform

$$w(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} N_{-}(\omega)e^{-i\omega x}d\omega=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-\vartheta_o}{\omega\sqrt{2\pi}}\left(\frac{(i)^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}}{\sqrt{\pi}}\right)e^{-i\omega x}d\omega$$

$$=\frac{2}{2\pi}\int_{0}^{\infty}\frac{-\vartheta_o}{\omega}\left(\frac{(i)^{-\frac{1}{2}}\left(\frac{\omega\eta h}{\pi}\right)^{-\frac{1}{2}}}{\sqrt{\pi}}\right)e^{-i\omega x}d\omega=\frac{-\vartheta}{\pi}(i)^{-\frac{1}{2}}(\eta h)^{-\frac{1}{2}}\int_{0}^{\infty}\omega^{-\frac{3}{2}}e^{-i\omega x}d\omega$$

But from table of integrals

$$\int_{0}^{\infty}\omega^{-\frac{3}{2}}e^{-i\omega x}d\omega=\frac{-2\sqrt{\pi}}{(ix)^{-\frac{1}{2}}}$$

Hence

$$w(x,0)=\frac{-\vartheta}{\pi}(i)^{-\frac{1}{2}}(\eta h)^{-\frac{1}{2}}\left[\frac{-2\sqrt{\pi}}{(ix)^{-\frac{1}{2}}}\right]=2\vartheta_0(\pi\eta h)^{-\frac{1}{2}}x^{\frac{1}{2}}$$

$$=2\vartheta_0\left(\frac{x}{\pi\eta h}\right)^{\frac{1}{2}}=\frac{2}{\sqrt{2}}\vartheta_0\left(\frac{2}{\pi\eta h}\right)^{\frac{1}{2}}x^{\frac{1}{2}}\qquad as\ x\to -0$$

$$\therefore\quad w(x,0)=\frac{\sqrt{2}}{\sqrt{\pi}}\vartheta_0\left(\frac{2}{\eta h}\right)^{\frac{1}{2}}x^{\frac{1}{2}}=\frac{\sqrt{2}}{\mu\sqrt{\pi}}K_{111}x^{\frac{1}{2}}$$

(4.11) This is the required displacement field near the crack tip.

12. CONCLUSION

The elastodynamic crack problem was solved in this present paper with the linear and homogeneous theory of elasticity. The problem is of practical interest since the cracked material have finite boundary, being of the form of long strips. As it can be evidently concluded the solution of the problem using the Wiener-Hopf technique is possible though cumbersome.With our technique we have been able to obtain the displacement and stress fields near the crack tip. The stress intensity factor is also obtained. The stress field possesses the inverse root singularity in line with crack problems. A similar conclusion was also stated by Georgiadis in his paper on complex variable and integral transform methods for elastodynamic solutions of cracked orthotropic strips.

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