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THE INFLUENCE OF A CIRCUMBINARY DISK ON THE COLLINEAR POINTS IN THE ELLIPTIC RESTRICTED THREE - BODY PROBLEM

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Abstract

The dynamics of an infinitesimal particle around spherical primaries moving in elliptic orbits around their common barycenter surrounded by a circumbinary disk in the neighborhood of collinear libration points is studied. The positions of these points are found to be remarkably affected by the gravitational potential from the circumbinary disk. In addition to the three collinear points $L_i(i = 1, 2, 3)$ in the classical restricted three – body problem, there appear two more collinear points (L_a and L_b) which are as a result of the gravitational potential from the circumbinary disk. It is observed graphically and numerically that, with the introduction of the gravitational potential from circumbinary disk, the stability behavior remain unchanged. The collinear points are linear unstable.

***Keywords*:** Circumbinary Disk – ER3BP – Collinear points

1. Introduction

The restricted problem of three bodies describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called the primaries, which move in circular or elliptic orbits around their centre of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. A prominent example of the classical restricted three-body problem is the movement of a satellite around the sun - earth system. The restricted three-body problem with perturbed forces such as oblateness, radiation pressure, triaxiality, solar wind and gravitational potential from the disk has been studied by many scientist. The R3BP posses five libration points; three such points L_1, L_2, L_3 are on the line joining the primaries called collinear points while L_4, L_5 (triangular points) form equilateral triangles with the primaries. The collinear points are however linearly unstable, while the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$ where the critical mass parameter $\mu_c = 0.03852 \ldots\ldots$ and mass ratio $\mu = \frac{M_2}{M_1+M_2} \leq \frac{1}{2}$ [1].

The classical CR3BP consider the bodies to be strictly spherical, but some bodies in the solar system (e.g; Earth, Jupiter and Saturn) and in the stellar system (e.g; Regulus, Peanut binary, Antares and Altair) are sufficiently oblate [2]. The oblateness of a body can produce deviation from the two-body motion. The orbits of celestial bodies are mostly elliptic not circular, Thus, the study of the ER3BP has significant effects. It has motivated many researchers [1] - [10] to study ER3BP in different perspectives.

The influence of the eccentricity of the orbits of the primaries bodies on the existence of the equilibrium points and their stability has been the subject of a number of communications [4,5,6,7]. Then, the effects of the luminosity and oblateness of both primary bodies on the collinear libration points of the binary systems Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB and Xi Bootis moving in elliptic orbits around common centre of mass was investigated by [6]. Later, the investigation of the resultant effect of radiation of the bigger primary, oblateness up to the zonal harmonic J_4 of the smaller primary and gravitational potential from the belt on the sites and stability of collinear points in CR3BP was carried out by [11].

The present work aims to investigate the effects of semi-major axis, eccentricity of the orbit and gravitational potential from the circumbinary disk on the locations and stability of the collinear libration points in the ER3BP.

The remaining parts of the work is organized as follows: Section 2 presents the equation of motion, section 3 & 4 describes the positions and investigate the linear stability of the collinear libration points; Section 5 contains numerical result of the problem, Finally, in Section 6 we have discussed the results obtained.

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2. Equations of motion

The equations of motion in the elliptic restricted three body problem (ER3BP) when the primaries are surrounded by a circumbinary disc in dimensionless - pulsating coordinate system (ξ, η, ς) are giving as:

$$\xi'' - 2\eta' = \Omega_\xi$$
$$\eta'' + 2\xi' = \Omega_\eta$$
$$\varsigma'' = \Omega_\varsigma \qquad (1)$$

With the force function

$$\Omega = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\frac{\xi^2+\eta^2}{2} + \frac{1}{n^2}\left\{\frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{M_b}{(r^2+T^2)^{\frac{1}{2}}}\right\}\right] \qquad (2)$$

The mean motion, n, is given by

$$n^2 = \frac{1}{a}\left(1 + \frac{3e^2}{2} + \frac{2M_b r_c}{(r_c^2+T^2)^{\frac{3}{2}}}\right) \qquad (3)$$

$$r_i^2 = (\xi-\xi_i)^2 + \eta^2 + \zeta^2 \qquad i = 1,2; \quad \xi_1 = -\mu \quad \xi_2 = (1-\mu)$$
$$\mu = \frac{m_2}{m_1+m_2} \qquad (4)$$

Where $\frac{M_b}{(r_c^2+T^2)^{\frac{1}{2}}}$ is the potential due to the Disk [13] and [11], where M_b is the total mass of the disk, r is the radial distance of the infinitesimal body and is given by $r^2 = \xi^2+\eta^2$, $T = b + d$, *b* and *d* are parameter which determine the density profile of the circular cluster of material points. The parameter b controls the flatness of the profile and is known as flatness parameter. The parameter d controls the size of the core of the density profile and is called the core parameter when *b = d = 0*, the potential equals to the one by a point mass, r_c is the radial distance of the infinitesimal body in the classical restricted and *n, a, e, A* are the mean motion, semi – major axis, eccentricities of the orbits, oblateness respectively in 3BP. And the prime represents differentiation with respect to the eccentric anomaly E which describes the position of a particle moving along an elliptic keplerian orbit.

3. Locations of the Collinear Libration Points

To obtain the collinear points, we obtain the first derivative of equation (2) with respect to ξ, η *and* ζ respectively and equate them to zero. That is $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$ since the collinear points lies only on the ξ axis, it implies that $\eta = \zeta = 0$ on the system i.e

$$\Omega_\xi = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi+\mu)}{r_1^3} + \frac{\mu(\xi+\mu-1)}{r_2^3} + \frac{M_b\xi}{(r^2+T^2)^{\frac{3}{2}}}\right\}\right] = 0$$
$$\Omega_\eta = \frac{1}{(1-e^2)^{\frac{1}{2}}}\eta\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}}\right\}\right] = 0$$
$$\Omega_\zeta = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\frac{-\zeta}{n^2}\left\{\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}}\right\}\right] = 0 \qquad (5)$$

From equation (4), when $\eta = \zeta = 0$

$$|r_1|^2 = |\xi+\mu|^2 \qquad |r_2|^2 = |\xi+\mu-1|^2 \qquad (6)$$
$$\xi n^2 - \frac{(1-\mu)(\xi+\mu)}{|\xi+\mu|^3} - \frac{\mu(\xi+\mu-1)}{|\xi+\mu-1|^3} - \frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}} = 0 \qquad (7)$$

The collinear points lie on the line joining the primaries. To locate them, the orbital plane is divided into three intervals $x < -\mu$, $-\mu < x < 1-\mu$ and $x > 1-\mu$ with respect to the primaries.

From equation (7) it is seen that whenever $T < \sqrt{2}\mu$ and

$n^2\left(-\frac{T}{\sqrt{2}}\right) - \frac{(1-u)\left(-\frac{T}{\sqrt{2}}+\mu\right)}{\left|-\frac{T}{\sqrt{2}}+\mu\right|^3} - \frac{\mu\left(-\frac{T}{\sqrt{2}}+\mu-1\right)}{\left|-\frac{T}{\sqrt{2}}+\mu-1\right|^3} - \frac{M_b\left(-\frac{T}{\sqrt{2}}\right)}{\left(\left(-\frac{T}{\sqrt{2}}\right)^2+T^2\right)^{\frac{3}{2}}} > 0$ in the interval $(-\mu, 0)$, Equation (7) will have five collinear points (figure 2).

Now, from equation (7) and using Mathematica Software, we obtain the collinear libration points (L_1, L_2, L_3, L_a and L_b) and is shown in Table 1 – 3.

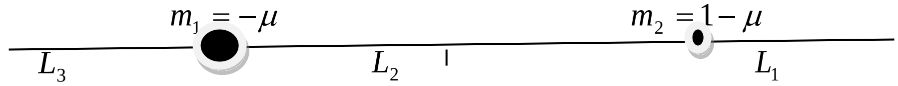


Figure 1: Positions of the collinear points in the classical case

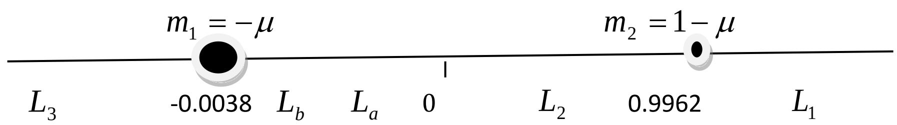


Figure 2: Positions of the collinear points under the effects of the disk

Table1. Effect of Semi-major axis on the collinear points when $\mu = 0.0038, M_b = 0.01, e = 0.3$ and $T = 0.01$.

A	L_1	L_2	L_3	L_a	L_b
0.85	1.07587	0.856639	-0.907452	-0.00378	-0.003818
0.80	1.07118	0.846504	-0.922379	-0.00378	-0.003818
0.75	1.06669	0.834706	-0.870463	-0.00378	-0.003818
0.70	1.06237	0.821100	-0.850725	-0.00378	-0.003818
0.65	1.05821	0.805576	-0.830022	-0.00378	-0.003818
0.60	1.05418	0.788060	-0.808229	-0.00379	-0.003818

Table 2. Effect of eccentricity on the collinear points when $\mu = 0.0038, M_b = 0.01, a = 0.9$ and $T = 0.01$.

e	L_1	L_2	L_3	L_a	L_b
0.10	1.09166	0.879413	-0.959224	-0.00378	-0.003818
0.15	1.08972	0.877318	-0.953514	-0.00378	-0.003818
0.20	1.08717	0.874313	-0.945744	-0.00378	-0.003818
0.25	1.08413	0.869518	-0.936117	-0.00378	-0.003818
0.30	1.08075	0.865291	-0.924865	-0.00378	-0.003818
0.35	1.07717	0.859123	-0.912238	-0.00378	-0.003818
0.40	1.07350	0.851769	-0.898491	-0.00378	-0.003818

Table 3. Effect of circumbinary disk on the collinear points when $\mu = 0.0038,\ a = 0.9, e = 0.3$ and $T = 0.01$.

M_b	L_1	L_2	L_3	L_a	L_b
0.01	1.08075	0.865291	-0.924865	-0.00378	-0.003818
0.10	1.07289	0.858612	-0.906472	-0.00377	-0.003830
0.20	1.06625	0.851801	-0.890507	-0.00376	-0.003842
0.30	1.06167	0.845661	-0.877825	-0.00374	-0.003856
0.40	1.05688	0.840166	-0.867500	-0.00374	-0.003859
0.50	1.05340	0.835263	-0.858925	-0.00362	-0.005570

4. Stability of the collinear equilibrium points

To study the stability of the collinear libration points L_i $(i=1,2,3)$; we consider the characteristic equation of the system in [4] given by

$$\lambda^4 - \left(\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\right)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 = 0 \tag{8}$$

Here, we obtain the points corresponding to the collinear by taking the second partial derivatives of equation (2), with $\eta = 0$. Thus we have

$$\Omega_{\xi\xi} = \frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[1 + \frac{2}{n^2}\left\{\frac{(1-\mu)}{|\xi+\mu|^3} + \frac{\mu}{|\xi+\mu-1|^3} - \frac{M_b}{2\left(\xi^2+T^2\right)^{\frac{3}{2}}} + \frac{3M_b\xi^2}{2\left(\xi^2+T^2\right)^{\frac{5}{2}}}\right\}\right] \quad \Omega_{\eta\eta} = \frac{1}{\left(1-e^2\right)^{\frac{1}{2}}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)}{|\xi+\mu|^3} + \frac{\mu}{|\xi+\mu-1|^3} + \frac{M_b}{\left(\xi^2+T^2\right)^{\frac{3}{2}}}\right\}\right]$$

$$\Omega_{\xi\eta} = \Omega_{\eta\xi} = 0$$

$$\Omega_{\zeta\zeta} = 0 \tag{9}$$

Since $\dfrac{3M_b\xi^2}{2\left(\xi^2+T^2\right)^{\frac{5}{2}}} > \dfrac{M_b}{2\left(\xi^2+T^2\right)^{\frac{3}{2}}}$ and the remaining terms are all positive implies that

$$\Omega^0_{\xi\xi} > 0$$

4.1 Stability of $L_1\left(\xi > \xi_2\right)$

In the first interval, we have; $r_1 = \xi + \mu \quad \Rightarrow \xi = r_1 - \mu$ and $r_2 = \xi + \mu - 1$ (10)

Substituting equation (10) in the equation (7) yields

$$\frac{(1-\mu)}{r_1^2}=\left(\xi n^2-\frac{\mu}{r_2^2}-\frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right) \tag{11}$$

Substituting equation (11) in the second equation of (9) gives;

$$\Omega_{\eta\eta}=\frac{1}{(1-e^2)^{\frac{1}{2}}}\left[1-\frac{1}{n^2}\left\{\frac{1}{r_1}\left(\xi n^2-\frac{\mu}{r_2^2}-\frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right)+\frac{\mu}{r_2^3}+\frac{M_b}{(\xi^2+T^2)^{\frac{3}{2}}}\right\}\right]$$

Implies that

$$\Omega_{\eta\eta}=\frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\frac{\mu}{r_1}+\frac{1}{n^2}\left\{\frac{\mu}{r_2^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{M_b(r_1-\mu)}{[(r_1-\mu)^2+T^2]^{\frac{3}{2}}}\left(\frac{1}{r_1}-\frac{1}{(r_1-\mu)}\right)\right\}\right]$$

Thus, $\Omega^0_{\eta\eta}<0$. Since $M_b << 1, \mu<\frac{1}{2}$ $, r_1>1$ and $r_2<1$

4.2 Stability of $L_2(\xi_1<\xi<\xi_2)$

In the second interval, we have; $r_1=(\xi+\mu)\Rightarrow\xi=(r_1-\mu)$ and $r_2=-(\xi+\mu-1)$. (12)

Substituting equation (12) in the equation (7) yields

$$\frac{(1-\mu)}{r_1^2}=\left(\xi n^2+\frac{\mu}{r_2^2}-\frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right) \tag{13}$$

Substituting equation (13) in the second equation of (9) gives;

$$\Omega_{\eta\eta}=\frac{1}{(1-e^2)^{\frac{1}{2}}}\left[1-\frac{1}{n^2}\left\{\frac{1}{r_1}\left(\xi n^2+\frac{\mu}{r_2^2}-\frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right)+\frac{\mu}{r_2^3}+\frac{M_b}{(\xi^2+T^2)^{\frac{3}{2}}}\right\}\right] \tag{14}$$

Implies that

$$\Omega_{\eta\eta}=\frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\frac{\mu}{r_1}-\frac{1}{n^2}\left\{\frac{\mu}{r_2^2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)-\frac{M_b(r_1-\mu)}{[(r_1-\mu)^2+T^2]^{\frac{3}{2}}}\left(\frac{1}{r_1}-\frac{1}{(r_1-\mu)}\right)\right\}\right]$$

Thus, $\Omega^0_{\eta\eta}<0$, Since $\mu<\frac{1}{2}$, $M_b << 1$, $r_1>1$ and $r_2>1$

4.3 Stability of $L_3(\xi_1>\xi)$

In the third interval, we have; $r_1=-(\xi+\mu)\Rightarrow\xi=-(r_1+\mu)$ and $r_2=-(\xi+\mu-1)$. (15)

Substituting equation (15) in the equation (7) yields

$$\frac{(1-\mu)}{r_1^2}=-\left(\xi n^2+\frac{\mu}{r_2^2}-\frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right) \tag{16}$$

Substituting equation (16) in the second equation of (9) gives;

$$\Omega_{\eta\eta}=\frac{1}{(1-e^2)^{\frac{1}{2}}}\left[1-\frac{1}{n^2}\left\{-\frac{1}{r_1}\left(\xi n^2+\frac{\mu}{r_2^2}-\frac{M_b\xi}{(\xi^2+T^2)^{\frac{3}{2}}}\right)+\frac{\mu}{r_2^3}+\frac{M_b}{(\xi^2+T^2)^{\frac{3}{2}}}\right\}\right] \tag{17}$$

Implies that

$$\Omega_{\eta\eta}=\frac{1}{(1-e^2)^{\frac{1}{2}}}\left[-\frac{\mu}{r_1}+\frac{1}{n^2}\left\{\frac{\mu}{r_2^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{M_b(r_1+\mu)}{[(r_1+\mu)^2+T^2]^{\frac{3}{2}}}\left(\frac{1}{r_1}-\frac{1}{(r_1+\mu)}\right)\right\}\right]$$

$\Omega^0_{\eta\eta}<0$, Since $\mu<\frac{1}{2}$, $M_b << 1$, $r_1>1$ and $r_2<1$

Clearly, $\Omega^0_{\xi\xi}>0$, $\Omega^0_{\eta\eta}<0$, and $\Omega^0_{\xi\eta}=0$.

Since, $\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}-\Omega^0_{\xi\eta}<0$, the discriminant of equation (13) is positive. and its characteristics roots can be expressed as: $\lambda_{1,2}=\pm h$ and $\lambda_{3,4}=\pm ih'$ Where h and h' are real.

Thus, the motion in the neighborhood of collinear points is unbounded and therefore unstable.

5. Numerical Results

The position of the collinear equilibrium points L_i ($i=1,2,3,a,b$) were obtained numerically in Table 1-3, the effects of eccentricity, semi-major axis and gravitational potential from the circumbinary disk on the stability of the collinear libration points are given in Table 4-18. The effects are also shown graphically in figure 3 -7.

Table 4: Stability of a libration Point of the semi-major axis on L₁ $M_b = 0.01,\ e = 0.3$ and $T = 0.01$.

a	**L₁**	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.85	1.07587	13.8777	-5.36641	-74.4733
0.80	1.07118	15.3122	-6.08367	-93.1543
0.75	1.06669	16.9326	-6.89388	-116.731
0.70	1.06237	18.7653	-7.81025	-146.562
0.65	1.05821	20.8432	-8.84916	-184.445
0.60	1.05418	23.2072	-10.0312	-232.795

Table 5: Stability of a libration point of the eccentricity on L₁ $M_b = 0.01,\ a = 0.9,$ and $T = 0.01$.

e	**L₁**	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.10	1.09166	9.97696	-3.48092	-34.7290
0.15	1.08972	10.3667	-3.66617	-38.0059
0.20	1.08717	10.9257	-3.93192	-42.9590
0.25	1.08413	11.6671	-4.28434	-49.9857
0.30	1.08075	12.6071	-4.73114	-59.6461
0.35	1.07717	13.7660	-5.28170	-72.7077
0.40	1.07350	15.1681	-5.94741	-90.2107

Table 6: Stability of a libration Point of the circumbinary disk on L₁ $e = 0.3,\ a = 0.9,$ and $T = 0.01$.

M_b	**L₁**	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.01	1.08075	12.6071	-4.73114	-59.6461
0.10	1.07289	14.1918	-5.52349	-783884
0.20	1.06625	15.8246	-6.33989	-100.326
0.30	1.06107	17.3399	-7.09756	-123.071
0.40	1.05688	18.7544	-7.80481	-146.375
0.50	1.05340	20.0819	-8.46855	-170.065
0.60	1.05046	21.3341	-9.09464	-194.026

Table 7: Stability of a libration Point of the semi-major axis on L₂ $M_b = 0.01,\ e = 0.3$ and $T = 0.01$.

a	**L₂**	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.85	0.856639	5.64233	-1.24874	-7.0458
0.80	0.846504	5.07008	-0.96262	-4.8806
0.75	0,834706	4.60023	-0.72769	-3.3475
0.70	0.821100	4.22553	-0.54034	-2.2832
0.65	0.805576	3.93479	-0.39497	-1.5541
0.60	0.788060	3.71448	-0.28482	-1.0579

Table 8: Stability of a libration Point of the eccentricity on L₂ $M_b = 0.01,\ a = 0.9,$ and $T = 0.01$.

e	**L₂**	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.10	0.879413	7.72739	-2.35615	-18.2069
0.15	0.877318	7.46025	-2.21296	-16.5092
0.20	0,874313	7.11799	-2.02806	-14.4357
0.30	0.865291	6.31964	-1.58740	-10.0318
0.35	0.859123	5.92227	-1.35986	-8.05343
0.40	0.851769	5.56134	-1.14404	-6.36238

Table 9: Stability of a libration Point of the circumbinary disk on L_2 $e = 0.3,\ a = 0.9,$ and $T = 0.01$.

M_b	L_2	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.01	0.865291	6.31964	-1.58740	-10.0318
0.10	0.858612	5.52635	-1.19077	-6.58060
0.20	0.851801	4.94878	-0.90200	-4.46381
0.30	0.845661	4.56206	-0.70866	-3.23294
0.40	0.840166	4.29226	-0.57377	-2.46278
0.50	0.835263	4.09721	-0.47626	-1.95135
0.60	0.830887	3.95178	-0.40355	-1.59476

Table 10: Stability of a libration Point of the semi-major axis on L_3 $M_b = 0.01,\ e = 0.3$ and $T = 0.01$.

a	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.85	-0.907452	3.15264	-0.003895	-0.012278
0.80	-0.922379	2.93388	-0.003873	-0.011363
0.75	-0.870463	3.15304	-0.004092	-0.012903
0.70	-0.850725	3.15326	-0.004205	-0.013258
0.65	-0.830022	3.15351	-0.004328	-0.013649
0.60	-0.808229	3.15378	-0.004465	-0.014083

Table 11: Stability of a libration Point of the eccentricity on L_3 $M_b = 0.01,\ a = 0.9,$ and $T = 0.01$.

e	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.10	-0.95922	3.02210	-0.003493	-0.010556
0.15	-0.95351	3.04141	-0.003541	-0.010769
0.20	-0.94574	3.06908	-0.003608	-0.011074
0.25	-0.93611	3.10578	-0.003697	-0.011481
0.30	-0.92486	3.15246	-0.003807	-0.012001
0.35	-0.91224	3.21044	-0.003941	-0.012653
0.40	-0.89849	3.28147	-0.004102	-0.013460

Table 12: Stability of a libration Point of the circumbinary disk on L_3 $e = 0.3,\ a = 0.9,$ and $T = 0.01$.

M_b	L_3	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.01	-0.92487	3.15246	-0.003806	-0.012001
0.10	-0.90647	3.15198	-0.003582	-0.011291
0.20	-0.89051	3.15150	-0.003355	-0.010575
0.30	-0.87783	3.15107	-0.003152	-0.009932
0.40	-0.86750	3.15067	-0.002969	-0.009354
0.50	-0.85893	3.15033	-0.002805	-0.008836

Table 13: Stability of a libration Point of the circumbinary disk on L_a $e = 0.3,\ a = 0.9,$ and $T = 0.01$.

M_b	L_a	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.01	-0.00378	2.79043×10^{14}	1.40592×10^{6}	3.92313×10^{20}
0.1	-0.00377	5.21214×10^{13}	7.21403×10^{6}	3.76005×10^{20}
0.2	-0.00376	1.9116×10^{13}	9.36631×10^{6}	1.79647×10^{20}
0.3	-0.00374	5.00954×10^{12}	8.21116×10^{6}	4.11341×10^{19}
0.4	-0.00374	4.49065×10^{12}	9.8142×10^{6}	4.40721×10^{19}
0.5	-0.00362	1.5071×10^{11}	3.50443×10^{6}	5.28151×10^{17}

Table 14: Stability of a libration Point of the Circumbinary disk on L_b $e = 0.3,\ a = 0.9,$ and $T = 0.01$.

M_b	L_b	$\Omega^0_{\xi\xi}$	$\Omega^0_{\zeta\zeta}$	$\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}$
0.01	-0.00382	2.79043×10^{14}	1.40592×10^{6}	3.92313×10^{20}
0.1	-0.00383	5.21214×10^{13}	7.28609×10^{6}	3.79761×10^{20}
0.2	-0.00384	1.65131×10^{13}	9.03799×10^{6}	1.49245×10^{20}
0.3	-0.00386	6.16151×10^{12}	8.98022×10^{6}	5.53317×10^{19}
0.4	-0.00386	4.72288×10^{12}	1.01846×10^{7}	4.81004×10^{19}
0.5	-0.00557	1.58461×10^{11}	315740	5.00325×10^{17}

Table 15: Stability of a libration Point of the eccentricity on L_a $M_b = 0.01,\ a = 0.9,$ and $T = 0.01$.

E	**L_a**	$\mathbf{\Omega^0_{\xi\xi}}$	$\mathbf{\Omega^0_{\zeta\zeta}}$	$\mathbf{\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}}$
0.10	-0.00378	2.17641×10^{14}	1.34462×10^{6}	2.92645×10^{20}
0.15	-0.00378	2.15131×10^{14}	1.32911×10^{6}	2.85933×10^{20}
0.20	-0.00378	2.11807×10^{14}	1.30857×10^{6}	2.77165×10^{20}
0.25	-0.00378	2.07839×10^{14}	1.28405×10^{6}	2.66876×10^{20}
0.30	-0.00378	2.03422×10^{14}	1.25676×10^{6}	2.55653×10^{20}
0.35	-0.00378	1.987666×10^{14}	1.22798×10^{6}	2.44082×10^{20}
0.40	-0.00378	1.94086×10^{14}	1.19906×10^{6}	2.32720×10^{20}

Table 16: Stability of a libration Point of the eccentricity on L_b $M_b = 0.01,\ a = 0.9,$ and $T = 0.01$.

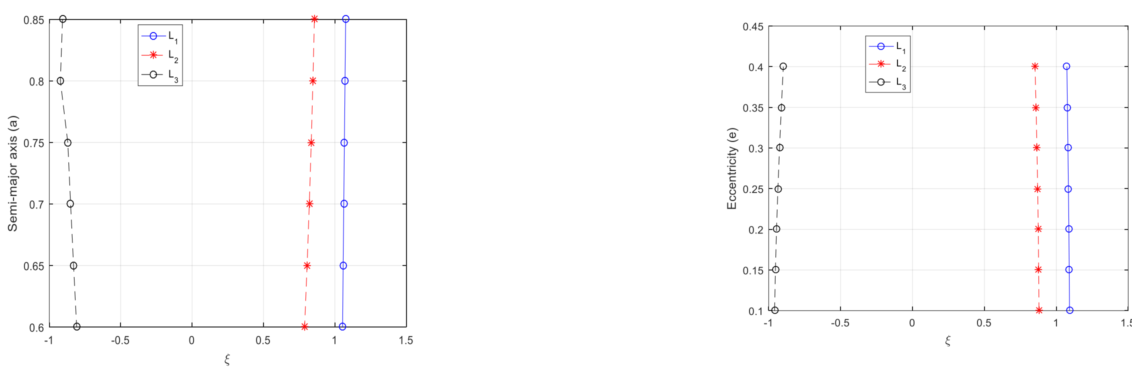
E	**L_b**	$\mathbf{\Omega^0_{\xi\xi}}$	$\mathbf{\Omega^0_{\zeta\zeta}}$	$\mathbf{\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}}$
0.10	-0.003818	2.9853×10^{14}	1.50422×10^{6}	4.49079×10^{20}
0.15	-0.003818	2.95104×10^{14}	1.48687×10^{6}	4.3878×10^{20}
0.20	-0.003818	2.90545×10^{14}	1.46389×10^{6}	4.25324×10^{20}
0.25	-0.003818	2.85102×10^{14}	1.43646×10^{6}	4.092313×10^{20}
0.30	-0.003818	2.79043×10^{14}	1.40592×10^{6}	3.92313×10^{20}
0.35	-0.003818	2.72656×10^{14}	1.37373×10^{6}	3.74557×10^{20}
0.40	-0.003818	2.66236×10^{14}	1.34137×10^{6}	3.57122×10^{20}

Table 17: Stability of a libration Point of the semi-major axis on L_a $M_b = 0.01,\ e = 0.3$ and $T = 0.01$.

A	**L_a**	$\mathbf{\Omega^0_{\xi\xi}}$	$\mathbf{\Omega^0_{\zeta\zeta}}$	$\mathbf{\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}}$
0.85	-0.00378	1.92121×10^{14}	1.18693×10^{6}	2.28034×10^{20}
0.80	-0.00378	1.8082×10^{14}	1.1171×10^{6}	2.01993×10^{20}
0.75	-0.00378	1.69519×10^{14}	1.04727×10^{6}	1.77531×10^{20}
0.70	-0.00378	1.58217×10^{14}	1.09345×10^{6}	2.37315×10^{20}
0.65	-0.00378	1.46916×10^{14}	907604	1.33342×10^{20}
0.60	-0.00378	1.35615×10^{14}	837773	1.13615×10^{20}

Table 18: Stability of a libration Point of the semi-major axis on L_b $M_b = 0.01,\ e = 0.3$ and $T = 0.01$.

A			**L_b**	$\mathbf{\Omega^0_{\xi\xi}}$	$\mathbf{\Omega^0_{\zeta\zeta}}$	$\mathbf{\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}}$
0.85			-0.003818	2.63541×10^{14}	1.3278×10^{6}	3.4993×10^{20}
0.80			-0.003818	2.48038×10^{14}	1.24968×10^{6}	3.0997×10^{20}
0.75			-0.003818	2.32536×10^{14}	1.17157×10^{6}	2.72431×10^{20}
0.70			-0.003818	2.17034×10^{14}	1.09345×10^{6}	2.37315×10^{20}
0.65	-0.003818	2.01531×10^{14}	1.01533×10^{6}	2.0462×10^{20}		
0.60	-0.003818	1.86029×10^{14}	937208	1.74348×10^{28}		



(a) (b)

Figure 3: Effects of Semi-major axis (a) and eccentricity (b) on the collinear libration points $L_i (i = 1,2,3)$

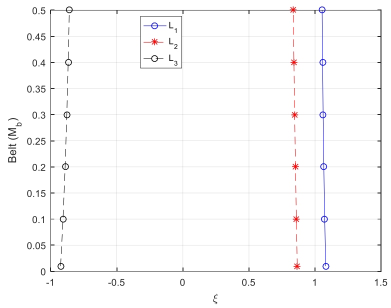
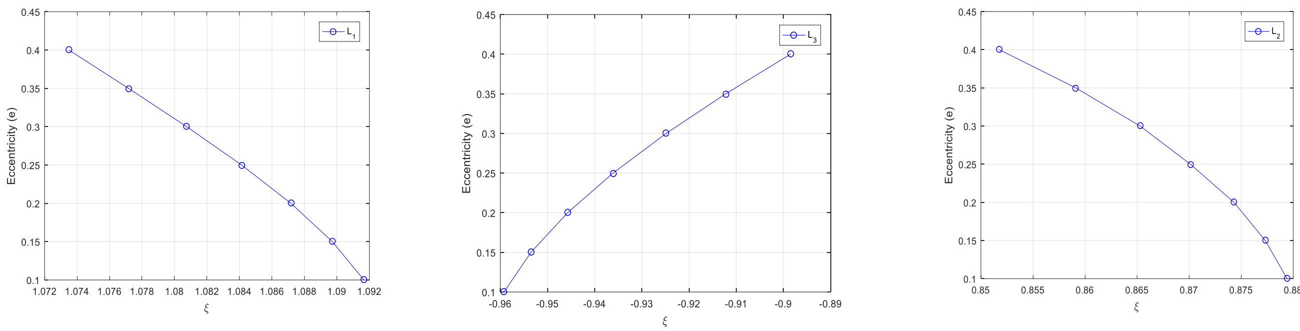
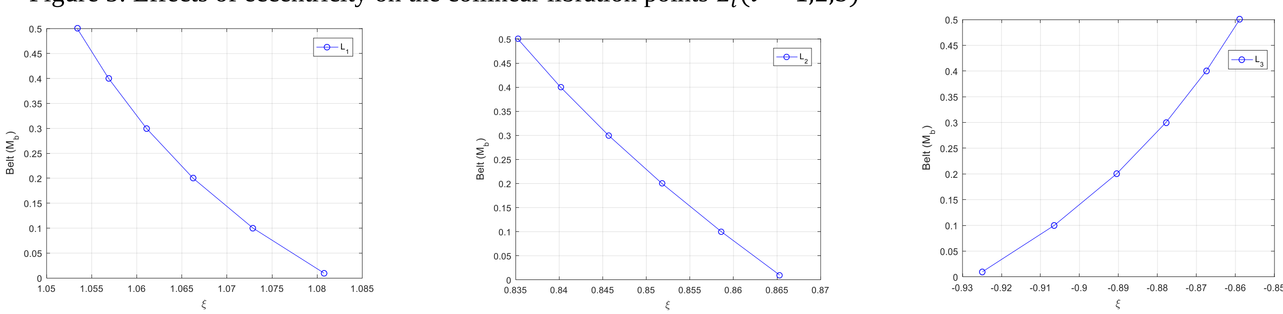


Figure 4: Effects of the gravitational potential from disk on the collinear libration points $L_i(i = 1,2,3)$



(a) (b) (c)

Figure 5: Effects of eccentricity on the collinear libration points $L_i(i = 1,2,3)$



(a) (b) (c)

Figure 6: Effects of the gravitational potential from disk on the collinear libration points $L_i(i = 1,2,3)$

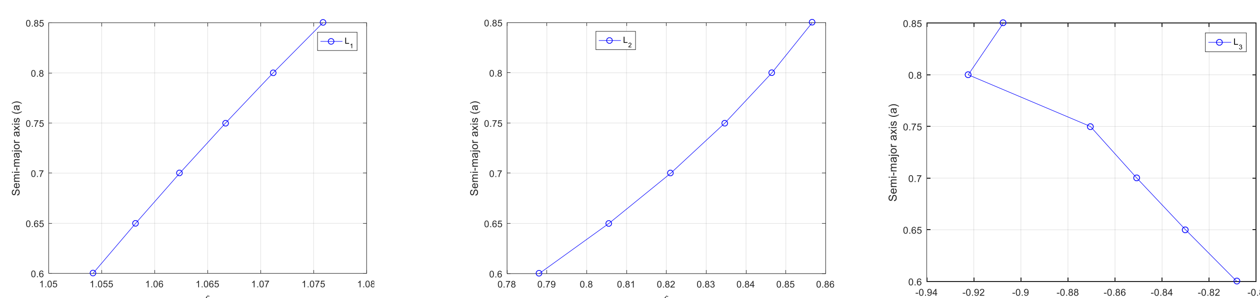


Figure 7: Effects of semi-major axis on the collinear libration points $L_i(i = 1,2,3)$

6. Discussions

We have studied the positions and linear stability of the collinear equilibrium points in elliptic restricted three – body problem when the primaries are spheroids and the bodies are surrounded by a circumbinary disk. The equations are affected by the eccentricity, semi-major axis and gravitational potential from the circumbinary disk. Numerically and analytically we have determined the positions of the collinear libration points. It is observed that, in Figure 5a and 6a and Table 2 and 3 that increase in eccentricity and gravitational potential from the circumbinary disk cause a shift towards the position of the bigger primary; while increase in semi-major axis causes a shift away from the smaller primary on the collinear libration points L_1; Figure 5b,c and 6c,b has shown that, an increase in eccentricity and circumbinary disk cause a shift towards the origin on $L_{2,3}$; and Figure 7c shows a move towards the position of the smaller primary on L_3 while Figure 7a,b move away from the position of the smaller primary on $L_{1,2}$ as a result of increase in semi - major axis. In addition to the three collinear points $L_i(i = 1,2,3)$ in the classical restricted three – body problem, there appear two more collinear points (L_a and L_b) which are as a result of the gravitational potential from circumbinary disk which agrees with the finding of [11] and [14]. With increase in circumbinary disk (Table 3) L_b moves away from the bigger primary while L_a draw closer to the origin.

The present work agree with [6] when $A_1 = A_2 = 0$ and $q_1 = q_2 = 1$ in their work and $M_b = 0$ in the presence study; and also when $A_2 = 0$ and $q_1 = 1$ in [7] and $M_b = 0$ ours; in the absence of gravitational potential from circumbinary disk turned to classical elliptic case of [1].

The stability behavior of the collinear points ,with the introduction of gravitational potential from circumbinary disk, the collinear points are unstable in the linear sense.

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