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EXTENDED HYBRID TAU METHOD FOR SECOND ORDER IVPS IN ODES

E. Amuseghan and K.O. Muka

Department of Mathematics, University of Benin, Benin City.

Abstract

The article is concerned with the development of extended Hybrid Tau methods that integrate directly second order differential equations without necessarily reducing to system of first order differential equations before integration. Extended Hybrid Tau methods are derived in this article via collocation of the perturbed differential system approximate solution of second order differential equations on grid and off grid points. The proposed methods enjoy relatively improved accuracy when compared to the approximate solution generated using two existing methods, in terms of absolute errors.

Keyword: Tau, Extended hybrid.

1. INTRODUCTION

Consider the second order initial value problems (IVPs) in ordinary differential equations (ODEs)

$$y'' = f(x,y,y'), y(a) = \eta,\ y'(a) = \xi; f: \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m, y: \mathbb{R} \to \mathbb{R}^m, x \in [a,b] \qquad (1)$$

Closed form solution of (1) are often cumbersome or impossible to obtain, thereby making validation and studies of most Mathematical models that are formulated using second order IVPs in ODEs (1) difficult. In order to overcome this difficulty occasioned by non-availability of closed form solution, numerical solution of (1) is desired using numerical method such as Tau method, Linear multistep methods, Runge- Kutta method [1 – 3].

The Tau method assumes an approximate polynomial solution $y_n(x)$, and a perturbation term $H_n(x)$ added to the right-hand side of equation of the linear differential equations as follows:

First consider the second order IVPs in ODEs of the form

$$y'' = f(x,y), y(a) = \eta,\ y'(a) = \xi, x \in [a,b] \qquad (2)$$

This can be rewritten as $D^2y(x) = f(x,y), \quad x \in [a,b],\ yE\mathbb{R}$ (3)

By adding the term $H_n(x)$ to (3) we obtain

$$D^2y_n(x) = f(x,y) + H_n(x),\ x \in [a,b], \qquad (4)$$

where

$$y_n(x) = \sum_{i=0}^{n} a_i x^i, \text{ and } H_n(x) = \sum_{i=0}^{n} r_i T_i(x).$$

The coefficients $a_i \in \mathbb{R}$,and $r_{i,}\ \ i = 0,1,2,3,\dots,n$ in (4) are Tau parameters to be determined. $T_i(x)$, is any of the orthogonal polynomial such as Chebyshev, Legendre or Hermite polynomials [4].

Considered Tau method for the numerical solution (1)
which has the solution of the form

$$y_n(x) = \sum_{j=0}^{n} a_j Q_j(x),\ a \le x \le b \qquad (5)$$

Define the linear differential operator, D, as:

$$D = \frac{d^2}{dx^2} + 1, \qquad (6)$$

D is assumed invertible [4].

The Tau method in a perturbed collocation approach is given as

$$\sum_{j=0}^{n} j(j-1)Q_j(x)^{j-2} = f(x,y) + \sum_{j=0}^{n} r_{j+1}p_{j+1}(x) \qquad (7)$$

where $\sum_{j=0}^{n} r_{j+1} p_{j+1}(x)$ is the perturbation term, $p_{j+1}(x)$is the *(j+1)th* degree Legendre or Chebyshev polynomial valid for $x \in$

Correspondence Author: Amuseghan E., Email: edamisanamu@yahoo.com, Tel: +2348056793532

$[a,b]$. $Q_j(x)$is a canonical polynomial, r_{j+1}, j = 0, 1, 2, …, n are parameters to be determined, and $Q_j(x)$ is a canonical polynomial obtained by the use of (6) on x^n, to have

$$Dx^n = n(n-1)DQ_{n-2}(x) + DQ_n(x) \qquad (8)$$

where

$$DQ_n(x) = x^n, \; n = 0,1,2,\dots$$

Which reduces to

$$Q_n(x) = x^n - n(n-1)Q_{n-2}(x), \; n = 0,1,\dots \qquad (9)$$

The recursive formular (9) is used to develop Tau method.
The case $n = 2$ in (5) is considered in [4] to have,

$$y_2(x) = a_0Q_0(x) + a_1Q_1(x) + a_2Q_2(x), \qquad (10)$$

Substituting the values of Q_0,$Q_1$and Q_2 into (9) yields the solution of the form

$$y_2(x) = a_0 + a_1x + a_2(x^2-2), \qquad (11)$$

differentiating (11) twice and substituting the result into (7) to obtain

$$y''_2(x) = 2a_2 = f(x,y) + \sum_{j=0}^{n} r_{j+1}p_{j+1}(x) \qquad (12)$$

For $n = 1$, (12) becomes

$$2a_2 = f(x,y) + r_1p_1(x) + r_2p_2(x) \,. \qquad (13)$$

The (13) is collocated at the grid points x_{n-1}, x_n and x_{n+1}within the interval $[-1,1]$ to have

$$\left.\begin{aligned} 2a_2 &= f_{n-1} + r_1p_1(x_{n-1}) + r_2p_2(x_{n-1}) \\ 2a_2 &= f_n + r_1p_1(x_n) + r_2p_2(x_n) \\ 2a_2 &= f_{n+1} + r_1p_1(x_{n+1}) + r_2p_2(x_{n+1}) \end{aligned}\right\} \qquad (14)$$

and interpolated (11) at $x_{n-1}$and x_n to have

$$\left.\begin{aligned} a_0 + a_1x_{n-1} + a_2(x_{n-1}^2 - 2) &= y_{n-1} \\ a_0 + a_1x_n + a_2(x_n^2 - 2) &= y_n \end{aligned}\right\} \qquad (15)$$

The equations (14) and (15), yield a system of linear equation.

$$\left.\begin{aligned} a_0 + a_1x_{n-1} + a_2(x_{n-1}^2 - 2) &= y_{n-1} \\ a_0 + a_1x_n + a_2(x_n^2 - 2) &= y_n \\ 2a_2 - r_1p_1(x_{n-1}) - r_2p_2(x_{n-1}) &= f_{n-1} \\ 2a_2 - r_1p_1(x_n) - r_2p_2(x_n) &= f_n \\ 2a_2 - r_1p_1(x_{n+1}) - r_2p_2(x_{n+1}) &= f_{n+1} \end{aligned}\right\} \qquad (16)$$

The values of p_n(x) in (16) are obtained through transforming the interval
$x \in [-1,1]$ for $x \in [x_{n-1}, x_{n+1}]$, to have

$$p_n(x) = \frac{2x-(a+b)}{b-a}, x \in [a,b], \qquad (17)$$

with $a = x_{n-1}$ and $b = x_{n+1}$, to have

$$p_n(x) = \frac{2x-(x_{n-1}+x_{n+1})}{x_{n+1}-x_{n-1}}, x \in [x_{n-1}, x_{n+1}]. \qquad (18)$$

$h = X_{n+1} - X_n$

Using the Legendre polynomial, $p_n(x)$, to have

$$p_1(x_{n-1}) = \frac{2x_{n-1}-(x_{n-1}+x_{n+1})}{x_{n+1}-x_{n-1}} = -1. \qquad (19)$$

Similarly, $p_1(x_n) = 0, p_1(x_{n+1}) = 1$, and from Legendre polynomial,

$$p_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$p_2(x) = \frac{3}{2}\left(\frac{2x_n-(x_{n+1}+x_{n-1})}{x_{n+1}+x_{n-1}}\right)^2 - \frac{1}{2} = -\frac{1}{2} \qquad (20)$$

Similarly, $p_2(x_{n-1}) = 1, p_2(x_{n+1}) = 1$. (21)

Substituting $p_1(x_n), p_1(x_{n+1}), p_1(x_{n-1}), p_2(x_{n-1})$, $p_2(x_n)$ and $p_2(x_{n+1})$ into (16) to obtain

$$\left.\begin{aligned} a_0 + a_1x_{n-1} + a_2(x_{n-1}^2 - 2) &= y_{n-1} \\ a_0 + a_1x_n + a_2(x_n^2 - 2) &= y_n \\ 2a_2 + r_1 - r_2 &= f_{n-1} \\ 2a_2 + \tfrac{1}{2}r_2 &= f_n \\ 2a_2 - r_1 - r_2 &= f_{n+1} \end{aligned}\right\} \qquad (22)$$

The system of linear equations (22) can be solved to obtain values for a_k, $k = 0,1,2$ and r_n, $n = 1,2$, substituting the values obtained into (9) gives

$$y(x) = y_n + \frac{(x-x_n)}{h}(y_{n+1} - y_n) + \frac{(x-x_{n+1})(x-x_n)}{12}[f_{n+2} + 4f_{n+1} + f_n] \qquad (23)$$

$x = x_{n+2}$, (23) becomes

$$y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{6}[f_{n+2}+4f_{n+1}+f_n] \quad (24)$$
Using Chebyshev polynomial, at $x=x_{n+2}$, to obtain
$$y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{4}[f_{n+2}+2f_{n+1}+f_n], \quad (25)$$
The (25) is derived in [5].

2. DERIVATION OF METHODS

The case n =3 in (5) is considered to have,
$$y_3(x)=a_0Q_0(x)+a_1Q_1(x)+a_2Q_2(x)+a_3Q_3(x) \quad (26)$$
From (9) $Q_0(x)=1$, $Q_1(x)=x, Q_2(x)=x^2-2$ and $Q_3(x)=x^3-6x$.
Therefore, (26) becomes
$$y_3(x)=a_0+a_1x+a_2(x^2-2)+a_3(x^3-6x), \quad x_n\le x\le x_{n+1} \quad (27)$$
Differentiating (27) twice and substituting the result into (7), gives
$$y_3''(x)=2a_2+6a_3x=f(x,y)+\sum_{j=0}^{n} r_{j+1}p_{j+1}(x) \quad (28)$$
For $n=3$, in (28),
$$2a_2+6a_3x=f(x,y)+r_1p_1(x)+r_2p_2(x)+r_3p_3(x) \quad (29)$$
Rearranging (29), to have:
$$2a_2+6a_3x-r_1p_1(x)-r_2p_2(x)-r_3p_3(x)=f(x,y) \quad (30)$$
Collocating (30) at,
$x_n, x_{n+1}, x_{n+3/2}, x_{n+2}$ and x_{n+3} to have
$$\left.\begin{array}{l}
2a_2+6a_3x_n-r_1p_1(x_n)-r_2p_2(x_n)-r_3p_3(x_n)=f_n\\
2a_2+6a_3x_{n+1}-r_1p_1(x_{n+1})-r_2p_2(x_{n+1})-r_3p_3(x_{n+1})=f_{n+1}\\
2a_2+6a_3x_{n+3/2}-r_1p_1(x_{n+3/2})-r_2p_2(x_{n+3/2})-r_3p_3(x_{n+3/2})=f_{n+3/2}\\
2a_2+6a_3x_{n+2}-r_1p_1(x_{n+2})-r_2p_2(x_{n+2})-r_3p_3(x_{n+2})=f_{n+2}\\
2a_2+6a_3x_{n+3}-r_1p_1(x_{n+3})-r_2p_2(x_{n+3})-r_3p_3(x_{n+3})=f_{n+3}
\end{array}\right\} \quad (31)$$

The values for $P_n(x)$ for (31) are obtained through
$p_n(x)=\dfrac{2x-[a+b]}{b-a} \quad x\in[a,b]$, and let a = x_n, b = x_{n+3}, to have
$$p_n(x)=\frac{2x-[x_n+x_{n+3}]}{x_{n+3}-x_n}, \qquad n=1,2,3 \quad x\in[x_n,x_{n+3}] \quad (32)$$
Simplifying (32), to have $p_n(x)=\dfrac{2x-2x_n}{3h}-1$,

Using the Legendre polynomial to obtained $p_1(x_n)=-1$ and $p_1(x_{n+1})=\dfrac{-1}{3}$, $P_2(x), P_3(x)$ are obtained at

$x=x_n, x_{n+1}, x_{n+3/2}, x_{n+2}, and\ x_{n+3}$

Substituting all the values of $p_n(x)'s$ required into (31), to obtain the system of linear equations
$$\begin{pmatrix}
2 & 6x_n & 1 & -1 & 1\\
2 & 6x_{n+1} & 1/3 & 1/3 & -11/27\\
2 & 6x_{n+3/2} & -1/3 & 1/3 & 11/27\\
2 & 6x_{n+2} & -1/3 & 1/3 & 11/27\\
2 & 6x_{n+3} & -1 & -1 & -1
\end{pmatrix}
\begin{pmatrix} a_2\\ a_3\\ r_1\\ r_2\\ r_3\end{pmatrix}=
\begin{pmatrix} f_n\\ f_{n+1}\\ f_{n+3/2}\\ f_{n+2}\\ f_{n+3}\end{pmatrix} \quad (33)$$
Interpolate (27) at $x=x_n$ and $x=x_{n+1}$ to have

$$\begin{pmatrix} 1 & x_n \\ 1 & x_{n+1} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} y_n - a_2\left(x_n^2 - 2\right) - a_3\left(x_n^3 - 6x_n\right) \\ y_{n+1} - a_2\left(x_{n+1}^2 - 2\right) - a_3\left(x_{n+1}^3 - 6x_{n+1}\right) \end{pmatrix} \quad (34)$$

the values of $a_0, a_1, a_2,$ and a_3 are determined using equation (33) – (34) substituting same into (27), simplifying and evaluating at some grid/off grid point to have the Methods:

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{8}\left[f_n + 3f_{n+1} + 8f_{n+3/2} - 5f_{n+2} + f_{n+3} \right] \quad (35)$$

Repeating the procedure using Chebyshev polymonial yields

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{32}\left[7f_n + 9f_{n+1} + 32f_{n+3/2} - 23f_{n+2} + 7f_{n+3} \right] \quad (36)$$

Method (35) and (36) are both of order 2.

Observe that to evaluate f_{n+3}, the point x_{n+3} and y_{n+3} are required. Method (35) and (36) are called extended methods. Therefore, for reference purposes (35) and (36) are also known to be Extended Hybrid Tau Methods.

3. STABILITY ANALYSIS

The roots of the First characteristics equation of each proposed methods (34) and (35) can be verified to be zero-stable. Applying the methods (35), (36) to the scalar test

$$y'' = -\lambda^2 y, \quad \lambda, y \in \mathbb{R} \quad (37)$$

Yields the stability polynomials

$$\Pi(r, z) = \rho(r) - z\sigma(r) = 0 \quad (38)$$

where $z = \lambda^2 h^2$

To determine the region of absolute stability R_A of the proposed methods with associated the first and second characteristic polynomials, adopt the boundary locus method [1], [6]:

Thus, redefine (38) in terms of z $=\exp(i\theta)$, as follows

$$\Pi\left(\exp(i\theta), z\right) = \rho\left(\exp(i\theta)\right) - z\sigma\left(\exp(i\theta)\right) = 0 \quad (39)$$

So that, the locus of the boundary is given by

$$y(\theta) = \frac{\rho\left(\exp(i\theta)\right)}{\sigma\left(\exp(i\theta)\right)} \quad (40)$$

The plots of stability domains are done with the aid of the boundary locus methods using MATHEMATICA 10 software. The Fig. 1 and Fig. 2 are the plots of the stability domains of the proposed methods(35) and (36).

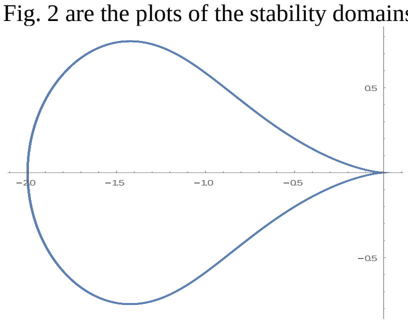

Fig. 1: R_A of Method (35)

The interval of absolute stability is (-2, 0)

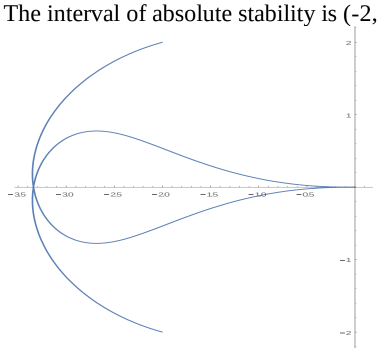

Fig. 2: R_A of Method (36)

The interval of absolute stability is (-3.4, 0)
The enclosed regions in Fig. 1and Fig 2 are the stability region R_A, while the exteriors are the unstable region. Methods (35) and (36) are both convergent.

4. NUMERICAL EXPERIMENTS

Proposed Methods (35) and (36) are tested on standard problems, and the results obtained are compared with those of two existing methods.
Well-known methods that are of order P=2 are given by

(a) $y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{4}[f_{n+2} + 2f_{n+1} + f_n],$ (41)

(b) $y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{6}[f_{n+2} + 4f_{n+1} + f_n],$ (42)

respectively.

The methods (35) and (36) with method (41), and (42) are applied to solve Linear and Non-Linear Second Order IVPs in ODEs. The numerical results are presented in Tables 1 – 5.
The proposed methods along with the two (2) existing methods are programmed in MATLAB 2007b for implementation.

Problem 1, [7]:

$y'' = y + e^{3x}$ $y(0) = -\frac{3}{2},$ $x \in [0,1]$ $y'(0) = -\frac{3}{32},$ $h = \frac{0.1}{40},$

Exact solution $y(x) = \frac{4x-3}{32\exp(-3x)}$

Table 1: Numerical solution to problem 1

$\lvert\bar{y}(x) - y(x)\rvert$				
Proposed method			Existing method	
Grid Points (x)	Method (35)	Method (36)	Method (41)	Method (42)
0.0025	5.70867495796050e-06	5.70866164778693e-06	5.70867941782072e-06	5.70867644454631e-06
0.25	1.24043066632162e-05	1.24043011995328e-05	1.24043163540477e-05	1.24043098934934e-05
0.5	2.71352479924991e-05	2.71352397916735e-05	2.71352691918469e-05	2.71352550589299e-05
0.70	5.07196105660676e-05	5.07196100032034e-05	5.07196501906698e-05	5.07196237742244e-05
1	1.29457663339161e-04	1.29457615047584e-04	1.29457764477703e-04	1.29457697051860e-04

Problem 2, [8].

$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0,$ $x \in [1,3],$ $y(1) = 1,$ $y'(1) = 1,$ $h = \frac{0.1}{32},$

Exact solution $y(x) = \frac{5}{3x} - \frac{2}{3x^2}$

Table 2: Numerical solution showing the absolute error to problem 2

$\lvert\bar{y}(x) - y(x)\rvert$				
Proposed method			Existing method	
Grid Points (x)	Method (35)	Method (36)	Method (41)	Method (42)
1.00315	9.75516448409319e-06	9.75500414815129e-06	9.75583550781924e-06	9.75584171523160e-06
1.3125	8.62147472968644e-06	8.62140813807195e-06	8.62494196916000e-06	8.68090131329000e-06
1.9375	6.66623797374033e-07	6.66623645262376e-07	7.79622907232103e-07	7.79622583357925e-07
2.6250	5.25557802144583e-07	5.25557727448778e-07	7.02724034629210e-07	7.02124366585845e-07
3	4.70197026347652e-07	4.70196980562054e-07	5.62627990308151e-07	5.62628193812031e-07

Problem 3, [5].

$y'' = 2Cos\,x^2 - 4x^2y$ $x \in [0,1],$ $y(0) = 0,$ $y'(0) = 0,$ $h = 0.001,$

Exact solution $y(x) = sinx^2$

Table 3: Numerical solution to problem 3

$\lvert\bar{y}(x) - y(x)\rvert$				
Proposed method			Existing method	
Grid Points (x)	Method (35)	Method (36)	Method (41)	Method (42)
0.0025	1.00009358325152e-09	1.00009348325150e-09	2.499997495001119e-07	2.49999749500108e-07
0.25	9.99992707736985e-09	9.99992607736980e-09	2.49974500838440e-07	2.49974500804017e-07
0.5	9.98335095842018e-08	9.98335094842010e-08	2.4700915483844e0-07	2.47009154935585e-07
0.75	8.41470810919454e-07	8.41450810919450e-07	3.47753803531248e-07	3.47753817964147e-07
1	9.09295873485760e-07	9.09293539175486e-07	1.77529941858268e-06	1.77529971445711e-06

Problem 4, [9]:

$y'' = -y,$ $y(0) = 1,$ $x \in [0,1],$ $y'(0) = 1,$ $h = 0.001,$

Exact solution $y(x) = \text{Cos}\,x + \text{Sin}\,x$

Table 4: Numerical solution to problem 4

$\lvert\bar{y}(x) - y(x)\rvert$				
Proposed method			Existing method	
Grid Points (x)	Method (35)	Method (36)	Method (41)	Method (42)
0.001	1.00099932372287e-06	1.00099918709518e-06	1.00099924948971e-06	1.00099933297848e-06
0.01	1.00994960742008e-06	1.00994951823807e-06	1.00994958129874e-06	1.00994966545365e-06
0.1	1.09483744514804e-06	1.09483723975679e-06	1.09483730814652e-06	1.09483739940686e-06
1	1.38177311792553e-06	1.38177285879948e-06	1.38177294517483e-06	1.38177306041598e-06

Problem 5, [7]:

$y'' = 2y^3, y(1) = 1, \quad x \in [1,2], \quad y'(1) = -1, \quad h = 0.001,$

Exact solution$y(x) = \frac{1}{x}$

Table 5: Numerical solution showing the absolute error to problem 5

$\lvert\bar{y}(x) - y(x)\rvert$				
Proposed method			Existing method	
Grid Points (x)	Method (35)	Method (36)	Method (41)	Method (42)
1.1	1.65239444031824e-006	1.65239331333085e-006	1.6523940464877e-006	1.65239356368614e-006
1.3	1.18343309063462e-006	1.18343240784746e-006	1.1834328630389e-006	1.18343255950393e-006
1.5	8.88889629679923e-007	8.88889185146624e-007	8.88889481465149e-007	8.88889283956493e-007
1.8	6.17284379278438e-007	6.17284122039763e-007	6.1728429356922e-007	6.17284179216249e-007
2	5.00000312486648e-007	5.00000124969979e-007	5.00000249981092e-007	5.0000166714365e-007

The computational results in Tables 1 – 5 show that the proposed methods are capable of solving standard Linear and Non-Linear second order IVPs in ODEs.

Tables 1 – 5 show the better performance of accuracy in term of numerical results of the absolute errors of proposed methods (35) and (36), over existing methods.

In tables 1 – 5 showing all problems 1 - 5 solved, the computed values are very close to the exact solutions via the small absolute error values; this indicating the excellent performance of all the methods computed, in term of accuracy. But the proposed methods, especially method (36), having the smallest absolute error values, performed better in term of accuracy.

5. CONCLUSION

Two Hybrid Tau Methods are developed. Tau methods that requires the use of future point in the approximation of second order IVP are developed, they have a blend of the characteristics of hybrid and extended Linear multistep methods. Numerical results shown in Tables 1-5 reveal that the proposed methods are suitable for integrating Linear and Non-Linear second order IVPs. The better performance of accuracy in terms of the numerical results of the absolute error of the proposed methods over two existing methods are presented. The proposed methods performed well in term of accuracy.

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