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EFFECT OF PERTURBATIONS IN CORIOLIS AND CENTRIFUGAL FORCES ON THE EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL RESTRICTED FOUR-BODY PROBLEM: OUT-OF-PLANE CASE

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Abstract

The photogravitational restricted four-body problem including the effect of perturbations in Coriolis and centrifugal forces is employed to describe the motion of an infinitesimal particle near the out-of-plane equilibrium points in the vicinity of three finite radiating bodies. The three bodies (P_1, P_2, P_3) are moving in circular orbits about their common centre of mass fixed at the origin of the coordinate system, according to the solution of Lagrange where they are always at the vertices of an equilateral triangle. The fourth body P_4 of infinitesimal mass does not affect the motion of the three bodies. We consider that two of the bodies (P_2 and P_3) have the same radiation and mass value μ while the dominant primary body P_1 is of mass $1-2\mu$. The equilibrium points (L_1^z, L_2^z) lying out of the orbital plane of the three bodies as well as the allowed regions of motion as determined by the zero velocity curves are studied numerically. It is observed that their positions as well as the allowed regions of motion while not affected by the small perturbation in the Coriolis force, are essentially varied under the joint effects of radiation pressure parameters and centrifugal force. Finally the stability of these points is studied and they are found to be unstable.

Keywords: Equilibrium points; Restricted four-body problem; Coriolis and centrifugal forces; Radiation; Zero velocity curves; Stability.

1. Introduction

The Restricted four-body problem (R4BP) is perhaps the simplest model after its famous predecessor, the restricted three-body problem (R3BP) and a natural generalization of it. It deals with the motion of an infinitesimal particle under the Newtonian gravitational attraction of three bodies, called primaries, whose trajectories are the solution of the three Newtonian body problems. It is known that in the planar general problem of three bodies attracting each other according to Newtonian gravitational law, there exist only two permanent central configurations, namely, the collinear (Eulerian) and the equilateral (Lagrangian). In the first case, the primaries of the problem lie on a single straight line while in the second one, the primary bodies lie at the vertices of an equilateral triangle. In analogy with the R3BP, we consider here the restricted problem of four bodies consisting of three primaries moving in circular orbits keeping an equilateral triangle configuration and a massless particle moving under the gravitational attraction of the primaries.

The R4BP has witnessed intensive research contributions in the scientific community resulting into several applications such as the Sun–Jupiter and Saturn system [1]; star, two massive planets and a massless Trojan [2]; star, brown dwarf, gas giant and a massless Trojan [3]; Jupiter, Trojan Asteroid, spacecraft [4]; Spacecraft orbit design in the circular restricted three-body problem [5]. Applications of Gravity Assists in the Bicircular and Bielliptic R4BP and references therein.

In the same vein, many authors and scientists have extended these ideas in the R4BP: The existence of equilibrium points and their stability was treated by [6, 7]. The radiation effect of the problem was carried out by [8—13]. Also, [14] studied the problem with oblate primaries. Positions of the equilibrium points of the problem with triaxiality of the primaries was studied [15—17] and effect of the Coriolis and centrifugal forces of the problem has been investigated [18, 19].

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On the other hand, in recent years many perturbing forces, such as oblateness, radiation forces of the primaries, Coriolis and centrifugal forces, variation of the masses of the primaries etc. have been included in the study of R3BP. The R3BP with oblateness effects has been studied by many investigators such as [20, 21]. The case where the photogravitational effect is taken into account was treated by [22—25], and the small perturbations in the Coriolis and centrifugal forces of the problem was studied by [26, 27].

The case of the R3BP where one or both the primaries are a source of radiation is a long-standing, well-known problem usually referred to as "the photogravitational problem of three bodies''. This is because this case was initially studied by Radzievskii [28, 29]. As it is known, out of plane points have no analogy in the classical R3BP. As we have already mentioned, the existence of the out of the orbital plane equilibrium points, in the photogravitational R3BP was first pointed out by Radzievskii. He found two equilibrium points on the (x, z) plane in symmetrical positions with respect to the (x, y) plane. Reference [30] proved the existence of two more out of plane equilibrium points, L_8 and L_9, which are always linearly unstable. They also established that L_6 and L_7 are linearly stable for a small range of radiation pressures provided both the primaries are luminous. The regions of stability for the above out of plane points ($L_{6,7,8,9}$) were discussed by [31]. Using the three dimensional equations of motion [32, 33] have shown the existence of out of plane points L_6 and L_7 and proved their instability when one or both primaries are oblate with or without radiation and the latter case when the orbits of the primaries are elliptic as well. Reference [34] examined the out of plane equilibrium points $L_{6,7}$ by considering the effect of a small change in Coriolis and centrifugal forces, when the primaries are both radiating and oblate spheroids.

In the present paper, we deal with the equilibrium points which exist out of the orbital plane by considering all the three primary bodies as radiation sources in the presence of small perturbations given in the Coriolis and the centrifugal forces. The primaries are modeled as a radiation sources with two of the bodies having the same radiation and mass value. Also, the infinitesimal body is assumed to have no influence on the motion of these primaries.

Here, our effort is geared towards investigating the effects produced by perturbations in Coriolis and centrifugal forces and the radiation pressure from the primaries on the existence of out of plane equilibrium points and their stability in the R4BP. More precisely, we study the equilibrium points, the zero velocity curves and the linear stability for this problem. This work may be applicable to the study of a test particle in the Sun-Jupiter- Trojan-spacecraft system and the results obtained in this study will have thus practical application in astrophysics.

It is known, that in the Lagrange central configuration the Routh's criterion for the linear stability of the configuration is the inequality,

$$\frac{m_1 m_2 + m_2 m_3 + m_1 m_3}{(m_1 + m_2 + m_3)^2} < \frac{1}{27},$$

where m_1, m_2 and m_3 are the three primary bodies.

In the photogravitational case, the necessary condition for the stability of the configuration is the inequality [8]

$$\frac{q_1 m_1 q_2 m_2 + q_2 m_2 q_3 m_3 + q_1 m_1 q_3 m_3}{(q_1 m_1 + q_2 m_2 + q_3 m_3)^2} < \frac{1}{27}$$

where q_1, q_2 and q_3 are the corresponding radiation pressure forces of the primaries.

In present work, we will consider sets of (m_i, q_i) which satisfy the above condition. So, throughout the paper we shall assume that we have a dominant primary body with mass $m_1 = 0.962$ and two small equal primaries with masses $m_2 = m_3 = \mu = 0.019$. All the development and computations, both algebraic and numerical, have been performed with *Mathematica* ® version 11 [35].

The paper is organized in six sections. Section 2 provides the equations of motion for the system under investigation. Section 3 locates the positions of the out of plane points. Section 4 is devoted to the surfaces and curves of zero velocity. The regions of allowed motion as determined by the zero velocity curves as well as the positions of out of plane points are given. Section 5 established their stability; while Section 6 discusses the obtained results and conclusion of the paper.

2. Equations of Motion

The system we consider is the motion of an infinitesimal object, e.g. a spacecraft, in the presence of three finite celestial objects (P_1, P_2, P_3) with masses m_1, m_2 and m_3, which we treat as point mass. We suppose that $m_1 >> m_2 = m_3$ always lie at the vertices of an equilateral triangle and one of them, m_1 (say), is on the negative *x*-axis at the origin of time. This system is also dimensionless, i.e., we normalize the units with the supposition such that the sum of the masses, the separation between the primaries, the Gaussian constant and unit of time are equal to unity. The equilateral configuration is possible for all distributions of the masses, whilst the fourth body of negligible mass moves in the same plane. Let us denote the small perturbations in Coriolis and centrifugal forces by ε_1 and ε_2 respectively, with $\varepsilon_1, \varepsilon_2 << 1$. The radiation pressure parameters of the primaries q_1, q_2, q_3, using the same notation as in the classical three-body problem [8], are given by the relations $q_i = 1 - b_i$, i = 1, 2, 3 where b_1, b_2 and b_3 are the ratios of the force F_r which is caused by radiation to the force F_g which results from gravitational due to the three primary bodies. For $i = 1,2,3$, it is clear that: If q_i =1, the radiation pressure has no effect. If $0 < q_1 < 1$, the gravitational force is greater than the radiational one. If q_i =0, the radiation force balances the gravitational one. If $q_i < 0$, the radiation pressure overrides the gravitational attraction. Let the coordinates of the infinitesimal mass be (*x*, *y*) and masses m_1, m_2 and m_3 are

$(-\sqrt{3}\mu, 0)$, $(\frac{\sqrt{3}}{2}(1-2\mu), -\frac{1}{2})$ and $(\frac{\sqrt{3}}{2}(1-2\mu), \frac{1}{2})$ respectively, relative to the rotating frame of reference $Oxyz$, where $\mu = \frac{m_2}{m_1+m_2+m_3} = \frac{m_3}{m_1+m_2+m_3} < \frac{1}{2}$ is the mass parameter.

Now we consider perturbations in the Coriolis and centrifugal forces with the help of the parameter α and β, respectively, such that $\alpha = 1+\varepsilon_1, |\varepsilon_1| << 1$ and $\beta = 1+\varepsilon_2, |\varepsilon_2| << 1$. The unperturbed value of each is unity.

The equations of motion of the infinitesimal mass (Spacecraft) of the photogravitational R4BP under small perturbations in the Coriolis and centrifugal forces, in the rotating coordinate system following [26, 8] are

$$\begin{aligned} \ddot{x} - 2\alpha\dot{y} &= \Omega_x \\ \ddot{y} + 2\alpha\dot{x} &= \Omega_y \\ \ddot{z} &= \Omega_z \end{aligned} \qquad (1)$$

where

$$\Omega = \frac{\beta}{2}(x^2+y^2) + \frac{q_1(1-2\mu)}{r_1} + \frac{q_2\mu}{r_2} + \frac{q_3\mu}{r_3} \qquad (2)$$

and

$$r_1^2 = (x+\sqrt{3}\mu)^2 + y^2 + z^2$$

$$r_2^2 = (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y+\frac{1}{2})^2 + z^2$$

$$r_3^2 = (x - \frac{\sqrt{3}}{2}(1-2\mu))^2 + (y-\frac{1}{2})^2 + z^2$$

where r_1, r_2 and r_3 are the distances of the infinitesimal body from the primaries, Ω is the photogravitational potential, dots denote time derivatives, the suffixes *x* and *y* indicate the partial derivatives of Ω with respect to *x* and *y* respectively.

The energy (Jacobi) integral of system (1) is given by the expression

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega - C \qquad (3)$$

where *C* is the Jacobi constant.

We note here that in the previous studies [8, 10, 12, 23, 29, 30] a necessary condition in order to exist critical points out of plane, is that the radiation parameter must be negative. As a consequence, we shall investigate in the next section the existence and location of out of plane equilibrium points in the case where $-1 < q_1 \le 0, 0 < q_2 \le 1$ (since $q_2 = q_3$), $\alpha = 1+\varepsilon_1, |\varepsilon_1| << 1$ and $\beta = 1+\varepsilon_2, |\varepsilon_2| << 1$.

3. Positions of out-of-plane equilibrium points

The positions of the out-of-plane equilibrium points are the solutions of equations (1) with

$$\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0 \qquad (4)$$

and considering z ≠0. Thus, equations (1) yield $\Omega_x = \Omega_y = \Omega_z = 0$ implying that

$$\beta x - \frac{(1-2\mu)(x+\sqrt{3}\mu)q_1}{r_1^3} - \frac{q_2(x-\frac{\sqrt{3}}{2}(1-2\mu))\mu}{r_2^3} - \frac{q_3(x-\frac{\sqrt{3}}{2}(1-2\mu))\mu}{r_3^3} = 0 \qquad (5)$$

$$\beta y - \frac{(1-2\mu)yq_1}{r_1^3} - \frac{(y+\frac{1}{2})\mu q_2}{r_2^3} - \frac{q_3(y-\frac{1}{2})\mu}{r_3^3} = 0 \qquad (6)$$

$$\left(\frac{(1-2\mu)q_1}{r_1^3} + \frac{\mu q_2}{r_2^3} + \frac{\mu q_3}{r_3^3}\right)z = 0 \qquad (7)$$

If $y = 0$, equation (6) is fulfilled (since $q_2 = q_3$) and we solve equations (5) and (7) for $y = 0$ and $z \neq 0$. When the effects of perturbations in the Coriolis and centrifugal forces as well as radiation of the primaries are neglected (classical case), the resulting system has not any real solution ($x, 0, z$) since its second equation (equation (7)) is fulfilled only for $z = 0$. Hence, as in the classical (gravitational) three-body problem, the classical four-body problem has not equilibrium points in the present problem. Now equations (5) and (7) are expressed, respectively, as

$$\beta x_0 - \frac{(1-2\mu)(x_0+\sqrt{3}\mu)q_1}{r_{10}^3} - \frac{2q_2(x_0 - \frac{\sqrt{3}}{2}(1-2\mu))\mu}{r_{20}^3} = 0 \qquad (8)$$

$$\frac{(1-2\mu)q_1}{r_{10}^3} + \frac{2q_2\mu}{r_{20}^3} = 0 \qquad (9)$$

where

$$r_{10}^2 = (x_0+\sqrt{3}\mu)^2 + z_0^2, \qquad r_{20}^2 = r_{30}^2 = (x_0 - \frac{\sqrt{3}}{2}(1-2\mu))^2 + \frac{1}{4} + z_0^2, \qquad q_2 = q_3$$

and the subscript '0' is used to denote the equilibrium values.
From equation (9) we have that

$$\frac{r_{20}}{r_{10}} = \left[\left(\frac{-q_2}{q_1}\right)\frac{2\mu}{1-2\mu}\right]^{\frac{1}{3}} \equiv k \qquad (10)$$

which means that if k = constant the locus of these points is an Apollonius circle . From equation (10) it can be seen that, for the existence of any real solution, one of the following conditions is necessary to hold:

$$q_1 q_2 < 0 \quad \text{or} \quad q_1 = q_2 = 0 \qquad (11)$$

The second condition means that the gravitational attractions balance the corresponding radiation pressure forces. This case will not be considered here.
The first condition means that the radiation pressure force of just one of the primaries exceeds its gravitational attraction.
The existence of these points is of particular astronomical interest in connection with planetary system formation, satellite motion, etc. These points are found to be located in the (x, z) plane in symmetrical positions with respect to the (x, y) plane. They are dynamically equivalent, which means that are characterized by the same Jacobian constant C and by the same state of stability. As it is known, such points do not appear if only gravitational forces are considered. For any given μ , the existence, position and stability of these points depend on α , β, q_1 and q_2. So, the value of radiation factors could be taken in $0 < q_2 \le 1$, $-1 \le q_1 \le 0$ and the effect of Coriolis and centrifugal forces in $\alpha = 1+\varepsilon_1, |\varepsilon_1| << 1$ and $\beta = 1+\varepsilon_2, |\varepsilon_2| << 1$ respectively.
Now, for $\mu = 0.019$, $\beta = 1+\varepsilon_2$, and $0 < q_2 \le 1$ there are intervals of q_1 of the form $-1 \le q_1 \le 0$ for which there exist out of the plane equilibrium points, which we call here L_1^z and L_2^z . The positions are shown numerically in Tables 1—3 under the joint effects of radiation pressure parameters and centrifugal force. Figures 1, 2 and 3 show the positions of out-of-plane points in (x, q_2) and (z, q_2), as q_2 varies, for fixed values of q_1 and β. Figure 1 shows the effect of increasing q_2 in the interval [1, 0.77] for fixed $q_1 = -0.03$ and $\beta = 1.01$, Figure 2 corresponds to q_2 in the interval [1, 0.26] for fixed $q_1 = -0.01$ and $\beta = 1.01$ while Figure 3 corresponds to q_2 in the interval [1, 0.77] for fixed $q_1 = -0.03$ and $\beta = 1.2$. From the results we conclude that the radiation effects from the primaries is more significant than the centrifugal force on the positions of out of plane equilibrium points. We also observe that when there are small perturbations ε_1 and ε_2 in the Coriolis and centrifugal forces in the present case, the number of equilibrium points are the same as that in the problem with no perturbations [10], but positions of the equilibrium points have changed. The small change in the Coriolis force does not affect the positions of the equilibrium points.

Table 1. Effect of radiation and centrifugal forces on the positions of out-of-plane equilibrium points for $\mu = 0.0190$, $0 < q_2 \leq 1$, $q_1 = -0.03$ and $\beta = 1.01$

q_2	x_0	$\pm z_0$
1.00	−0.00242476	2.16888
0.99	−0.00228217	2.21315
0.98	−0.00214194	2.26044
0.97	−0.00200415	2.31110
0.96	−0.00186892	2.36556
0.95	−0.00173633	2.42431
0.94	−0.00160651	2.48794
0.93	−0.00147959	2.55715
0.92	−0.00135568	2.63280
0.91	−0.00123495	2.71595
0.90	−0.00111755	2.80792
0.89	−0.00100365	2.91037

Table 2. Effect of radiation and centrifugal forces on the positions of out-of-plane equilibrium points for $\mu = 0.0190$, $0 < q_2 \leq 1$, $q_1 = -0.01$ and $\beta = 1.01$

q_2	x_0	$\pm z_0$
1.00	−0.0158335	0.804461
0.99	−0.0155804	0.808791
0.98	−0.0153275	0.813211
0.97	−0.0150749	0.817725
0.96	−0.0148225	0.822336
0.95	−0.0145704	0.827048
0.94	−0.0143187	0.831864
0.93	−0.0140672	0.836788
0.92	−0.0138160	0.841825
0.91	−0.0135652	0.846979
0.90	−0.0133148	0.852253
0.89	−0.0130647	0.857654

Table 3. Effect of radiation and centrifugal forces on the positions of out-of-plane equilibrium points for $\mu = 0.0190$, $0 < q_2 \leq 1$, $q_1 = -0.03$ and $\beta = 1.2$

q_2	x_0	$\pm z_0$
1.00	−0.00204298	2.16812
0.99	−0.00192272	2.21242
0.98	−0.00180447	2.25974
0.97	−0.00168829	2.31043
0.96	−0.00157428	2.36492
0.95	−0.00146251	2.42370
0.94	−0.00135309	2.48736
0.93	−0.00124612	2.55660
0.92	−0.00114171	2.63228
0.91	−0.00103997	2.71545
0.90	−0.00094106	2.80747
0.89	−0.00084510	2.90995

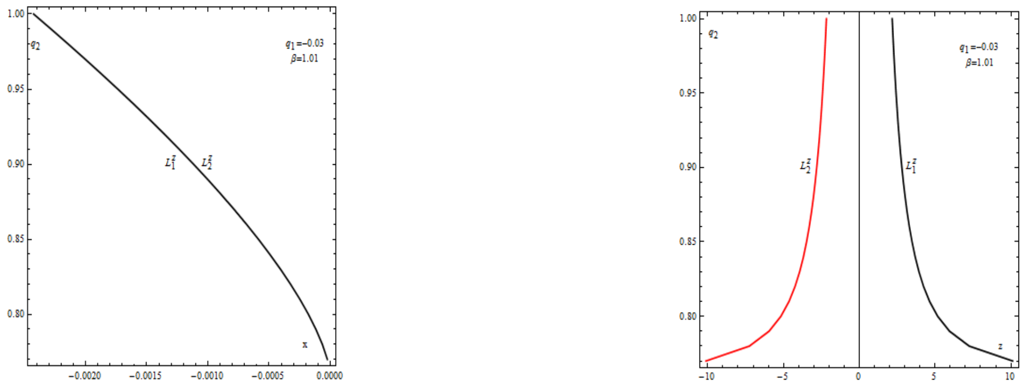

Fig. 1. Position of L_1^z and L_2^z in the ($x-z$) plane as a function of q_2 in the interval [1, 0.77], for $q_1=-0.03$, $\beta=1.01$, $\mu=0.019$

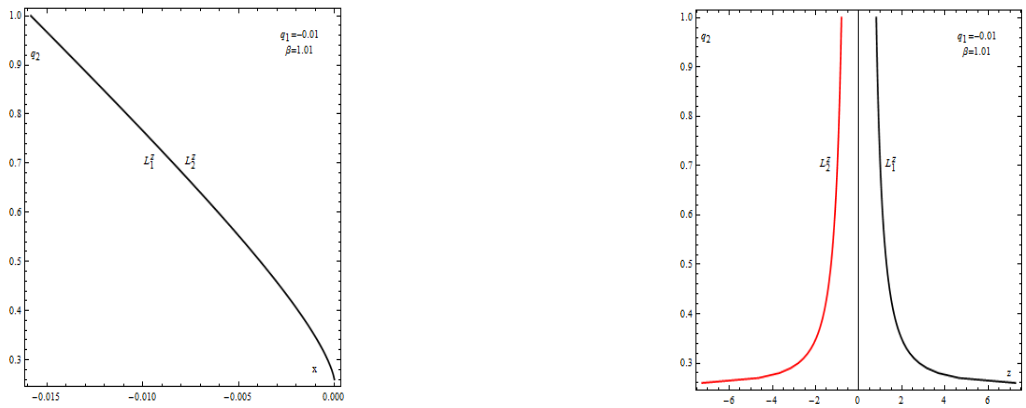

Fig. 2. Position of L_1^z and L_2^z in the ($x-z$) plane as a function of q_2 in the interval [1, 0.26], for $q_1=-0.01$, $\beta=1.01$, $\mu=0.019$

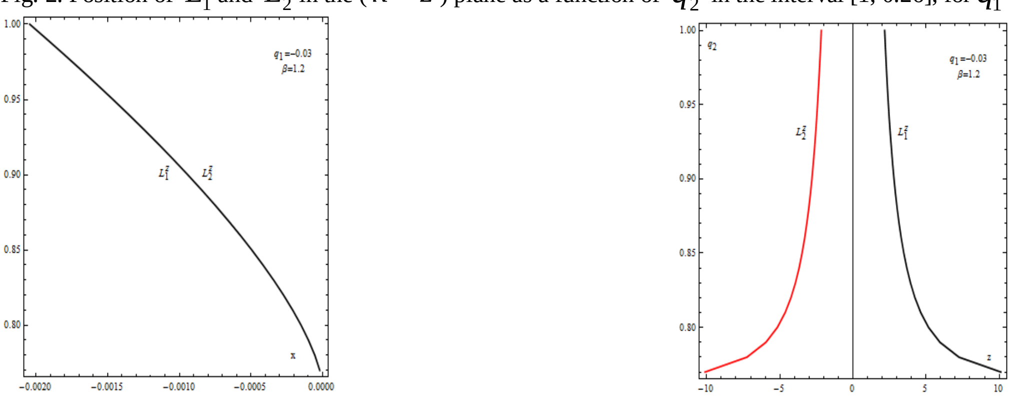

Fig. 3. Position of L_1^z and L_2^z in the ($x-z$) plane as a function of q_2 in the interval [1, 0.77], for $q_1=-0.03$, $\beta=1.2$, $\mu=0.019$

4. Zero velocity curves in the ($x-z$) plane

The usefulness of the Jacobi constant integral in clarifying certain general properties of the relative motion of a small body by the construction and investigation of zero velocity curves in every problem of celestial dynamics was pointed out by many investigators in the past. In this section we present the contours of the surface (3) on the ($x-z$) plane, for zero velocity, which provide the zero velocity curves. In Fig. 4 we present the zero velocity curves under the joint effects of radiation and centrifugal forces for fixed value of $\mu=0.019$. We have plotted only the curves for Jacobi constant values corresponding to the out of plane equilibrium point L_1^z and L_2^z . In Fig. 4 panel (a) we consider $q_1=-0.03$, q_2 =0.99 and $\beta=1.01$; panel

(b) $q_1 = -0.03$, q_2 =0.99 and $\beta = 1.2$ and panel (c) $q_1 = -0.01$, q_2 =0.5 and $\beta = 1.01$. Large (black) dots indicate the primary bodies, while the small (black) ones are the out of plane equilibrium points of the problem. We note here that between centre of the dominant primary P_1 and its companion out of plane equilibrium points, the zero velocity curves form small rhombus of regions not allowed to motion which shrink very fast under the variation of radiation pressure and the particle comes very close to P_1 (Fig. 4 panel (c)). In Fig. 4 panel (d) zero velocity curves which correspond to combined effect of panels (a), (b) and (c) are represented on figure as blue, green and red respectively.
From these figures, it is obvious that the radiation pressure of the primary bodies effect significantly the structure of the regions allowed to motion of the particle as determined by the zero velocity surface compare to the effects of perturbations in Coriolis and centrifugal forces.

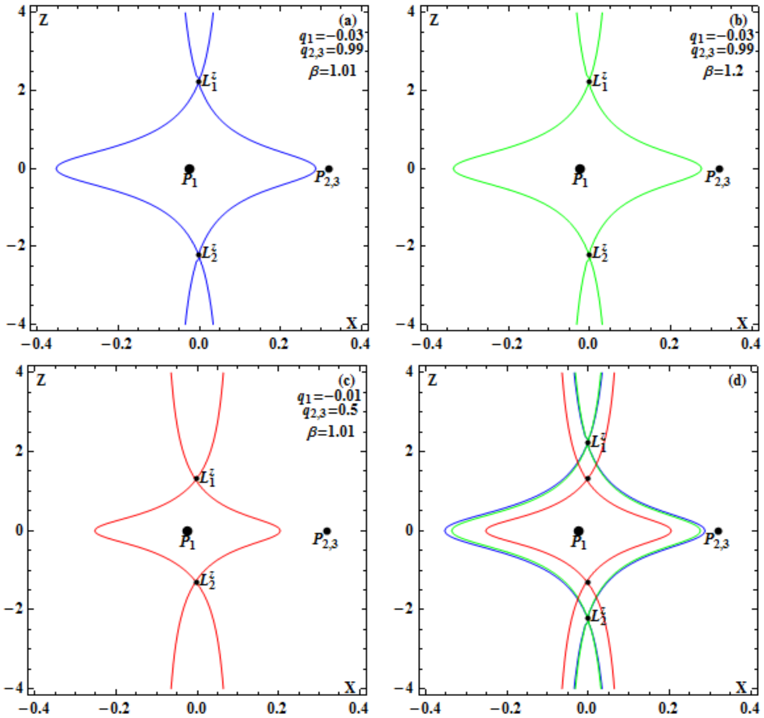

Fig. 4. Zero velocity curves in the ($x - z$) plane for (a) $q_1 = -0.03, q_2 = q_3 = 0.99, \beta = 1.01$, C=0.00504603876 (b) $q_1 = -0.03,\ q_2 = q_3 = 0.99, \beta = 1.2$, C=0.00504687247 (c) $q_1 = -0.01,\ q_2 = q_3 = 0.5, \beta = 1.01$, C=0.00860476470 (d) the zero velocity curves which correspond to combined effect of (a), (b) and (c) are represented on figure as blue, green, red, respectively. Large (black) dots are the primary bodies and small (black) ones are the equilibrium points of the problem. The mass parameter for all cases is $\mu = 0.019$.

5. Linear stability of out-of-plane equilibrium points

In order to study the linear stability of the out of plane equilibrium point ($x_0, 0, z_0$) we displace the infinitesimal body to the point ($x_0 + \xi, \eta, z_0 + \zeta$) where $\xi\ \eta, \zeta$, are the corresponding perturbations along the axes *Ox*, *Oy* and *Oz*. So we linearize the system (1) to obtain the variational equations of motion as

$$\begin{aligned} &\ddot{\xi} - 2\alpha\dot{\eta} = \Omega^0_{xx}\xi + \Omega^0_{xy}\eta + \Omega^0_{xz}\zeta \\ &\ddot{\eta} + 2\alpha\dot{\xi} = \Omega^0_{yx}\xi + \Omega^0_{yy}\eta + \Omega^0_{yz}\zeta \qquad (12) \\ &\ddot{\zeta} = \Omega^0_{zx}\xi + \Omega^0_{zy}\eta + \Omega^0_{zz}\zeta \end{aligned}$$

where the superscript 'o' indicates that the partial derivatives are to be evaluated at out of plane points ($x_0, 0, z_0$)

Explicitly, the partial derivatives of system (12) are

$\Omega^0_{xy} = \Omega^0_{yx} = \Omega^0_{yz} = \Omega^0_{zy} = 0,$

$$\Omega^0_{xx}=\beta-\frac{q_1(1-2\mu)}{r^3_{10}}[1-\frac{3(x_0+\sqrt{3}\mu)^2}{r^2_{10}}]-\frac{2q_2\mu}{r^3_{20}}[1-\frac{3(x_0-\frac{\sqrt{3}}{2}(1-2\mu))^2}{r^2_{20}}]$$

$$\Omega^0_{zx}=3z_0[\frac{q_1(1-2\mu)(x_0+\sqrt{3}\mu)}{r^5_{10}}+\frac{2q_2\mu(x_0-\frac{\sqrt{3}}{2}(1-2\mu))}{r^5_{20}}]$$

$$\Omega^0_{yy}=\beta-\frac{q_1(1-2\mu)}{r^3_{10}}+\frac{2\mu q_2}{r^5_{20}}[\frac{3}{4}-r^2_{20}]$$

$$\Omega^0 zz=\frac{(1-2\mu)q_1}{r^3_{10}}[\frac{3z^2_0}{r^2_{10}}-1]+\frac{2\mu q_2}{r^3_{20}}[\frac{3z^2_0}{r^2_{20}}-1]$$

with

$$r^2_{10}=(x_0+\sqrt{3}\mu)^2+z^2_0,\quad r^2_{20}=r^2_{30}=(x_0-\frac{\sqrt{3}}{2}(1-2\mu))^2+\frac{1}{4}+z^2_0,\quad q_2=q_3$$

The characteristic equation corresponding to system (12) is

$$\lambda^6+a\lambda^4+b\lambda^2+c=0 \qquad (13)$$

with

$$a=4\alpha^2-\Omega^0xx-\Omega^0yy-\Omega^0zz$$

$$b=\Omega^0xx\Omega^0yy+\Omega^0yy\Omega^0zz+\Omega^0zz\Omega^0xx-4\Omega^0zz-(\Omega^0xz)^2$$

$$c=(\Omega^0xz)^2\Omega^0yy-\Omega^0xx\Omega^0yy\Omega^0zz$$

which is a polynomial of sixth degree in λ

The eigenvalues of the characteristic equation (13) determine the stability or instability of the respective equilibrium points. An equilibrium point will be stable if the characteristic equation has six imaginary roots or complex roots with non-positive real parts.

We have computed the characteristic roots $\lambda_i, i=1,2,\ldots 6$ as the perturbation parameters increase ($0<q_2\le 1$, $-1\le q_1\le 0$, $\alpha=1+\varepsilon_1,|\varepsilon_1|<<1$ and $\beta=1+\varepsilon_2,|\varepsilon_2|<<1$) with an arbitrary small step and found no case in which all the roots are all imaginary. This lead to the instability of the out of plane points.

6. Discussion and conclusion

In this paper we have studied the existence, location and the stability of the out of plane equilibrium points for particles moving in the vicinity of three massive bodies which emit light radiation under the influence of small perturbations in the Coriolis and centrifugal forces formulated on the basis of Lagrange equilateral triangle configuration. As it is known, such points do not appear if only the gravitational forces are considered. Solutions of equations (8) and (9) give the positions of the out of plane points, they are affected by the radiation pressure and the perturbation in centrifugal force, but not by that of Coriolis force because equations (8) and (9) are independent of parameter α There are two out of plane equilibrium points that lie in the ($x-z$) plane in symmetrical positions with respect to the (x, y) plane. The effects of the parameters involved on the positions of the out of plane points are shown numerically (Tables 1─3) and graphically in Figures 1─3. The existence of these points is of particular astronomical interest in connection with planetary system formation, satellite motion, etc. The involved parameters is also seen to have significant effects on the topology of the zero velocity curves in the ($x-z$) plane (Fig. 4 panels (a)—(c)). In particular, between the centre of the dominant primary body and its companion out of plane equilibrium points, the zero velocity curves form small rhombus of regions not allowed to motion which shrink very fast under the variation of radiation pressures and the particle comes very close to P_1. Finally, the stability of these points has been achieved numerically by determining the roots of the characteristic equation. The numerical investigation of these roots shows no case in which the roots are all imaginary in spite of the introduction of aforementioned parameters. Consequently, the motion is unbounded, and thus unstable, which agree with [8] when only the dominant primary body is a radiation source and with [10] in the absence of Coriolis and centrifugal forces.

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