*Transactions of the Nigerian Association of Mathematical Physics*
*Volume 8, (January, 2019), p113 –120*
*© Trans. of NAMP*

# EFFECT OF CIRCUMBINARY DISC ON THE STABILITY OF TRIANGULAR EQUILIBRIUM POINTS IN THE ER3BP

*Umar Aishetu[1], Bashir Umar[2] and Kamfa A. Salisu[3]*

[1]**Department of Mathematics, Faculty of Physical sciences, Ahmadu Bello University, Zaria**
[2]**Department of IJMB, School of Liberal Studies, Nuhu Bamalli Polytechnic, Zaria**
[3] **Department of Mathematics, Faculty of Physical sciences, Ahmadu Bello University, Zaria**

## *Abstract*

***The motion of a test particle in the vicinity of triangular equilibrium points $L_{4,5}$ in the ER3BP is studied when both spherical primaries are surrounded by a circular cluster of material points. The existence and positions of these points are established and found to be affected by the eccentricity, semi-major axis and circumbinary disc. It is observed that the triangular points are stable for $0 < \mu < \mu_c$. While the eccentricity and semi-major axis are destabilizing, resulting in a decrease in the size of the region of stability. The disc has a desired stabilizing tendency.***

***Keywords***: Elliptic restricted three body problem, Eccentricity, Semi-major axis, Potential from the disc

## 1. Introduction

The restricted three-body problem (R3BP) describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The study of the R3BP is of great theoretical, practical, historical and educational relevance, and in its many form has had important implications in several scientific fields, including among others, celestial mechanics, galactic dynamics, chaos theory and molecular physics, dynamical astronomy, celestial and space mechanics, lunar and planetary sciences as can be found in the works of [1 -8] is the subject of interest. In such a co-rotating frame, an effective potential that includes the gravitational potential of the two stars is defined. This potential has five Lagrangian points where the gradient of the effective potential is zero, three of the collinear points $L_i (i = 1,2,3)$ lie on the $\xi$-axis, the line joining the primaries and two are off the $\xi$-axis on the $\xi - \eta$ plane, forming equilateral triangle called the triangular Lagrangian points $L_{4,5}$ . Over the years; several authors from [9-14] have found the study of the locations and stability of these equilibrium points important and interesting because of its diverse applications. The 3B problem in general relativity has also been the subject of several researches [15-17].

When the orbits of the primaries are elliptic, called the elliptic restricted three- body problem (ER3BP), a non-uniformly rotating-pulsating coordinate system is commonly used. The orbits of the celestial bodies are mostly elliptic not circular. It is therefore more accurate to study the ER3BP, because it has a significant effect which the CR3BP neglects. The ER3BP analyses the dynamical systems more accurately. It possesses five coplanar equilibrium points; three collinear and two triangular, where the gravitational and centrifugal forces just balance each other. The collinear points are generally unstable, while the triangular points are conditionally stable. Their stability occurs in spite of the fact that the potential energy has a maximum rather than a minimum at the latter points. The ER3BP generalizes the original CR3BP, and improves its applicability, while some outstanding and useful properties of the circular model still hold true or can be adapted to the elliptic case. In particular, possible positions of the equilibrium occur when the three bodies form equilateral triangles.

There are ring-type belts of dust particles in the stellar systems which are regarded as the young analogues of the Kuiper belt [18-19] detected debris rings in many main–sequence stellar binary systems using the Spitzer Space Telescope. Among the observed 69 A3-F8 main sequence binary star systems, nearly 60% showed debris dust rings surrounding the binary stars. This inspired many scientists to study the CR3BP by taking into consideration the additional gravitational potential from the disc [20-21] studied the CR3BP by considering the influence from a disc for planetary systems and found that the chance to

Correspondence Author: Umar A., Email: umaraishetu33@yahoo.com, Tel: +2348036786146

have libration points around inner part of the disc is higher than near the outer part. In their research, [20], [22] have incorporated additional gravitational potential from the disc in their studies of the CR3BP. They found that the gravitational potential from the disc makes the structure of the CR3BP quite different such that new libation points exist only under certa in conditions. Due to the recent important application [23], these new equilibrium points are now called Jiang-Yeh points. The orbital motion of a test particle around a primary is greatly affected in the presence of a massive disc [24-25]. Of late, a bi-dimensional ring model for the minor asteroids of the solar system has been implemented and modeled in the latest EPM2013 planetary ephemerides [26]. In his research [27] studied analytically and numerically the effects of radiation pressure of th e more massive primary, oblateness of the less massive primary and gravitational potential from the disc on the linear stabilit y of libration points in the R3BP. In their work [28] examined the combined effect of radiation and oblateness of both primaries, together with additional gravitational potential from the belt on the motion of an infinitesimal body in the CR3BP. They [29] studied the effects of oblateness up to J4 of the less massive primary and gravitational potential from a belt on the linear stability of triangular points in the photogravitational CR3BP. They [30] examined the effect of triaxiality of the mo re massive primary, oblateness up to the zonal harmonics J4 of the smaller primary and gravitational potential from a belt on the linear stability of the triangular libration points in the CR3BP.

Figure 1: Artists impression of HD 98800

In this study, we have examined the effect of the potential from the disc on the location of $L_{4,5}$ and their linear stability in the ER3BP on the binary system HD98800B. This study can be useful in the investigation of motion of a test particle near binary stars surrounded by a cluster of material points.

This paper is arranged as follows: Section 2 presents equations of motion of the infinitesimal body, locations of $L_{4,5}$ and their linear stability are established in section 3 and 4, while section 5 contains discussion.

## 2. Equations of motion

The equation of motion of the elliptic restricted three body problem (ER3BP) surrounded by a circumbinary disc in a dimensionless-pulsating coordinate system $(\xi, \eta, \zeta)$ following [31] is given as:

$$\ddot{\xi} - 2\dot{\eta} = \Omega_\xi$$
$$\ddot{\eta} + 2\dot{\xi} = \Omega_\eta$$
$$\ddot{\zeta} = \Omega_\zeta \tag{1}$$

with

$$\Omega = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\frac{\xi^2+\eta^2}{2} + \frac{1}{n^2}\left\{\frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{M_b}{(r^2+T^2)^{\frac{1}{2}}}\right\}\right] \tag{2}$$

$$r_1^2 = (\xi+\mu)^2 + \eta^2 + \zeta^2$$
$$r_2^2 = (\xi+\mu-1)^2 + \eta^2 + \zeta^2 \tag{3}$$

and

$$n^2 = \frac{1}{a}\left(1 + \frac{3}{2}e^2 + \frac{2M_b r_c}{(r_c^2+T^2)^{\frac{3}{2}}}\right) \tag{4}$$

Where *n, a, e,* are the mean motion, semi-major axis, eccentricity of the orbits of the primaries respectively; $\frac{M_b}{(r^2+T^2)^{\frac{1}{2}}}$ is the potential due to the circular cluster of material points, $r$ is the radial distance of the infinitesimal body and is given by $r^2 = \xi^2 + \eta^2$ , $T = b + d$, $b$ and $d$ are parameters which determine the density profile of the circular cluster of material points. The parameter b controls the flatness of the profile and is known as the flatness parameter. The parameter d controls the size of the core of the density profile and is called the core parameter. When $b = d = 0$, the potential equals to the one by a point mass, $r_c$ is the radial distance of the infinitesimal body in the classical restricted 3BP [33], [23], [22], [34] and [31].

### 3. Locations of triangular equilibrium points

The libration points represent stationary solutions of the ER3BP. All the derivatives of the coordinates with respect to time are zero at these points [35]. Thus, the libration points of the ER3BP if they exist a re determined by setting all velocities and accelerations of the equations of motion (1) to zero, i.e.

$$\Omega_\xi = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi+\mu)}{r_1^3} + \frac{\mu(\xi+\mu-1)}{r_2^3} + \frac{M_b\xi}{(r^2+T^2)^{\frac{3}{2}}}\right\}\right] = 0 \tag{5}$$

$$\Omega_\eta = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\eta - \frac{1}{n^2}\left\{\frac{(1-\mu)\eta}{r_1^3} + \frac{\mu\eta}{r_2^3} + \frac{M_b\eta}{(r^2+T^2)^{\frac{3}{2}}}\right\}\right] = 0 \tag{6}$$

$$\Omega_\zeta = -\frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\frac{1}{n^2}\left\{\frac{(1-\mu)\zeta}{r_1^3} + \frac{\mu\zeta}{r_2^3}\right\}\right] = 0 \tag{7}$$

From equation (5) and equation (6) respectively we have

$$n^2\xi - \frac{(1-\mu)(\xi+\mu)}{r_1^3} - \frac{\mu(\xi+\mu-1)}{r_2^3} - \frac{M_b\xi}{(r^2+T^2)^{\frac{3}{2}}} = 0 \tag{8}$$

And

$$n^2\eta - \frac{(1-\mu)\eta}{r_1^3} - \frac{\mu\eta}{r_2^3} - \frac{M_b\eta}{(r^2+T^2)^{\frac{3}{2}}} = 0 \tag{9}$$

Now, equations, (8)and (9) gives the position of the triangular points. Expanding equation (8) and collecting terms, we have

$$\xi\left(n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}}\right) - \frac{\mu(1-\mu)}{r_1^3} + \frac{\mu(1-\mu)}{r_2^3} = 0 \tag{10}$$

From equation (10), we have

$$n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}} = 0$$

This implies that

$$-\frac{\mu(1-\mu)}{r_1^3} + \frac{\mu(1-\mu)}{r_2^3} = 0$$

Therefore

$$\mu(1-\mu)\left\{\frac{q_1}{r_1^3} - \frac{q_2}{r_2^3}\right\} = 0$$

This indicate

$$\mu(1-\mu) = 0 \quad \Rightarrow \mu = 0 \quad or \quad \mu = 1$$

Substituting these values, when

$$\mu = 0$$

$$n^2 - \frac{1}{r_1^3} - \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}} = 0$$

When

$$\mu = 1$$

$$n^2 - \frac{1}{r_2^3} - \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}} = 0 \tag{11}$$

When the potential from the circular cluster is neglected

$$n^2 - \frac{1}{r_1^3} = 0 \quad and \quad n^2 - \frac{1}{r_2^3} = 0 \tag{12}$$

$$\Rightarrow r_1 = r_2$$

From equation (12)

$$r_1 = \left(\frac{1}{n^2}\right)^{\frac{1}{3}}$$

$$r_2 = \left(\frac{1}{n^2}\right)^{\frac{1}{3}}$$

Substituting the mean motion, the above equation will change slightly by $\varepsilon_i (i = 1,2)$, where $\varepsilon_i << 1$

$$r_1 = (a)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3(r_c^2 + T^2)^{\frac{3}{2}}}\right) + \varepsilon_1 \qquad (13)$$

$$r_2 = (a)^{\frac{1}{3}}\left(1 - \frac{1}{2}e^2 - \frac{2M_b r_c}{3(r_c^2 + T^2)^{\frac{3}{2}}}\right) + \varepsilon_2$$

Where $\varepsilon_1$ *and* $\varepsilon_2$ are small quantities, which on substituting into equation (12) and solving we obtain

$$\varepsilon_1 = \frac{M_b a^{\frac{4}{3}}}{3(r^2 + T^2)^{\frac{3}{2}}}$$

$$\varepsilon_2 = \frac{M_b a^{\frac{4}{3}}}{3(r^2 + T^2)^{\frac{3}{2}}} \qquad (14)$$

Now since $r^2 = \xi^2 + \eta^2 \Rightarrow r_c^2 = 1 - \mu + \mu^2$

Substituting the values of ε1 and ε2 into equation (12) respectively we obtain

$$r_1^2 = (a)^{\frac{2}{3}}\left(1 - e^2 - \frac{2M_b(2r_c - a)}{3(r_c^2 + T^2)^{\frac{3}{2}}}\right)$$

$$r_2^2 = (a)^{\frac{2}{3}}\left(1 - e^2 - \frac{2M_b(2r_c - a)}{3(r_c^2 + T^2)^{\frac{3}{2}}}\right) \qquad (15)$$

Substituting the values of $r_1^2$ and $r_2^2$ into

$$\xi = \frac{1}{2} - \mu + \frac{r_1^2 + r_2^2}{2} \quad and \quad \eta^2 = r_1^2 - (\xi^2 + \mu)^2$$

obtained from equation (3), we have

$$\xi = \frac{1}{2} - \mu \qquad (16)$$

and

$$\eta = \pm\left((a)^{\frac{2}{3}}\left(1 - e^2 - \frac{2M_b(2r_c - a)}{3(r_c^2 + T^2)^{\frac{3}{2}}} - \frac{1}{4}\right)\right)^{\frac{1}{2}} \qquad (17)$$

Therefore, the Lagrangian points denoted by L4,5 (ξ,±η)are given by equations (16) and (17).

4. Stability of the triangular equilibrium points

Denoting the positions of equilibrium points by $(\xi_0, \eta_0)$ and displacements $\alpha$ and $\beta$ we obtain equations leading to the derivation of equations in $(\Omega^0_{\xi\xi}),\ (\Omega^0_{\eta\eta})$ and $(\Omega^0_{\eta\eta})^2$.

$$\Omega^0_{\xi\xi} = (1 - e^2)^{-\frac{1}{2}}\left[\frac{3}{4(a)^{\frac{2}{3}}} + \frac{3}{4(a)^{\frac{2}{3}}}e^2 + \frac{M_b(4r_c - 5a)}{4(r_c^2 + T^2)^{\frac{3}{2}}(a)^{\frac{2}{3}}} + \frac{3M_b a(\frac{1}{4} - \mu + \mu^2)}{(r_c^2 + T^2)^{\frac{5}{2}}}\right] \qquad (18)$$

$$\Omega^0_{\eta\eta} = (1 - e^2)^{-\frac{1}{2}}\left[3 - \frac{3}{4(a)^{\frac{2}{3}}} - \frac{3}{4(a)^{\frac{2}{3}}}e^2 - \frac{3M_b a}{(r_c^2 + T^2)^{\frac{3}{2}}} - \frac{3M_b a(\frac{1}{4} - (a)^{\frac{2}{3}})}{(r_c^2 + T^2)^{\frac{5}{2}}}\right] \qquad (19)$$

$$\Omega^0_{\xi\eta} = \left(1-e^2\right)^{-\frac{1}{2}}\eta\left[\frac{3}{2(a)^{\frac{2}{3}}} - \frac{3\mu}{(a)^{\frac{2}{3}}} - \frac{3\mu}{(a)^{\frac{2}{3}}}e^2 + \frac{3}{2(a)^{\frac{2}{3}}}e^2 - \frac{3\mu M_b a}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{M_b(4r_c-5a)}{2(r_c^2+T^2)^{\frac{3}{2}}(a)^{\frac{2}{3}}} - \frac{\mu M_b(4r_c-5a)}{(r_c^2+T^2)^{\frac{3}{2}}(a)^{\frac{2}{3}}} + \frac{3M_b a(\frac{1}{2}-\mu)}{(r_c^2+T^2)^{\frac{5}{2}}}\right] \tag{20}$$

The characteristic equation is given by

$$\lambda^4 - (\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - (\Omega^0_{\xi\eta})^2 = 0 \tag{21}$$

Substituting equations (18), (19) and (20) in equation (21) and restricting ourselves to only linear terms in $e^2$ , $0<\mu<\mu_c$ we have

$$4(\lambda^2)^2 + 4(4-3\psi_1)\lambda^2 + 27\mu(1-\mu) + 4\psi_2 = 0 \tag{22}$$

Where

$$\psi_1 = (1-e^2)^{-\frac{1}{2}}\left(1 - \frac{M_b}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{M_b r_c^2}{(r_c^2+T^2)^{\frac{5}{2}}}\right)$$

And

$$\psi_2 = 3\mu(1-\mu)\alpha + \frac{45\mu(1-\mu)}{4}e^2 + \frac{3\mu M_b(1-\mu)(4r_c-11)}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{27\mu M_b(1-\mu)}{4(r_c^2+T^2)^{\frac{5}{2}}}$$

$$\psi_2 = 3\mu(1-\mu)\left[\alpha + \frac{15}{4}e^2 + \frac{M_b(4r_c-11)}{2(r_c^2+T^2)^{\frac{3}{2}}} + \frac{9M_b}{4(r_c^2+T^2)^{\frac{5}{2}}}\right]$$

The characteristic equation is a quadratic equation in

$$\lambda^2 = \frac{-4(4-3\psi_1) \pm \left[(4-3\psi_1)^2 - 27\mu(1-\mu) - 4\psi_2\right]^{\frac{1}{2}}}{2} \tag{23}$$

For stable motion, we require $\lambda$ to be purely imaginary

i.e. $3\psi_1 - 4 \le 0$

and the discriminant

$$\Delta = (4-3\psi_1)^2 - 27\mu(1-\mu) - 4\psi_2 > 0 \tag{24}$$

This gives

$$0 < e \le \left[1 - \frac{9}{16}\left(1 - \frac{M_b}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{M_b r_c^2}{(r_c^2+T^2)^{\frac{5}{2}}}\right)^2\right]^{\frac{1}{2}} \tag{25}$$

When $M_b = 0$ equation (25) becomes

$$0 < e \le \frac{\sqrt{7}}{4} \tag{26}$$

From equation (24), we have

$$\Delta = \left(27 + 12\alpha + 45e^2 + \frac{6M_b(4r_c-11)}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{27M_b}{(r_c^2+T^2)^{\frac{5}{2}}}\right)\mu^2 - \left(27 + 12\alpha + 45e^2 + \frac{6M_b(4r_c-11)}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{27M_b}{(r_c^2+T^2)^{\frac{5}{2}}}\right)\mu + \left(1 - 3e^2 + \frac{6M_b}{(r_c^2+T^2)^{\frac{3}{2}}} - \frac{6M_b r_c^2}{(r_c^2+T^2)^{\frac{5}{2}}}\right) > 0 \tag{27}$$

The value of the critical mass parameter $\mu_c$ is obtained when $\Delta = 0$, and is given by

$$\mu_c = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right) - \frac{4}{27\sqrt{69}}\alpha - \frac{14}{9\sqrt{69}}e^2 + \left[\frac{(76-8r_c)(r_c^2+T^2)^{\frac{5}{2}} - 9(1+6r_c^2)(r_c^2+T^2)^{\frac{3}{2}}}{27\sqrt{69}(r_c^2+T^2)^4}\right]M_b$$

The value of the critical mass parameter to fifteen decimal places taking Mathematica 10.3 software is

$(T = 0.01, \quad r_c = 0.9999)$

$\mu_c = 0.0385208965\,045514 - 0.0624222941\,926211 e^2 - 0.0178349411\,978917\,\alpha + 0.0223206578\,823944\,M_b$

Now, since $b > 0; \Delta > 0$ in the interval $0 < \mu < \mu_c$, it follows that the roots of equation (23) are distinct imaginary numbers, hence, the triangular point is stable in this region. If $\mu_c < \mu < \frac{1}{2}, \ \Delta < 0$ the real parts of two of the roots of equation (23) are positive, therefore, the triangular point is unstable. If $\mu = \mu_c, \Delta = 0$, the roots of equation (23) are double roots, which gives instability of the point. Hence, the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$, where the critical mass parameter $\mu_c$ depends on the gravitational potential from the circular cluster of material points.

Table1: Relevant numerical data

| Binary system | Masses ($M_0$) | | Luminosity ($L_0$) | | Eccentricity |
|---|---|---|---|---|---|
| | $M_1$ | $M_2$ | $L_1$ | $L_2$ | $(e)$ |
| HD 98800 B | 0.70 | 0.58 | 0.33 | 0.17 | 0.5 |

Table 2: Effect of eccentricity on $L_{4,5}$ in the binary system HD98800B

| $e$ | ξ | ±η |
|---|---|---|
| 0.1 | 0.0467152 | 1.01834 |
| 0.3 | 0.0467283 | 0.966228 |
| 0.5 | 0.0467546 | 0.8525 |
| 0.7 | 0.0467939 | 0.645365 |
| 0.9 | 0.0468463 | 0.0530516 |

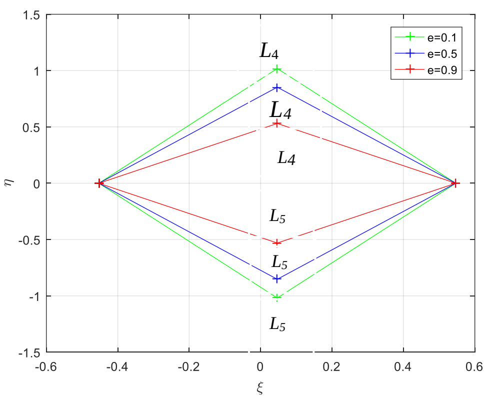


Figure1: Effect of eccentricity on $L_{4,5}$ in the binary system HD98800B

Table 3: Effect of semi-major axis on $L_{4,5}$ in the binary system HD98800B

| $a$ | ξ | ±η |
|---|---|---|
| 0.95 | 0.0467828 | 0.708602 |
| 0.85 | 0.0467645 | 0.805021 |
| 0.75 | 0.0467529 | 0.860137 |
| 0.65 | 0.0467418 | 0.910042 |
| 0.55 | 0.046731 | 0.955855 |

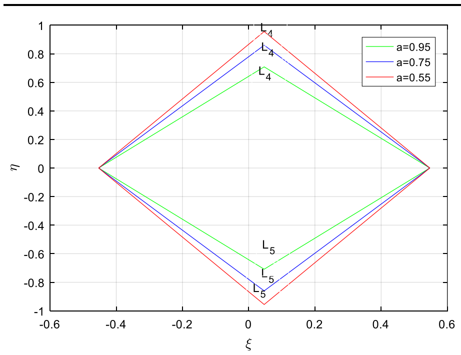


Figure 2: Effect of semi-major axis on $L_{4,5}$ in the binary system HD98800B

Table 4: Effect of potential from the disc on $L_{4,5}$ in the binary system HD98800B

| $M_b$ | ξ | ±η |
|---|---|---|
| 0 | 0.0467521 | 0.848272 |
| 0.01 | 0.0467525 | 0.846217 |
| 0.02 | 0.0467530 | 0.844158 |
| 0.03 | 0.0467653 | 0.784297 |
| 0.1 | 0.0467565 | 0.827497 |
| 0.2 | 0.0467609 | 0.806186 |
| 0.4 | 0.0467697 | 0.761779 |
| 0.5 | 0.0467742 | 0.738574 |

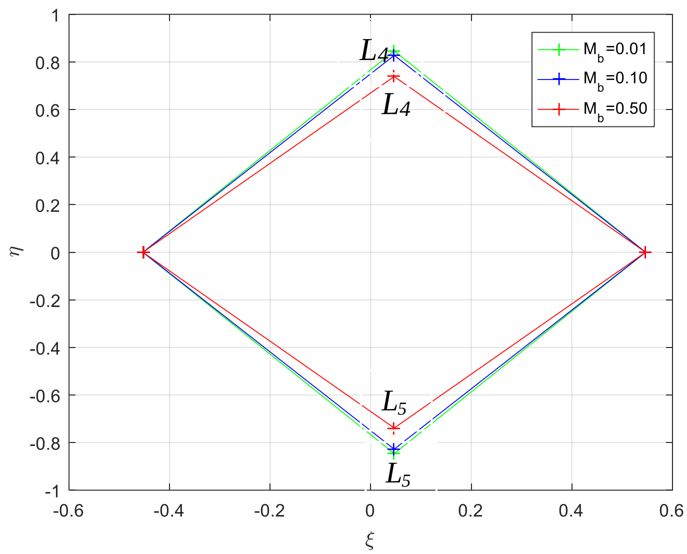

Figure 3: Effect of potential from the disc on $L_{4,5}$ in the binary system HD98800B

**5. Discussion**

Equations (1) are the equations of motion of the infinitesimal body in the ER3BP and are affected by the gravitational potential from the circumbinary disc. They are different from those obtained by [31] due to the oblateness and radiation of the primaries in their work and in the presence of the potential from the circumbinary disc in ours. In the absence of the oblateness coefficient, $A_1$ *and* $A_2$ of the bigger primary and smaller primary and absence of the potential respectively $(i.e \quad A_1 = A_2 = M_b = 0)$. Equations (2) agree with [31] when the primaries are non-radiating. However, in the absence of semi-major axis, eccentricity and potential from the circular disc $(i.e \quad a = e = M_b = 0)$, $\mu_c$ becomes $\mu_0$ which corresponds to the classical CR3BP [13].

We have studied the motion of an infinitesimal body under the influence of the gravitational potential from the disc. We have found the equations that govern the motion of the infinitesimal body and the positions of the triangular equilibrium points. The triangular points are stable for

$0 < \mu < \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$, where $\mu_c$ is the critical mass parameter influenced by the potential from the circumbinary disc.

**References**


[1] Topputo, F., Vasile, M., Bernelli-Zazzara, F.: Earth-to-Moon low energy transfers targeting $L_1$ hyperbolic transit orbits. Ann. N.Y.Acad. Sci. 1065, 55–76 (2005).

[2] Belbruno, E., Topputo, F., Gidea, M.: Resonance transitions associated to weak capture in the restricted three-body problem. Adv. Space Res. 42, 330–1351 (2008). 2008. doi:10.1016/j.asr.2008.01.018.

[3] Romagnoli, D., Circi, C.: Earth-Moon weak stability boundaries in the restricted three and four-body problem. Celest. Mech. Dyn. Astron.103, 79–103 (2009).

[4] Bazso, A. (2012). 43rd lunar and planetary Science Conference.

[5] Bruno,A.D.(1994). *Therestricted3-body problem: Plane periodicorbits*, Walterde Gruyter.

[6] Gutzwiller, M.(1998). *Rev. Modern Phys.*, **70(2)**.

[7] Valtonen, M., Karttunen, H.: The Three-Body Problem. Cambridge University Press, Cambridge (2006)

[8] Chenciner, A. (2007). *Scholarpedia*, **2(10)**, 2111.

[9] Lagrange, J. L. **(**1772). Collected works Paris, vol. VI, pp, 229.

[10] Laplace and Delaunay (1867). Memoire sur la theorie de la Lune, Mem. De l'des Sci. 28-29.

[11] Poincare, H. (1892-1899) *Les methods nouvelles de la mecanique celeste in 3 volumes*, Paris:Gauthier-Villars, reprinted by Dover, New York.

[12] Birhoff, (1927). Dynamical system American Mathematical Society, New-York.

[13] Szebehely, V. (1967). Theory of Orbits: The Restricted Problem of Three Bodies. Academic Press, New York.

[14] Danby, J.M.A.: Fundamentals of Celestial Mechanics, 2nd edn. Willmann-Bell, Virginia (1988)

[15] Fokker, A.D.: A consequence of Einstein's theory of gravitation. The geodesic precession. Ned. Akad. Wet. Proc. 23(1), 729–738 (1921).

[16] Renzetti, G.: Exact geodetic precession of the orbit of a two-body gyroscope in geodesic motion about a third mass. Earth Moon Planets 109(1–4), 55–59 (2012).

[17] Iorio, L. (2014). Orbital motions as gradiometers for post-Newtonian tidal effects, Front. Astron. Space Sci. 1, id 3. Doi:10.3389/fspas.2014.00003.

[18] Greaves, J.S., Holland W.S., Moriarty-Schieven, G., Jenness, T., Dent, (1988). Astrophy. J. 506.

[19] Trilling, D.E., Stansberry, J. A., Stapelfeldt, K.R., Rieke, G.H., Su, K.Y.L., Gray, R.O., Corbally, C.J., Bryden, G., Chen, C.H., Boden, A. and Beichman, C.A.: (2007). Debris disks in main-sequence binary systems. Astrophysical Journal, 658, 1289-1311.

[20] Jiang, I.G., Yeh, L.C. (2003): Int. J. Bifurc. Chaos **13**, 534–539.

[21] Jiang, I.G., Yeh, L.C. (2004a): Rev. Mex. Astron. Astrofís. **21**, 152–155.

[22] Jiang, I.G., Yeh, L.C. (2004b): Astron. J. **128**, 923–932.

[23] Yeh, L.C., Jiang, I.G.: (2006). On the Chermnykh-Like Problems: The Equilibrium Points. Astrophysics and Space Science, 306, 189-200.

[24] Jiang I.G., and Yen L.C. (2014). Galaxies with supermassive binary black holes: A possible model for the centers of galaxies. Astrophys and Space sd. 349-881.

[25] lorio, L. (2007). Dynamicaldetermination of the quadruple mass moment of a white dwarf. *Astrophysics and Space Science,* 310.

[26] lorio, L. (2012). Orbital Perturbations Due to Massive rings: Earth Moon Planet., *Springer* 108(3-4), 189-217.

[27] Pitjeva, E. V., Pitjev, N. P. (2014). Development of planetary ephemerides EPM and their Applications *Celestial Mechanics and Dynamical Astronomy*, 119 ( 3-4), 237-256.

[28] Kushvah, B.S. (2008): Astrophys. Space Sci. **318**, 41–50 .

[29] Singh, J. and Taura, J.J.:(2013). Motion in the generalized restricted three-body problem. Astrophysics and Space Science, 343,95-106.

[30] Singh, J. and Taura, J.J. (2014a). Combined Effect of Oblateness, Radiation and a Circular

[31] Singh, J. and Taura, J.J. (2014b). Effects of Zonal Harmonics and a Circular Cluster of Material Points on the Stability of Triangular Equilibrium Points in the R3BP.*Astrophysics and Space Science*, **350**, 127-132. http://dx.doi.org/10.1007/s10509-013-1719-4.

[32] Singh, I & Umar, A. (2012). On the stability of triangular equilibrium points in the elliptic R3BP under radiating and oblate primaries. *Astrophysics and Space Science*, 341, 349-358. http://dx.doi.org/10.1007/s 10509-012-1109-3.

[33] Miyamoto, M., Nagai, R. (1975): Publ. Astron. Soc. Jpn. **27**, 533–543.