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STRATEGIC OPTIMAL PORTFOLIO MANAGEMENT FOR A DC PENSION SCHEME WITH RETURN OF PREMIUM CLAUSES

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Abstract

The optimal investment policy for a member in a defined contribution (DC) pension with return of premium clause was studied. This clause allows death members of the scheme to reclaim their contributions during the accumulation phase. A continuous time mean-variance stochastic optimal control problem was formulated with the help of the actuarial symbol. Investments in one risk-free asset (treasury) and two risky assets namely equity and loan were considered to help increase the accumulated fund of the remaining pension members in order to meet up with their retirement needs. Next, we established an optimization problem from the extended Hamilton Jacobi Bellman equation using the variational inequalities method and solved the optimization problem for the optimal investment policies of the three assets, efficient frontier of the pension members and the corresponding optimal fund size for investments. Furthermore, we present a numerical simulation of the optimal investment policies of the three assets with respect to time. We observed that the pension member prefers investment in stock as compared to loan; secondly we observed that optimal investment policy of the risky assets is inversely proportional to risk averse level, predetermined interest rate and the initial wealth.

Keywords: DC Pension Fund, HJB, Optimal Investment Policy, Variational InequalitiesMethods, Return of PremiumsClauses
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1. Introduction

The optimization theory in a stochastic frame work is a very vital instrument in the field of mathematical finance since it can be used in solving a large range of problems involving stochastic optimization. Most financial institutions such as insurance companies, banks, pension schemes have applied stochastic optimal control methods in solving optimization problems. In pension, a good number of authors have tried solving optimization problems for the optimal investment policies for a pension member with different portfolios. The study of portfolio optimization in pension fund system has grown over the years since it has a lot to do with in determining the old age income adequacy of retirees.

Presently, we have two types of pension schemes; namely are the defined benefit (DB) pension scheme and the defined contribution (DC) pension scheme. In the former scheme, members benefits are predetermined based on some basic requirements such as age, salary histories, years in service etc. their benefits is often times depends mostly on the efficiency of the employers contributions and based of the mode of contributions in this scheme, most private organization found it challenging in having a pension plan for their members as a result, this plan was only workable for members in government sectors. Since the contributions in this plans are employers dependent most members are prefers this plan but over the years, it has generate controversies and delay in payment of retirement benefits to its members and these has led to the introduction of the later plan which is mostly members dependent. This plan mandate members of the scheme to contribute a certain percentage of their earnings into their retirement serving account (RSA). In this plan, members are involved in the investment

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process and their retirement benefits depend upon the returns of the investments made during the accumulation phase and the expected return can be determined by some factors which include mortality risk, inflation, investment efficiency, government policy etc. Although the DC plan is very transparent, attractive and reliable compared to the DB plan, there is need for members to get acquainted with the financial market on how investments in different assets are carried out. This has led to the study of optimal investment policies by financial institutions and this study explains the proportion of the members' wealth to be invested among the available assets involved to give an optimal return with minimal risk.

Optimal investment policies with bounded risks, general utilities, and goal achieving were investigated in [1].In [2], the optimal investment strategy to DC pension members with asset, salary and interest rate risk, and propose a novel form of terminal utility function by incorporating habit formulation were investigated. it was proposed and investigated in [3] a model of optimal allocation for DC pension plan with a minimum guarantee in the continuous-time setting. Asset allocation problem under a stochastic interest rate was studied in [4]. In [5],it is shown that optimal investment decisions when time horizon is uncertain.the study of constant elasticity of Variance (CEV) model in DC pension fund investment strategies have taken centre stage in modelling the stock price. The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts was studied in [6]. Explicit solutions of the optimal investment strategy for investor with CRRA and CARA utility function were obtained in [7]. The optimal investment strategies in DC pension fund with multiple contributors using Legendre transformation method to obtain the explicit solution for CRRA and CARA were studied in [8]. In [9],stochastic strategies of optimal investment for DC pension fund with multiple contributors where they considered the rate of contribution to be stochastic was studied.In [10], portfolio problem by maximizing utility from terminal wealth with respect to a power utility function by studying the Heston model was investigated. In [11], the Heston's SV model to develop the reinsurance and investment problem under the mean-variance criterion was used. According to [12], The Mean-variance condition was first developed by Markowitz to investigate portfolio selection problem but the optimal investment policies under mean-variance criterion are not time consistent, because the mean-variance condition does not have the iterated expectation property hence the Bellman's principle of optimality does not hold. They stated that in most cases time consistency of strategies is a critical requirement for rational decision makers. The general theory of Markovian time inconsistent stochastic control problems was studied in [13]. The portfolio optimization with state-dependent risk aversion in the mean-variance framework was studied in [14].

Recently, some researchers have contributed to the study of optimal investment policies with refund of premium clause with some of them including[15], studied optimal investment strategy for a defined contribution pension scheme with the return of premiums clauses in a mean-variance utility function; in their work, they considered investment in bond and equity only and determined the optimal investment strategy for the risky asset as well as the efficient frontier. In Equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under constant elasticity of variance model had been investigated in [16]; here investment in a risk free asset ant two risky asset were consideredand assumed that the price process of the risky assets were modelled by constant elasticity of variance (CEV) model.In [12] the optimal time-consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts were investigated.Here investment in two assets, a risk free asset and a risky asset (stock) were considered and assumed the stock market price to follow Heston volatility model. Optimization problem with return of premium in a DC pension with multiple contributors was studied in [17]; where the fund manager deals with more than one contributor per time and also investment in two asset where the price of the risky asset followed CEV model. DC pension plan with the return of premium clauses under inflation risk and volatility risk was studied as in [18]; In their work, they assume investment in a risk free asset, the inflation index bond and the stock whose price is modelled by Heston volatility.

Throughout the literatures, no work has been done on optimal investment policy with refund of premium clause that considers investment in a treasury, equity and loan and as such this form the bedrock of our research work where we investigatestrategic optimal portfolio management for a DC pension scheme with return of premium clauses under mean variance utility function and assume the price of the risky assets follow the geometric Brownian motion.

2. Financial Market Model

Consider a financial market which is complete and frictionless and is continuously open over a predetermined time interval$t \in [0, T]$. T is the time frame of the accumulation phase.Assume(Ω, F, P) is a complete probability space whereΩ is a real space and P a probability measure, $\{B_0(t) : t \geq 0\}$ is a standard Brownian motion. $F$is the filtration and represent the information generated by the Brownian motion $\{W_1(t), W_2(t), W_3(t)\}$. Such that

$dW_1(t)\,dW_2 = dW_1(t)\,dW_3 = dW_2(t)\,dW_3 = 0$

Let $E_t^1(t)$, $E_t^2(t)$ and $E_t^3(t)$represent the price of the risk-free asset (cash) and the (equity) and that of loan respectively, and their models are given as follows:

$$\frac{dE_t^1(t)}{E_t^1(t)} = r_1 dt, \quad (1)$$
$$\frac{dE_t^2(t)}{E_t^2(t)} = (r_1 + k_1)dt + \sigma_1 dW_1(t). \quad (2)$$
$$\frac{dE_t^3(t)}{E_t^3(t)} = (r_1 + k_2)dt + \sigma_2 dW_2(t) + \sigma_3 dW_3(t) \quad (3)$$

See [19]

Where $r_1, k_1, k_2, \sigma_1, \sigma_2, \sigma_3$ are constant and r_1 is the risk-free interest rate, $(r_1 + k_1)$ is the expected instantaneous rate of return of equity and $(r_1 + k_2)$ is the expected instantaneous rate of return of loan and σ_1, σ_2 and σ_3 are the instantaneous volatility of equity and loan.

Also, let b be the contribution paid to the members retirement servings account (RSA) at a given time, ϑ_0 the initial age of accumulation phase and $\vartheta_0 + T$ is the end age, $\frac{1}{i}K_{\vartheta_0+t}$ is the mortality rate from time t to $t+\frac{1}{i}$, tb is the premium accumulated at time t, $tb\frac{1}{i}K_{\vartheta_0+t}$ is the returned premium to the death member's family.

Let μ_1, μ_2, and μ_3 represent the proportion of the members pension wealth to be invested in cash, stock and loan respectively such that $\mu_1 = 1 - \mu_2 - \mu_3$.

Since the surviving members will want to maximize the fund size and at the same time minimize the volatility of the accumulated wealth. There is need for the pension fund managers to formulate an optimal investment problem under the mean-variance criterion as follows:

$$\sup_{\mu}\{E_{t,l}L^{\mu}(T) - Var_{t,l}L^{\mu}(T)\} \quad (4)$$

Considering the time interval $[t, t+\frac{1}{i}]$, the differential form associated with the fund size when the remaining wealth is equally shared among the remaining members is given as:

$$L\left(t+\frac{1}{i}\right) = \left(L(t)\left(\mu_1\frac{E^1_{t+\frac{1}{i}}(t)}{E_t^1} + \mu_2\frac{E^2_{t+\frac{1}{i}}(t)}{E_t^2} + \mu_3\frac{E^1_{t+\frac{1}{i}}(t)}{E_t^1}\right) + b\left(\frac{1}{i}\right) + tb\frac{1}{i}K_{\vartheta_0+t}\right)\frac{1}{1-\frac{1}{i}K_{\vartheta_0+t}} \quad (5)$$

$$L\left(t+\frac{1}{i}\right) = \left(L(t)\left(1 + (1-\mu_2-\mu_3)\left(\frac{E^1_{t+\frac{1}{i}} - E_t^1}{E_t^1}\right) + \mu_2\left(\frac{E^2_{t+\frac{1}{i}} - E_t^2}{E_t^1}\right) + \mu_3\left(\frac{E^3_{t+\frac{1}{i}} - E_t^3}{E_t^2}\right)\right) + b\left(\frac{1}{i}\right) - tb\frac{1}{i}K_{\vartheta_0+t}\right)\left(1 + \frac{\frac{1}{i}K_{\vartheta_0+t}}{1-\frac{1}{i}K_{\vartheta_0+t}}\right) \quad (6)$$

The conditional death probability ${}_tq_x = 1 - {}_tp_x = 1 - e^{-\int_0^t \pi(\vartheta_0+t+s)\,ds}$, where $\pi(t)$ is the force function of the mortality at time t, and for $i \to \infty$,

$$\frac{1}{i}K_{\vartheta_0+t} = 1 - \exp\{-\int_0^{\frac{1}{i}}\pi(\vartheta_0+t+s)ds\} \approx \pi(\vartheta_0+t)\frac{1}{i} = O(\frac{1}{i})$$

$$\frac{\frac{1}{i}K_{\vartheta_0+t}}{1-\frac{1}{i}K_{\vartheta_0+t}} = \frac{1-\exp\{-\int_0^{\frac{1}{i}}\pi(\vartheta_0+t+s)\,ds\}}{\exp\{-\int_0^{\frac{1}{i}}\pi(\vartheta_0+t+s)\,ds\}} = \exp\{\int_0^{\frac{1}{i}}\pi(\vartheta_0+t+s)ds\} - 1 \approx \pi(\vartheta_0+t)\frac{1}{i} = O(\frac{1}{i})$$

$$i\to\infty, \frac{\frac{1}{i}K_{\vartheta_0+t}}{1-\frac{1}{i}K_{\vartheta_0+t}} = \pi(\vartheta_0+t)dt, \frac{1}{i}K_{\vartheta_0+t} = \pi(\vartheta_0+t)dtb\left(\frac{1}{i}\right) \to bdt, \frac{E^1_{t+\frac{1}{i}} - E_t^1}{E_t^1} \to \frac{dE_t^1(t)}{E_t^1(t)}, \frac{E^2_{t+\frac{1}{i}} - E_t^2}{E_t^2} \to \frac{dE_t^2(t)}{E_t^2(t)}, \frac{E^3_{t+\frac{1}{i}} - E_t^3}{E_t^3} \to \frac{dE_t^3(t)}{E_t^3(t)} \quad (7)$$

Substituting (7) into (6) we have

$$L\left(t+\frac{1}{i}\right) = \left(L(t)\left(1 + (1-\mu_2-\mu_3)\frac{dE_t^1(t)}{E_t^1(t)} + \mu_2\frac{dE_t^2(t)}{E_t^2(t)} + \mu_3\frac{dE_t^3(t)}{E_t^3(t)}\right) + bdt - tb\pi(\vartheta_0+t)dt\right)(1+\pi(\vartheta_0+t)dt) \quad (8)$$

$$dL(t) = L(t)\Big(\big((1-\mu_2-\mu_3)r_1dt + \mu_2\big((r_1+k_1)dt + \sigma_1dW_1(t)\big) + \mu_3\big((r_1+k_2)dt + \sigma_2dW_2(t) + \sigma_3dW_3(t)\big) + \pi(\vartheta_0+t)dt\Big) + bdt - tb\pi(\vartheta_0+t)dt \quad (9)$$

$$dL(t) = \left\{L(t)\left(\mu_2k_1 + \mu_3k_2 + r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right\}dt + L(t)\left(\mu_3\big(\sigma_2dW_2(t) + \sigma_3dW_3(t)\big) + \mu_2\sigma_1dW_1(t)\right) L(0) = l_0 \quad (10)$$

Where $\pi(t)$ is the force function and ϑ is the maximal age of the life table and are related as follows

$$\pi(t) = \frac{1}{\vartheta - t} \quad 0 \le t < \vartheta \quad (11)$$

If we apply the variational inequality method cited in [13], the mean-variance control problem in (4) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $A(t,l)$. Our interest here is to determine the optimal portfolio policy for the three assets using the mean-variance utility function.

$$\begin{cases} B(t,l,\mu) = E_{t,l}[L^{\mu}(T)] - \frac{\gamma}{2}Var_{t,l}[L^{\mu}(T)] \\ B(t,l,\mu) = E_{t,l}[L^{\mu}(T)] - \frac{\gamma}{2}\left(E_{t,l}[L^{\mu}(T)^2] - (E_{t,l}[L^{\mu}(T)])^2\right) \\ A(t,l) = \sup_{\mu} B(t,l,\mu) \end{cases} \quad (12)$$

Following the procedure in[13],the optimal investment strategy μ^* satisfies:

$$A(t,l) = \sup_{\mu} B(t,l,\mu) \tag{13}$$

$\gamma$is a constant representing risk aversion coefficient of the members.

Let $u^\mu(t,l) = E_{t,l}[L^\mu(T)]$, $v^\mu(t,l) = E_{t,l}[L^\mu(T)^2]$ then

$$A(t,l) \quad = \sup_{\mu} x\big(t,l,u^\mu(t,l),v^\mu(t,l)\big)$$

Where,

$$x(t,l,u,v) = u - \frac{\gamma}{2}(v - u^2) \tag{14}$$

Theorem 1 (verification theorem). If there exist three real functions X, Y, Z $[0,T]\times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\begin{cases} \sup_{\mu} \left\{ \begin{array}{c} X_t - x_t + (X_l - x_l)\left[l\left(\mu_2 k_1 + \mu_3 k_2 + r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right] \\ +\frac{1}{2}(X_{ll} - U_{ll})({\mu_3}^2({\sigma_2}^2+{\sigma_3}^2) + {\mu_2}^2{\sigma_1}^2) \end{array} \right\} = 0 \\ X(T,l) = x(t,l,l,l^2) \end{cases} \tag{15}$$

Where,

$$U_{ll} = x_{ll} + 2x_{lu}u_l + 2x_{lv}v_l + x_{uu}u_l^2 + 2x_{uv}u_l v_l + x_{vv}v_l^2 = \gamma u_l^2$$

$$\begin{cases} \left\{Y_t + Y_l\left[l\left(\mu_2 k_1 + \mu_3 k_2 + r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right] + \frac{1}{2}Y_{ll}({\mu_3}^2({\sigma_2}^2+{\sigma_3}^2) + {\mu_2}^2{\sigma_1}^2)\right\} = 0 \\ Y(T,l) = l \end{cases} \tag{16}$$

$$\begin{cases} \left\{Z_t + Z_l\left[l\left(\mu_2 k_1 + \mu_3 k_2 + r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right] + \frac{1}{2}Z_{ll}({\mu_3}^2({\sigma_2}^2+{\sigma_3}^2) + {\mu_2}^2{\sigma_1}^2)\right\} = 0 \\ Y(T,l) = l^2 \end{cases} \tag{17}$$

Then $A(t,l) = X(t,l), u^{\mu^*} = Y(t,l), v^{\mu^*} = Z(t,l)$for the optimal investment strategy μ^*

Proof:

The details of the proof can be found in [20-22].

Our focus now is to obtain the optimal investment strategies for both risky and riskless asset as well as the efficient frontier by solving (15), (16), (17).

Recall that $x(t,l,u,v) = u - \frac{\gamma}{2}(v - u^2)$

$$x_t = x_l = x_{ll} = x_{lu} = x_{lv} = x_{uv} = x_{vv} = 0,\ x_u = 1 + \gamma u,\ x_{uu} = \gamma,\ x_v = -\frac{\gamma}{2} \tag{18}$$

Substituting (18) into (15) and differentiating (15) with respect to $\mu_2$and $\mu_3$and solving for $\mu_2$and $\mu_3$we have

$${\mu_2}^* = -\left[\frac{k_1 X_l}{(X_{ll} - \gamma Y_l^2) l {\sigma_1}^2}\right] \tag{19}$$

$${\mu_3}^* = -\left[\frac{k_2 X_l}{(X_{ll} - \gamma Y_l^2) l ({\sigma_2}^2+{\sigma_3}^2)}\right] \tag{20}$$

Substituting (19) and (20) into (15) we have

$$X_t + X_l\left[\left(r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right)l + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right] - \frac{X_l^2}{2(X_{ll} - \gamma Y_l^2)}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right) = 0 \tag{21}$$

$$Y_t + Y_l\left[\left(r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right)l + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right] - \frac{X_l Y_l}{(X_{ll} - \gamma Y_l^2)}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)$$
$$+\frac{Y_{ll}}{2}\left[\frac{X_l^2}{(X_{ll} - \gamma Y_l^2)}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)\right] = 0 \tag{22}$$

Proposition 1

The optimal investment policy for the three assets are given as

$${\mu_1}^* = 1 - {\mu_2}^* - {\mu_3}^* = 1 - \frac{k_1\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l {\sigma_1}^2} - \frac{k_2\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l ({\sigma_2}^2+{\sigma_3}^2)} \tag{23}$$

$${\mu_2}^* = \frac{k_1\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l {\sigma_1}^2} \tag{24}$$

$${\mu_3}^* = \frac{k_2\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l ({\sigma_2}^2+{\sigma_3}^2)} \tag{25}$$

proof

We assume a solution for $X(t,l)$ and $Y(t,l)$ as follows:

$$\begin{cases} X(t,l) = F(t)l + G(t) \quad F(T) = 1,\ G(T) = 0 \\ Y(t,l) = H(t)l + I(t) \quad H(T) = 1,\ I(T) = 0 \\ X_t = lF(t) + G(t), X_l = F(t), X_{ll} = 0,\ Y_t = lH(t) + I(t), Y_l = H(t), Y_{ll} = 0 \end{cases} \quad (26)$$

Substituting (26) into (21) and (22)

$$\begin{cases} F_t(t) + \left(r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right)F(t) = 0 \\ G_t(t) + F(t)b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + \frac{F^2(t)}{2\gamma H^2(t)}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right) = 0 \end{cases} \quad (27)$$

$$\begin{cases} H_t(t) + \left(r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right)H(t) = 0 \\ I_t(t) + H(t)b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + \frac{F(t)}{\gamma H(t)}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right) = 0 \end{cases} \quad (28)$$

Solving (27) and (28), we have

$$F(t) = \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} \quad (29)$$

$$H(t) = \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} \quad (30)$$

$$G(t) = \frac{1}{2\gamma}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)(T-t) + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{{r_1}^2} - \frac{2}{{r_1}^2}e^{r_1(T-t)} + \frac{\vartheta-\vartheta_0-2t}{r_1}e^{r_1(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r_1}\right] \quad (31)$$

$$I(t) = \frac{1}{\gamma}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)(T-t) + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{{r_1}^2} - \frac{2}{{r_1}^2}e^{r_1(T-t)} + \frac{\vartheta-\vartheta_0-2t}{r_1}e^{r_1(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r_1}\right] \quad (32)$$

$$X(t,l) = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} + \frac{1}{2\gamma}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)(T-t) + \frac{b}{\vartheta-\vartheta_0-T}\begin{bmatrix} \frac{2}{{r_1}^2} - \frac{2}{{r_1}^2}e^{r_1(T-t)} + \\ \frac{\vartheta-\vartheta_0-2t}{r_1}e^{r_1(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r_1} \end{bmatrix} \quad (33)$$

$$Y(t,l) = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} + \frac{1}{\gamma}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)(T-t) + \frac{b}{\vartheta-\vartheta_0-T}\begin{bmatrix} \frac{2}{{r_1}^2} - \frac{2}{{r_1}^2}e^{r_1(T-t)} + \\ \frac{\vartheta-\vartheta_0-2t}{r_1}e^{r_1(T-t)} - \frac{\vartheta-\vartheta_0-2T}{r_1} \end{bmatrix} \quad (34)$$

From (26), we have

$$X_l = F(t) = \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)}, X_{ll} = 0, Y_l = H(t) = \left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} \quad (35)$$

Substituting (35) into (19) and (20), we have

$${\mu_2}^* = \frac{k_1\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l{\sigma_1}^2}$$

$${\mu_3}^* = \frac{k_2\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l({\sigma_2}^2+{\sigma_3}^2)}$$

$${\mu_1}^* = 1 - \frac{k_1\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l{\sigma_1}^2} - \frac{k_2\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l({\sigma_2}^2+{\sigma_3}^2)}$$

Proposition 2

The optimal fund size $L^{\mu^*}(t)$ corresponding to the optimal investment policy μ^* is given as

$$L^{\mu^*}(t) = \frac{1}{\gamma}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)te^{r_1(T-t)} - \frac{b}{r_1}\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + \frac{2b}{{r_1}^2}\left(\frac{1}{\vartheta-\vartheta_0-t}\right) + \left(l_0(\vartheta-\vartheta_0) + \frac{b(\vartheta-\vartheta_0)}{r_1} - \frac{2b}{{r_1}^2}\right)\frac{e^{r_1 t}}{\vartheta-\vartheta_0-t} \quad (36)$$

Proof

Recall that (10), (24), (25) are given respectively as

$$dL(t) = \left\{L(t)\left(\mu_2 k_1 + \mu_3 k_2 + r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right)\right\}dt + L(t)\left(\mu_3(\sigma_2 dW_2(t) + \sigma_3 dW_3(t)) + \mu_2\sigma_1 dW_1(t)\right) L(0) = l_0$$

$${\mu_2}^* = \frac{k_1\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l{\sigma_1}^2}$$

$${\mu_3}^* = \frac{k_2\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right)e^{r_1(t-T)}}{\gamma l({\sigma_2}^2+{\sigma_3}^2)}$$

Dividing (10) through by dt and substituting (24) and (25) into (10), we have

$$L_t(t) - \left(r_1 + \frac{1}{\vartheta-\vartheta_0-t}\right)L = \frac{1}{\gamma}\left(\frac{{k_2}^2}{{\sigma_2}^2+{\sigma_3}^2} + \frac{{k_1}^2}{{\sigma_1}^2}\right)e^{r(T-t)}\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) + b\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) \qquad L(0) = l_0 \quad (37)$$

Solving the O.D.E (37) for $L(t)$ with initial condition using any method we have

$$L^{\mu^*}(t) = \frac{1}{\gamma}\left(\frac{k_2{}^2}{\sigma_2{}^2+\sigma_3{}^2}+\frac{k_1{}^2}{\sigma_1{}^2}\right)\left(\frac{\vartheta-\vartheta_0-T}{\vartheta-\vartheta_0-t}\right) t e^{r_1(T-t)} - \frac{b}{r_1}\left(\frac{\vartheta-\vartheta_0-2t}{\vartheta-\vartheta_0-t}\right) + \frac{2b}{r_1{}^2}\left(\frac{1}{\vartheta-\vartheta_0-t}\right) + \left(l_0(\vartheta-\vartheta_0)+\frac{b(\vartheta-\vartheta_0)}{r_1}-\frac{2b}{r_1{}^2}\right)\frac{e^{r_1 t}}{\vartheta-\vartheta_0-t}$$

Proposition 3

The efficient frontier of the pension fund is given as

$$E_{t,l}[L^{\mu^*}(T)] = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r_1{}^2}-\frac{2}{r_1{}^2}e^{r(T-t)}+\frac{\vartheta-\vartheta_0-2t}{r_1}e^{r(T-t)}-\frac{\vartheta-\vartheta_0-2T}{r_1}\right] + \sqrt{\left(\frac{k_2{}^2}{\sigma_2{}^2+\sigma_3{}^2}+\frac{k_1{}^2}{\sigma_1{}^2}\right)(T-t)\left(Var_{t,l}[L^{\mu^*}(T)]\right)} \quad (38)$$

Proof

Recall that

$$Var_{t,l}[L^{\mu^*}(T)] = E_{t,l}[L^{\mu}(T)^2] - (E_{t,l}[L^{\mu}(T)])^2$$

$$Var_{t,l}[L^{\mu^*}(T)] = \frac{2}{\gamma}(Y(t,l) - X(t,l)) \quad (39)$$

Substituting (33) and (34) into (39), we have

$$Var_{t,l}[L^{\mu^*}(T)] = \frac{1}{\gamma^2}\left(\frac{k_2{}^2}{\sigma_2{}^2+\sigma_3{}^2}+\frac{k_1{}^2}{\sigma_1{}^2}\right)(T-t) \quad (40)$$

$$\frac{1}{\gamma} = \frac{1}{\left(\frac{k_2{}^2}{\sigma_2{}^2+\sigma_3{}^2}+\frac{k_1{}^2}{\sigma_1{}^2}\right)}\sqrt{\frac{Var_{t,l}[L^{\mu^*}(T)]]}{(T-t)}} \quad (41)$$

$$E_{t,l}[L^{\mu^*}(T)] = Y(t,l) \quad (42)$$

Substituting (34) into (42), we have

$$E_{t,l}[L^{\mu^*}(T)] == l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r_1{}^2}-\frac{2}{r_1{}^2}e^{r(T-t)}+\frac{\vartheta-\vartheta_0-2t}{r_1}e^{r(T-t)}-\frac{\vartheta-\vartheta_0-2T}{r_1}\right] + \frac{1}{\gamma}\left(\frac{k_2{}^2}{\sigma_2{}^2+\sigma_3{}^2}+\frac{k_1{}^2}{\sigma_1{}^2}\right)(T-t) \quad (43)$$

Substitute (41) in (43), we have:

$$E_{t,l}[L^{\mu^*}(T)] = l\left(\frac{\vartheta-\vartheta_0-t}{\vartheta-\vartheta_0-T}\right)e^{r_1(T-t)} + \frac{b}{\vartheta-\vartheta_0-T}\left[\frac{2}{r_1{}^2}-\frac{2}{r_1{}^2}e^{r(T-t)}+\frac{\vartheta-\vartheta_0-2t}{r_1}e^{r(T-t)}-\frac{\vartheta-\vartheta_0-2T}{r_1}\right] + \sqrt{\left(\frac{k_2{}^2}{\sigma_2{}^2+\sigma_3{}^2}+\frac{k_1{}^2}{\sigma_1{}^2}\right)(T-t)\left(Var_{t,l}[L^{\mu^*}(T)]\right)}$$

Numerical Simulations

In this section we present numerical simulations of the optimal investment policy with respect to time using the following parameters. $\vartheta = 100; \vartheta_0 = 20; \gamma = 0.05; r_1 = 0.02; k_1 = 0.035; k_2 = 0.045; \sigma_1 = 0.85; \sigma_2 = 1; \sigma_3 = 0.60;$ $l = L(t); l_0 = 1;$ $T = 40;; t = 0{:}5{:}20;$

Unless otherwise stated

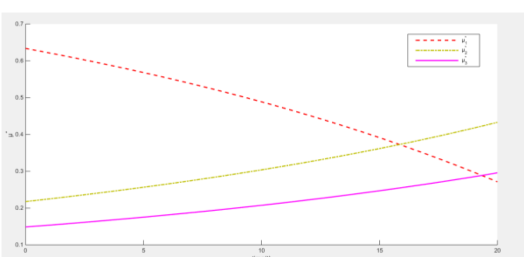

Fig 1: Time evolution of the optimal investment policies $\mu_1{}^*, \mu_2{}^*$, and whenl $l = l_0$

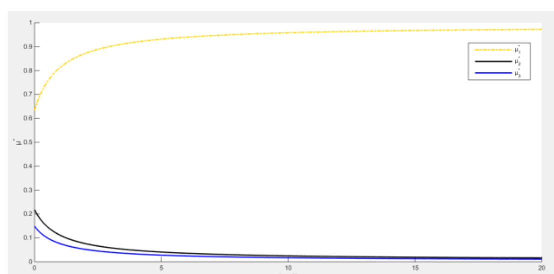

Fig 2: Time evolution of the optimal investment policies $\mu_1{}^*, \mu_2{}^*$, and $\mu_3{}^*$ when $\mu_3{}^*$ $l = L^{\mu^*}(t)$

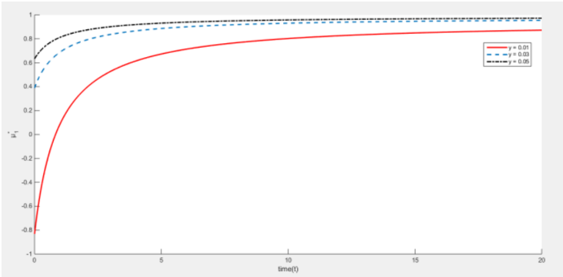

Fig 3: Time evolution of the optimal investment policy $\mu_1{}^*$ when $l = L^{\mu^*}(t)$ with different risk averse γ

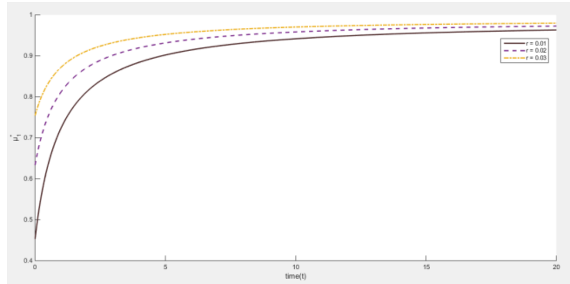

Fig 4: Time evolution of the optimal investment policy $\mu_1{}^*$ when $l = L^{\mu^*}(t)$ with different risk averse r_1

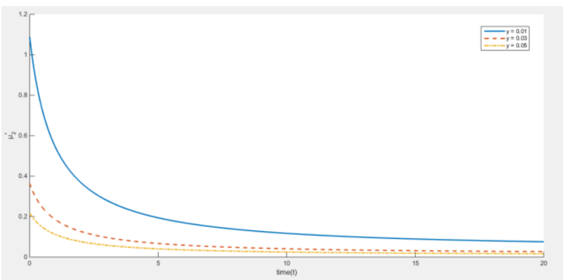

Fig 5: Time evolution of the optimal investment policy $\mu_2{}^*$ when $l = L^{\mu^*}(t)$ with different risk averse γ

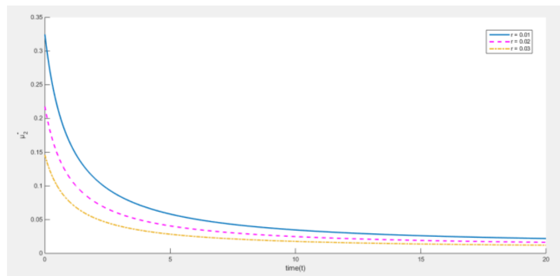

Fig 6; Time evolution of the optimal investment policy $\mu_2{}^*$ when $l = L^{\mu^*}(t)$ with different risk averse r_1

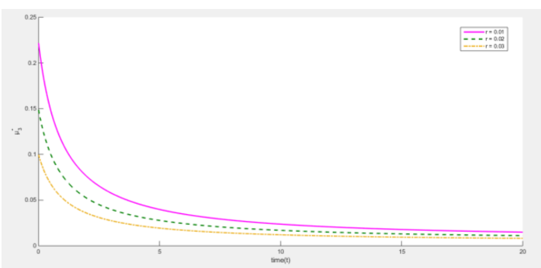

Fig 7; Time evolution of the optimal investment policy $\mu_3{}^*$ when $l = L^{\mu^*}(t)$ with different risk averse r_1

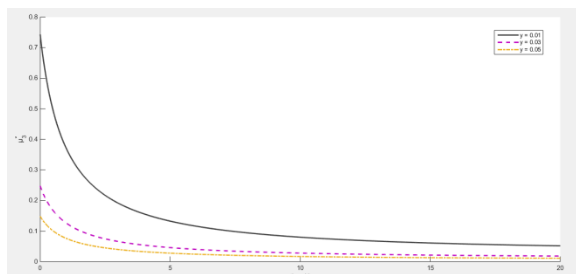

Fig 8; Time evolution of the optimal investment policy $\mu_3{}^*$ when $l = L^{\mu^*}(t)$ with different risk averse γ

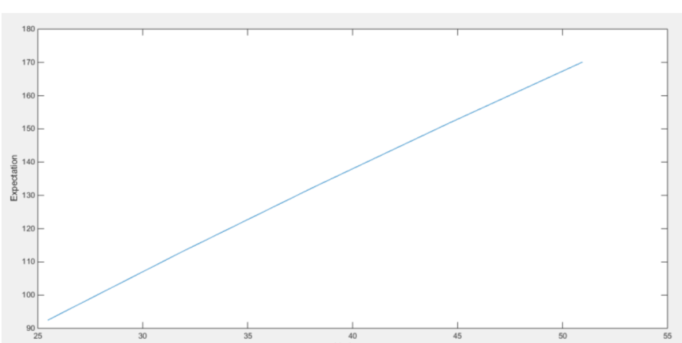

Fig 9; Time evolution of the expectation with variance when $l = L^{\mu^*}(t)$

6. Discussion

Figure 1 shows the plot of the optimal investment policy for the three assets with respect to time. It was observed that when the initial wealth was used at the early stage of the investment, the optimal investment policy for the risk free asset decreases with time while that of equity and loan increases with time. This is because as the retirement age draws closer, the fund manager will want to invest more in risky asset to increase the returns of his members.In figure 2, we observed that the optimal investment policy for the risk free asset increase continuously while that of equity and loan decreases continuously.

This is because the optimal fund size was used at the early stage of the investment and as retirement age draws near, the pension manager will prefer to invest more in risk free asset rather than taking risk of investing in more in risky assets. We observed from figure 3 and 4 that the optimal investment policy of the risk free asset is directly proportional to risk averse level of the pension member and inversely proportional to the predetermined interest rate of the risk free asset. This is because members with high risk averse are scared of taking risk hence prefers to invest more in risk free assets unlike members with less risk averse who prefers to take risk hence reduction in the proportion invested in risk free asset. On the other hand, when the predetermined interest rate is high, members will prefer to invest where there is more interest and little or no risk hence a reduction in investment in risky assets and vice versa.

From figure 5 and 6, we observed that an investor with high risk averse level will invest less in equity and loan and vice versa as retirement age draws closer. Figure 6 and 7 shows that as the predetermined interest increases, there is a decrease in investments in two risky assets. Figure 9, shows that the there is a linear relationship between the expectation and the variance; this implies that when more is taken by the member, he or she is expecting more returns and vice versa. In general, from all the graphs above, we observed that the pension member will prefer to invest more in stock as compared to loan; the implication here is that either the investment in equity is more profitable than the loan or the investment in loan is more risky as compared to equity.

Conclusion

In conclusion, the optimal investment policy for a member in a DC pension withreturn of premium clause was studied. A continuous time mean-variance stochastic optimal control problem was formulated with the help of the actuarial symbol. Investments in one risk-free (cash) and two risky assets namely equity and loan were considered to help increase the accumulated fund of the remaining pension members in order to meet up with the retirement needs of the remaining members. Next, we established an optimized problem from the extended Hamilton Jacobi Bellman equations and solved the optimized problem for the optimal investment policies of the three assets and the efficient frontier of the pension members. Furthermore, we present a numerical simulation of the optimal investment policy of the three assets with respect to time. We observed that the pension member prefers investment stock as compared to loan, secondly we observed that optimal investment policy of the risky assets is inversely proportional to risk averse level, predetermined interest rate and initial wealth.

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