Transactions of the Nigerian Association of Mathematical Physics
***Volume 8, (January, 2019*), p67 – 78**
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INTUITIONISTIC FUZZY SOFT SET: PROPERTIES AND APPLICATION

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Abstract

Since the introduction of the concept of intuitionistic fuzzy soft sets as a generalization of fuzzy soft set, many researchers have contributed in one way or the other in the development of the intuitionistic fuzzy soft sets. In this paper, we recall the definitions of fuzzy set, intuitionistic fuzzy set and soft set. We discuss properties of intuitionistic fuzzy soft set operations. We prove some theorems and propositions in the background of intuitionistic fuzzy soft set. We also state and prove various De Morgan's types of results. Finally, we present an adjustable approach to intuitionistic fuzzy soft set based decision making problems using level soft sets with concrete example and illustration.

2010 AMS Classification: 03E70, 03E75

***Keywords*:** fuzzy set, soft set, intuitionistic fuzzy set, intuitionistic fuzzy soft set.

1. Introduction

To find solution to complicated problems in economics, engineering, environment, medical and social sciences, we cannot successfully use traditional approaches due to various uncertainties associated with these problems. In an attempt to solve the problems, many theories were developed. These theories among others include theory of probability [1], theory of fuzzy sets [2], theory of interval mathematics [3], rough sets [4], vague sets [5] which we can consider as mathematical tools for dealing with uncertainties. However, all these theories have their own inherent limitations in dealing with uncertainties. One major problem common to these theories is their incompatibility with the parameterization tools. To overcome these limitations, Molodtsov [6] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties and imprecision that is free from the difficulties that have troubled the classical mathematical theories. Molodtsov pointed out the application of soft set in several directions. This theory has proven useful in many different fields such as decision making [7], data analysis [8], forecasting and so on.

Research on soft sets has been very active, since its introduction by Molodtsov in [6] up to the present and several important results have been achieved both in theory and application. Maji *et al.* [9] defined different algebraic operations in soft set theory and published a detailed theoretical study on soft sets. Ali *et al*., [10] further presented and investigated some new algebraic operations for soft sets. Sezgin and Atagun [11] proved that certain De Morgan's law holds in soft set theory with respect to different operations on soft sets and discuss the basic properties of operations on soft sets such as intersection, extended intersection, restricted union and restricted difference. Maji *et al.,* [12] extended crisp soft set to fuzzy soft set. Maji *et al*., [9] extended classical soft set to intuitionistic fuzzy soft sets, which were further discussed in Maji *et al*., [9] and Yin *et al*., [13].

As a generalization of fuzzy soft set theory, intuitionistic fuzzy soft set theory makes descriptions of the objective world more realistic, practical and accurate in some cases, making it very promising. We basically study the properties of the operations of intuitionistic fuzzy soft set and De Morgan's laws. Also, we use the approach introduced by Jiang *et al.,* [14] and construct some practical problems involving decision making. The definitions of the level soft sets are found in Jiang *et al.,* [14].

1. Some Preliminary Concepts

1.1. Fuzzy Set

We recall the definition of the notion of fuzzy set by Zadeh [2]:

Definition 2.1.1. Let U be a universe. A fuzzy set X over U is a set defined by a function μ_X representing a mapping,
$\mu_X: U \longrightarrow [0,1]$

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μ_X is called the membership function of X, and the value $\mu_X(u)$ is called the grade of membership of $u \in U$and represents the degree of u belonging to the fuzzy set X. Thus a fuzzy set X over U, can be represented as follows:
$X = \left\{\frac{u}{\mu_X(u)} : u \in U, \mu_X(u) \in [0,1]\right\}$ or $X = \left\{\frac{\mu_X(u)}{u} : u \in U, \mu_X(u) \in [0,1]\right\}$ or
$X = \{\langle u, \mu_X(u) \rangle : u \in U, \mu_X(u) \in [0,1]\}$.
Example 2.1.1. Let $U = \{h_1, h_2, h_3, h_4\}$. A fuzzy set X over U can be represented by
$X = \left\{\frac{h_1}{0.4}, \frac{h_2}{0.6}, \frac{h_3}{0.2}, \frac{h_4}{0.7}\right\}$

1.2. Intuitionistic Fuzzy Set

Definition 2.2.1. [15]. Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form
$A = \{\langle x,\ \mu_A(x), \lambda_A(x)\rangle : x \in X\}$, where the functions $\mu_A(x), : X \longrightarrow [0,1]$ and
$\lambda_A(x) : X \longrightarrow [0,1]$ defined respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X and for every element $x \in X,\ 0 \leq \mu_A(x) + \lambda_A(x) \leq 1$.
Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the intuitionistic fuzzy set A and $\pi_A(x) \in [0,1]$, that is, $\pi_A(x) : X \longrightarrow [0,1]$ and $0 \leq \pi_A \leq 1$ for every $x \in X$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to intuitionistic fuzzy set A or not.
For instance, let A be an intuitionistic fuzzy set with $\mu_A(x) = 0.55$ and $\lambda_A(x) = 0.25, \Longrightarrow \pi_A(x) = 1 - (0.55 + 0.25) = 0.2$. It can be interpreted as the degree that the object x belongs to intuitionistic fuzzy set A is 0.55, the degree that the object x does not belong to the intuitionistic fuzzy set A is 0.25 and the degree of hesitancy is 0.2.

1.2.1. Basic Operations on Intuitionistic Fuzzy Set

Let $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \lambda_B(x)\rangle : x \in X\}$ be two intuitionistic fuzzy sets over X.
(i) [**Inclusion**] $A \subseteq B \leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x), \forall x \in X$.
(ii) [**Equality**] $A = B \leftrightarrow \mu_A(x) = \mu_B(x)$ and $\lambda_A(x) = \lambda_B(x), \forall x \in X$.
(iii) **[Complement]** $A^C = \{\langle x, \lambda_A(x), \mu_A(x)\rangle : x \in X\}$.
(iv) **[Union]** $A \cup B = \left\{\langle x, max\left(\mu_A(x), \mu_B(x)\right), min\left(\lambda_A(x), \lambda_B(x)\right)\rangle : x \in X\right\}$.
(v) **[Intersection]** $A \cap B = \left\{\langle x, min\left(\mu_A(x), \mu_B(x)\right), max\left(\lambda_A(x), \lambda_B(x)\right)\rangle : x \in X\right\}$.

1.3. Soft Set

We first recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters or attributes with respect to U, $P(U)$ be the power set of U and $A \subseteq E$.
Definition 2.3.1[6]. A pair (F, A) is called a **soft set** over U, where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the soft set (F, A). The soft set (F, A) can be represented as a set of ordered pairs as follows:$(F, A) = \{(\ x, F(x)), x \in A, F(x)\ \in P(U)\}$
Definition 2.3.2 [9]. Let (F, A) and (G, B) be two soft sets over U. Then
(i) (F, A) is said to be a **soft subset** of (G, B), denoted by
$(F, A) \tilde{\subseteq} (G, B)$, if $A \subseteq B$ and $F(x) \subseteq G(x), \forall x \in A$
(ii) (F, A) and (G, B) are said to be **soft equal**, denoted by
$(F, A) = (G, B)$, if $(F, A) \tilde{\subseteq} (G, B)$ and $(G, B) \tilde{\subseteq} (F, A)$.

Definition 2.3.3 [10]. Let (F, A) be a soft set over U. Then the support of (F, A) written **supp**(F, A) is defined as **supp**$(F, A) = \{x\ \in\ A : F(x) \neq \emptyset\}$.
(ii) (F, A) is called a **non-null** soft set if supp $(F, A) \neq \emptyset$.
(ii) (F, A) is called a **relative null** soft set denoted by $\emptyset_A\ if\ F(x) = \emptyset, \forall x \in A$
(iii) (F, A) is called a **relative whole** soft set, denoted by $U_A\ if\ F(x) = U, \forall x \in A$.
Definition 2.3.4.[16]. Let (F, A) be a soft set over U. If $F(x) \neq \emptyset$ for all $x \in A$, then (F, A) is called a non-empty soft set.
Definition 2.3.5 [10]. Let (F, A) and (G, B) be two soft sets over U. Then the **union** of (F, A) and (G, B), denoted by $(F, A) \tilde{\cup} (G, B)$ is a soft set defined as $(F, A) \tilde{\cup} (G, B) = (H, C), where\ C = A \cup B\ and\ \forall x \in C,$

$$H(x) = \begin{cases} F(x), & if\ x \in A - B \\ G(x), & if x \in B - A \\ F(x) \cup G(x), & if\ x \in A \cap B \end{cases}$$

Definition 2.3.6[16]. Let (F, A) and (G, B) be two soft sets over U. Then the **restricted union** of (F, A) and (G, B), denoted by $(\Gamma, A) \tilde{\cup}_R (G, B)$ is a soft set defined as;

$(F,A)\ \tilde{\cup}_R\ (G,B) = (H,C)$, where $C = A \cap B \neq \emptyset$ and $\forall\, x \in C$, $H(x) = F(x) \cup G(x)$.

Definition 2.3.7 [10]. Let (F,A) and (G,B) be two soft sets over U. Then the **extended intersection** of (F,A) and (G,B), denoted by $(F,A)\ \tilde{\cap}_E\ (G,B)$, is a soft set defined as $(F,A)\ \tilde{\cap}_E\ (G,B) = (H,C)$ where $C = A \cup B$ and $\forall\, x \in C$,

$$H(x) = \begin{cases} F(x), & if\ x \in A - B \\ G(x), & if x \in B - A \\ F(x) \cap\ G(x), & if\ x \in A \cap B \end{cases}$$

Definition 2.3. 8 [10]. Let (F,A) and (G,B) be two soft sets over U. Then the **restricted intersection** of (F,A) and (G,B) denoted by $(F,A) \Cap (G,B)$, is a soft set defined as $(F,A) \Cap (G,B) = (H,C)$ where $C = A \cap B$ and $\forall\, x \in C$, $H(x) = F(x) \cap G(x)$.

Definition 2.3.9 [9]. Let (F,A) and (G,B) be two soft sets over U. Then the **AND-product** or **AND-intersection** of (F,A) and (G,B) denoted by $(F,A)\tilde{\wedge}\ (G,B)$ is a soft set defined as

$(F,A)\tilde{\wedge}\ (G,B) = (H,C), where\ C = A \times B\ and\ \forall\ (x,y) \in A \times B$, $H(x,y) = F(x) \cap G(y)$.

Definition 2.3.10 [9]. Let (F,A) and (G,B) be two soft sets over U. Then the **OR-product** or **OR-union** of (F,A) and (G,B), denoted by $(F,A)\tilde{\vee}\ (G,B)$ is a soft set defined as

$(F,A)\tilde{\vee}\ (G,B) = (H,C), where\ C = A \times B\ and\ \forall\ (x,y) \in A \times B$, $H(x,y) = F(x) \cup G(y)$.

2.4 Fuzzy Soft Set

Let U be an initial universe set and E be a set of parameters (which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denotes the set of all fuzzy subsets of U, and $A \subseteq E$.

Definition 2.4.1 [12]. A pair (F,A) is called a fuzzy soft set over U, where F is a mapping given by $F\colon A \longrightarrow P(U)$. In other words, a fuzzy soft set over U is a parameterized family of fuzzy subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the fuzzy soft set (F,A). Therefore, a fuzzy soft set (F,A) over U can be represented by the set of ordered pairs

$(F,A) = \{(x, F(x))\colon x \in A, F(x) \in P(U)\}$.

Example 2.4.1. Suppose that $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe set and $E = \{a_1, a_2, a_3, a_4\}$ be a set of parameters $A = \{a_1, a_2, a_3\} \subseteq E$, $F(a_1) = \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}$, $F(a_2) = U$ and $F(a_3) = \left\{\frac{h_1}{0.3}, \frac{h_3}{0.4}, \frac{h_5}{0.9}\right\}$, then the fuzzy soft set (F,A) is written as

$$(F,A) = \left\{\left(a_1, \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}\right), (a_2, U), \left(a_3, \left\{\frac{h_1}{0.3}, \frac{h_3}{0.4}, \frac{h_5}{0.9}\right\}\right)\right\}.$$

3. Intuitionistic Fuzzy Soft Set

Definition 3.1 [13]**.** Let U be an initial universe set, E a set of parameters, $I(U)$ denotes the set of all intuitionistic fuzzy subsets of U and $A \subseteq E$. Then a pair $(\hat{F}, A)$ is called an **intuitionistic fuzzy soft set** over U, where $\hat{F}$ is a mapping given by $\hat{F}\colon A \longrightarrow I(U)$.

In general, for every $e \in A$, $\hat{F}(e)$ is an intuitionistic fuzzy set of U and it is called intuitionistic fuzzy value set of parameter e. Obviously, $\hat{F}(e)$ can be written as an intuitionistic fuzzy set such that $\hat{F}(e) = \{\langle x, \mu_{\hat{F}(e)}(x), \lambda_{\hat{F}(e)}(x)\rangle\colon x \in U\}$. Where $\mu_{\hat{F}(e)}$ and $\lambda_{\hat{F}(e)}$ are the membership and non-membership functions, respectively. The set of all intuitionistic fuzzy soft sets over U with parameters from E is called an intuitionistic fuzzy soft class and it is denoted by $I\hat{F}(U,E)$.

Definition 3.2. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U. We say that $(\hat{F}, A)$ is an intuitionistic fuzzy **soft subset** of $(\hat{G}, B)$ and written as $(\hat{F}, A) \tilde{\subseteq} (\hat{G}, B)$ if,

(i) $A \subseteq B$,

(i) For any $e \in A$, $\hat{F}(e) \subseteq \hat{G}(e)$, that is, for all $x \in U$ and $e \in A$, $\mu_{\hat{F}(e)}(x) \leq \mu_{\hat{G}(e)}(x)$ and $\lambda_{\hat{F}(e)}(x) \geq \lambda_{\hat{G}(e)}(x)$.

Definition 3.3. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U. Then $(\hat{F}, A)$ and $(\hat{G}, B)$ are said to be intuitionistic fuzzy **soft equal**, denoted by $(\hat{F}, A) = (\hat{G}, B)$ if $(\hat{F}, A) \tilde{\subseteq} (\hat{G}, B)$ and $(\hat{G}, B) \tilde{\subseteq} (\hat{F}, A)$.

Definition 3.4. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U. Then, the **Union** of $(\hat{F}, A)$ and $(\hat{G}, B)$ is written as $(\hat{F}, A)\ \tilde{\cup}\ (\hat{G}, B)$ and is defined as $(\hat{F}, A)\ \tilde{\cup}\ (\hat{G}, B) = (\hat{H}, C)$, where $C = A \cup B$ and $\forall\ e \in C$,

$$\hat{H}(e) = \begin{cases} \hat{F}(e), & if\ e \in A/B \\ \hat{G}(e), & if\ e \in B/A\ . \\ \hat{F}(e) \cup \hat{G}(e), & if\ e \in A \cap B \end{cases}$$

Definition 3.5. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U, such that $A \cap B \neq \emptyset$. The **restricted intersection** of $(\hat{F}, A)$ and $(\hat{G}, B)$ is defined to be the intuitionistic fuzzy soft set $(\hat{H}, C)$, where $C = A \cap B$ and $\hat{H}(e) = \hat{F}(e) \cap \hat{G}(e), \forall e \in C$. This is written as $(\hat{F}, A) \Cap (\hat{G}, B) = (\hat{H}, C)$.

Definition 3.6. Let U be an initial universe set, E be the universe set of parameters and $A \subset E$. The intuitionistic fuzzy soft set $(\hat{F}, A)$ is called a **relative null intuitionistic fuzzy soft set** with respect to the parameter set A denoted by $\emptyset_A$, if $\hat{F}(e) =$ null intuitionistic fuzzy set of U, for all $e \in A$. The relative null intuitionistic fuzzy soft set $\emptyset_E$ with respect to the universe set of parameters E is called the **absolute null intuitionistic fuzzy soft set** over U.

Definition 3.7. Let U be an initial universe set, E be a universe set of parameters and $A \subseteq E$. The intuitionistic fuzzy soft set $(\hat{F}, A)$ is called a **relative whole intuitionistic fuzzy soft set** with respect to the parameter set A denoted by U_A, if $\hat{F}(e) = U$, for all $e \in A$. The relative whole intuitionistic fuzzy soft set U_E with respect to the universe set of parameters E is called the **absolute intuitionistic fuzzy soft set** over U.

Definition 3.8. The r**elative complement** of an intuitionistic fuzzy soft set $(\hat{F}, A)$ over U is denoted by $(\hat{F}, A)^r$ and is defined by $(\hat{F}^r, A)$, where $\forall e \in A, \mu_{\hat{F}^r(e)} = \lambda_{\hat{F}(e)}$ and $\lambda_{\hat{F}^r(e)} = \mu_{\hat{F}(e)}$, that is, $\hat{F}^r(e) = (\lambda_{\hat{F}(e)}, \mu_{\hat{F}(e)})$. Clearly, $((\hat{F}, A)^r)^r = (\hat{F}, A)$.

Definition 3.9. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over a common universe U, such that $A \cap B \neq \emptyset$. The **restricted difference of** $(\hat{F}, A)$ and $(\hat{G}, B)$ is denoted by $(\hat{F}, A) \sim_R (\hat{G}, B)$ and is defined as $(\hat{F}, A) \sim_R \left(\hat{G}, B\right) = (\hat{K}, P)$, where $P = A \cap B$ and $\forall\, p \in P, \hat{K}(p) = \hat{F}(p) - \hat{G}(p)$ (the intuitionistic fuzzy difference of two intuitionistic fuzzy sets $\hat{F}(p)$ and $\hat{G}(p)$ is denoted by $\hat{F}(p) - \hat{G}(p)$ and is defined as $\hat{F}(p) - \hat{G}(p) = \hat{F}(p) \cap \hat{G}^c(p)$.

Definition 3.10. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U.The **extended intersection of** $(\hat{F}, A)$ and $(\hat{G}, B)$ denoted by $(\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)$ is defined as

$$\left(\hat{F}, A\right) \tilde{\cap}_E \left(\hat{G}, B\right) = (\hat{H}, C),\ C = A \cup B, \text{ and } \forall e \in C, \hat{H}(e) = \begin{cases} \hat{F}(e), & if\ e \in A/B \\ \hat{G}(e), & if\ e \in B/A\ . \\ \hat{F}(e) \cap \hat{G}(e), & if\ e \in A \cap B \end{cases}$$

Definition 3.11. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U, such that $A \cap B \neq \emptyset$. The **restricted union** of $(\hat{F}, A)$ and $(\hat{G}, B)$ denoted by $(\hat{F}, A) \tilde{\cup}_R (\hat{G}, B)$ is defined by $\left(\hat{F}, A\right) \tilde{\cup}_R \left(\hat{G}, B\right) = (\hat{H}, C)$ where $C = A \cap B$ and $\hat{H}(e) = \hat{F}(e) \cup \hat{G}(e), \forall e \in C$.

4. De Morgan's Laws on Intuitionistic fuzzy soft set

It is well known from standard set that De Morgan's Laws interrelate union and intersection via complements. In theorem 4.1 and theorem 4.2, we shall show that De Morgan's Laws interrelate union ($\tilde{\cup}$) and extended intersection $\tilde{\cap}_E$, restricted union ($\tilde{\cup}_R$) and restricted intersection ($\Cap$) operations respectively.

Theorem 4.1. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U. Then the following holds:

(i) $\left((\hat{F}, A) \tilde{\cup} (\hat{G}, B)\right)^r = (\hat{F}, A)^r\ \tilde{\cap}_E\ (\hat{G}, B)^r$,

(ii) $\left((\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)\right)^r = (\hat{F}, A)^r\ \tilde{\cup}\ (\hat{G}, B)^r$.

Proof:

(i) Suppose that $\left(\hat{F}, A\right) \tilde{\cup} \left(\hat{G}, B\right) = (\hat{H}, C)$, where $C = A \cup B$, then for all $e \in C$,

$$\hat{H}(e) = \begin{cases} \hat{F}(e), & if\ e \in A/B \\ \hat{G}(e), & if\ e \in B/A\ . \\ \hat{F}(e) \cup \hat{G}(e), & if\ e \in A \cap B \end{cases}$$

Now, $\left((\hat{F}, A) \tilde{\cup} (\hat{G}, B)\right)^r = (\hat{H}, C)^r = (\hat{H}^r, C)$. For all $e \in C$, we have

$$\hat{H}^r(e) = \begin{cases} \hat{F}^r(e), & if\ e \in A/B \\ \hat{G}^r(e), & if\ e \in B/A\ . \\ \hat{F}^r(e) \cap \hat{G}^r(e), & if\ e \in A \cap B \end{cases}$$

Also, let $(\hat{F}, A)^r\ \tilde{\cap}_E\ (\hat{G}, B)^r = (\hat{T}, D)$, where $D = A \cup B$, then for all $e \in D$, we obtain

$$\hat{T}(e) = \begin{cases} \hat{F}^r(e), & if\ e \in A/B \\ \hat{G}^r(e), & if\ e \in B/A\ . \\ \hat{F}^r(e) \cap \hat{G}^r(e), & if\ e \in A \cap B \end{cases}$$

Clearly, $\hat{H}^r(e) = \hat{T}(e)$. Hence, the result has been established.

(i) Let $\left(\hat{F}, A\right) \tilde{\cap}_E \left(\hat{G}, B\right) = \left(\hat{W}, C\right)$, where $C = A \cup B$, then for all $e \in C$,

$$\hat{W}(e) = \begin{cases} \hat{F}(e), & if\ e \in A/B \\ \hat{G}(e), & if\ e \in B/A\ . \\ \hat{F}(e) \cap \hat{G}(e), & if\ e \in A \cap B \end{cases}$$

Now, $\left((\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)\right)^r = \left(\hat{W}, C\right)^r = (\hat{W}^r, C)$. For all $e \in C$,

$$\widehat{W}^r(e) = \left(\widehat{W}, C\right)^r = \begin{cases} \hat{F}^r(e), & if\ e \in A/B \\ \hat{G}^r(e), & if\ e \in B/A \\ \hat{F}^r(e) \cup \hat{G}^r(e), & if\ e \in A \cap B \end{cases}.$$

Also, $(\hat{F}, A)^r\ \tilde{\cup}\ (\hat{G}, B)^r = (\hat{F}^r, A)\ \tilde{\cup}\ (\hat{G}^r, B)$,
$= (\widehat{M}, C)$, (say), where $C = A \cup B$. For all $e \in C$,

$$\widehat{M}(e) = \begin{cases} \hat{F}^r(e), & if\ e \in A/B \\ \hat{G}^r(e), & if\ e \in B/A \\ \hat{F}^r(e) \cup \hat{G}^r(e), & if\ e \in A \cap B \end{cases}.$$

Obviously, $\widehat{W}^r(e) = \widehat{M}(e)$. Hence, $\left((\hat{F}, A)\ \tilde{\cap}_E\ (\hat{G}, B)\right)^r = (\hat{F}, A)^r\ \tilde{\cup}\ (\hat{G}, B)^r$.

Theorem 4.2. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic fuzzy soft sets over U. Then the following holds:

(i) $\left((\hat{F}, A)\ \tilde{\cup}_R\ (\hat{G}, B)\right)^r = (\hat{F}, A)^r \Cap (\hat{G}, B)^r$, if $A \cap B \neq \emptyset$,

(ii) $\left((\hat{F}, A) \Cap (\hat{G}, B)\right)^r = (\hat{F}, A)^r\ \tilde{\cup}_R\ (\hat{G}, B)^r$, if $A \cap B \neq \emptyset$.

Proof:

(i) $\left(\hat{F}, A\right)\ \tilde{\cup}_R\ \left(\hat{G}, B\right) = (\widehat{H}, C)$, where $C = A \cap B \neq \emptyset$. For all $e \in C$,
$\widehat{H}(e) = \hat{F}(e) \cup \hat{G}(e)$.

Now, let $\left(\left(\hat{F}, A\right)\ \tilde{\cup}_R\ \left(\hat{G}, B\right)\right)^r = \left(\widehat{H}, C\right)^r = \left(\widehat{H}^r, C\right)$, where $C = A \cap B \neq \emptyset$. For any $e \in C$, we have the following, $\widehat{H}^r(e) = \left(\widehat{H}(e)\right)^r = \left(\hat{F}(e) \cup \hat{G}(e)\right)^r$
$= \hat{F}^r(e) \cap \hat{G}^r(e)$.

Also, let $(\hat{F}, A)^r \Cap (\hat{G}, B)^r = (\hat{I}, C)$, where $C = A \cap B \neq \emptyset$. For any $e \in C$,
$\hat{I}(e) = \hat{F}^r(e) \cap \hat{G}^r(e)$.

Since, $\widehat{H}^r(e) = \hat{I}(e)$. Therefore, the result is established.

(ii) Let $\left(\hat{F}, A\right) \Cap \left(\hat{G}, B\right) = (\widehat{H}, C)$, where $C = A \cap B \neq \emptyset$ and $\widehat{H}(e) = \hat{F}(e) \cap \hat{G}(e)$.

Now, $\left((\hat{F}, A) \Cap (\hat{G}, B)\right)^r = (\widehat{H}, C)^r = (\widehat{H}^r, C)$.

For all $e \in C$, $\widehat{H}^r(e) = (\widehat{H}(e))^r = (\hat{F}(e) \cap \hat{G}(e))^r = \hat{F}^r(e) \cup \hat{G}^r(e)$.

Also, $(\hat{F}, A)^r\ \tilde{\cup}_R\ (\hat{G}, B)^r = \left(\hat{F}^r, A\right)\ \tilde{\cup}_R\ \left(\hat{G}^r, B\right) = (\hat{J}, D)$, (say), where $D = A \cap B$.

For all $e \in D$, $\hat{J}(e) = \hat{F}^r(e) \cup \hat{G}^r(e)$. Clearly, $\widehat{H}^r(e) = \hat{J}(e)$. Hence,
$\left((\hat{F}, A) \Cap (\hat{G}, B)\right)^r = (\hat{F}, A)^r\ \tilde{\cup}_R\ (\hat{G}, B)^r$, if $A \cap B \neq \emptyset$.

5. Properties of Intuitionistic Fuzzy Soft Set Operations

Proposition 5.1. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two intuitionistic soft sets over U. Then, the following holds.

(i) $(\hat{F}, A) \Cap (\hat{G}, B)$ is an intuitionistic fuzzy soft set over U, if it is non-null.

(ii) $(\hat{F}, A)\ \tilde{\cap}_E\ (\hat{G}, B)$ is an intuitionistic fuzzy soft set over U, if it is non-null.

(iii) $(\hat{F}, A)\ \tilde{\cup}_R\ (\hat{G}, B)$ is an intuitionistic fuzzy soft set over U, whenever, it is non-null and if $\hat{F}(x)$ and $\hat{G}(x)$ are ordered by inclusion relation for all $x \in supp((\hat{F}, A)\ \tilde{\cup}_R\ (\hat{G}, B))$.

(iv) $(\hat{F}, A)\widetilde{\wedge}(\hat{G}, B)$ is an intuitionistic fuzzy soft set over U, if it is non-null.

(v) $(\hat{F}, A)\ \tilde{\cup}\ (\hat{G}, B)$ is an intuitionistic fuzzy soft set over U, if it is non-null and if A and B are disjoint.

(vi) $\left(\hat{F}, A\right)\widetilde{\vee}\left(\hat{G}, B\right) = (\widehat{N}, A \times B)$ is an intuitionistic fuzzy soft set over U, if it is non-null and if $\hat{F}(x)$ and $G(y)$ are ordered by inclusion relation for all $(x, y) \in supp(\widehat{N}, A \times B)$.

(vii) $\left(\hat{F}, A\right)\mp(\hat{G}, B)$ is an intuitionistic fuzzy soft set over U, if it is non-null.

Proof:

(i) Let $\left(\hat{F}, A\right) \Cap \left(\hat{G}, B\right) = (\widehat{K}, C)$ where $\widehat{K}(x) = \hat{F}(x) \cap \hat{G}(x)$ for all $x \in C = A \cap B \neq \emptyset$. By hypothesis, $\left(\widehat{K}, C\right)$ is a non-null intuitionistic fuzzy soft set over U. If $x \in supp(\widehat{K}, C)$, then $\widehat{K}(x) = \hat{F}(x) \cap \hat{G}(x) \neq \emptyset$. It follows that $\hat{F}(x) \neq \emptyset$ and $\hat{G}(x) \neq \emptyset$ are both intuitionistic fuzzy set over U. Hence, $K(x)$ is an intuitionistic fuzzy set over U, for all $x \in supp(K, C)$.Thus, (K, C) is an intuitionistic fuzzy soft set over U.

(ii) Let $(\hat{F},A)\,\widetilde{\cap}_E\,(\hat{G},B) = (\widehat{M}, A\cup B)$, where $\widehat{M}(x) = \begin{cases} \hat{F}(x), & if\ x\in A-B \\ \hat{G}(x), & if x\in B-A \\ \hat{F}(x)\cap G(x), & if\ x\in A\cap B \end{cases}$

For all $x\in A\cup B$. Then by the hypothesis$(\widehat{M}, A\cup B)$ is a non-null intuitionistic fuzzy soft set over U. Let $x\in supp(\widehat{M}, A\cup B)$. If $x\in A-B$, then $\emptyset\neq\widehat{M}(x)=\hat{F}(x)$. If $x\in B-A$, then $\emptyset\neq\widehat{M}(x)=\hat{G}(x)$, and if $x\in A\cap B$, then $\widehat{M}(x)=\hat{F}(x)\cap\hat{G}(x)\neq\emptyset$. Since, $\hat{F}(x)\neq\emptyset$ and $\hat{G}(x)\neq\emptyset$, are both intuitionistic fuzzy set over U then $\widehat{M}(x)$ is an intuitionistic fuzzy set over U for all $x\in supp(\widehat{M}, A\cup B)$. Therefore, $(\hat{F},A)\,\widetilde{\cap}_E\,(\hat{G},B)=(\widehat{M},A\cup B)$ is an intuitionistic fuzzy soft set over U.

(iii) Let $(\hat{F},A)\,\widetilde{\cup}_R\,(\hat{G},B)=(\hat{R},A\cap B)$ where $\hat{R}(x)=\hat{F}(x)\cup\hat{G}(x)$ for all $x\in A\cap B\neq\emptyset$. Then by hypothesis $(\hat{R},A\cap B)$ is a non-null intuitionistic fuzzy soft set over U. if $x\in supp(\hat{R},A\cap B)$, $\hat{R}(x)=\hat{F}(x)\cup\hat{G}(x)\neq\emptyset$. Since, $\hat{F}(x)$ and $\hat{G}(x)$ are ordered by inclusion relation for all $x\in supp(\hat{R},A\cap B)$, $\hat{F}(x)\cup G(x)=\hat{F}(x)$ or $\hat{F}(x)\cup\hat{G}(x)=\hat{G}(x)$. Since, $\hat{F}(x)\neq\emptyset$ and $\hat{G}(x)\neq\emptyset$ are both intuitionistic fuzzy set over $U, then\ R(x)$ is an intuitionistic fuzzy set over U for all $x\in supp(\hat{R},A\cap B)$. Therefore, $(\hat{R},A\cap B)$ is an intuitionistic fuzzy soft set over U.

(iv) Let $(\hat{F},A)\widetilde{\wedge}(\hat{G},B)=(\hat{Q},A\times B)$, where $\hat{Q}(x,y)=\hat{F}(x)\cap G(y)$, for all $(x,y)\in A\times B$. Then by hypothesis, $(\hat{Q},A\times B)$ is a non-null intuitionistic fuzzy soft set over U. If $(x,y)\in supp(\hat{Q},A\times B)$, then $\hat{Q}(x,y)=\hat{F}(x)\cap\hat{G}(y)\neq\emptyset$. It follows that $\hat{F}(x)\neq\emptyset$ and $\hat{G}(y)\neq\emptyset$ are both intuitionistic fuzzy set over U. Hence, $\hat{Q}(x,y)$ is an intuitionistic fuzzy set over U for all $(x,y)\in supp(\hat{Q},A\times B)$. Therefore, $(\hat{F},A)\widetilde{\wedge}(\hat{G},B)$ is an intuitionistic fuzzy soft set over U.

(v) Let $(\hat{F},A)\,\widetilde{\cup}\,(\hat{G},B)=(\hat{V},A\cup B)$, where $\hat{V}(x)=\begin{cases}\hat{F}(x), & if\ x\in A-B, \\ \hat{G}(x), & if x\in B-A, \\ \hat{F}(x)\cup\hat{G}(x), & if\ x\in A\cap B,\end{cases}$

For all $x\in A\cup B$,and $A\cap B=\emptyset$, it follows that either $x\in A-B$ or $x\in B-A$, for all $x\in A\cup B$. If $x\in A-B$, then $\hat{V}(x)=\hat{F}(x)$ is an intuitionistic fuzzy set over U and if $x\in B-A$, then $\hat{V}(x)=\hat{G}(x)$ is an intuitionistic fuzzy set over U. Therefore, $(\hat{F},A)\,\widetilde{\cup}\,(\hat{G},B)$ is an intuitionistic fuzzy soft set over U.

(vi) Let $(\hat{F},A)\widetilde{\vee}(\hat{G},B)=(\widehat{N},A\times B)$, where $\widehat{N}(x,y)=\hat{F}(x)\cup\hat{G}(y)$, for all $(x,y)\in A\times B$. Then by hypothesis, $(\widehat{N},A\times B)$ is a non-null intuitionistic fuzzy soft set over U. If $(x,y)\in supp(\widehat{N},A\times B)$, then $\widehat{N}(x,y)=\hat{F}(x)\cup\hat{G}(y)\neq\emptyset$. Since $\hat{F}(x)$ and $\hat{G}(y)$ are ordered by inclusion relation for all $(x,y)\in supp(\widehat{N},A\times B)$, $\hat{F}(x)\cup\hat{G}(y)=\hat{F}(x)$ or $\hat{F}(x)\cup\hat{G}(y)=\hat{G}(y)$. Since, $\hat{F}(x)\neq\emptyset$ and $\hat{G}(y)\neq\emptyset$ are both intuitionistic fuzzy set over U for all $(x,y)\in supp(\widehat{N},A\times B)$. Therefore, $(\hat{F},A)\widetilde{\vee}(\hat{G},B)$ is an intuitionistic fuzzy soft set over U.

(vii) $(\hat{F},A)\widetilde{\mp}(\hat{G},B)=(\widehat{H},A\times B)$, where $\widehat{H}(x,y)=\hat{F}(x)+\hat{G}(y)$, for all $(x,y)\in A\times B$. Then by the hypothesis, $(\widehat{H},A\times B)$ is a non-null intuitionistic fuzzy soft set over U. Suppose $(x,y)\in supp(H,A\times B)$, then $\widehat{H}(x,y)=\hat{F}(x)+\hat{G}(y)\neq\emptyset$. It means that $\hat{F}(x)\neq\emptyset$ and $\hat{G}(y)\neq\emptyset$ are both intuitionistic fuzzy set over U. Hence, $\widehat{H}(x,y)$ is an intuitionistic fuzzy set over U for all $(x,y)\in supp(\widehat{H},A\times B)$. Therefore, $(\hat{F},A)\widetilde{\mp}(\hat{G},B)$ is an intuitionistic fuzzy soft set over U.

Some of the theorems below are due to Yin, *et al.*, [13].

Theorem 5.1. Let $(\hat{F},A)$ be an intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A)\,\widetilde{\cup}\,U_E=U_E$

(ii) $(\hat{F},A)\Cap U_E=(\hat{F},A)$

(iii) $(\hat{F},A)\,\widetilde{\cup}\,(\hat{F},A)=(\hat{F},A)$

(iv) $(\hat{F},A)\Cap(\hat{F},A)=(\hat{F},A)$.

Proof. The proof is trivial, hence omitted.

The following results can be easily deduced.

Theorem 5.2. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\widehat{H},C)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A)\,\widetilde{\cup}\,(\hat{G},B)=(\hat{G},B)\,\widetilde{\cup}\,(\hat{F},A)$.

(ii) $((\hat{F},A)\,\widetilde{\cup}\,(\hat{G},B))\,\widetilde{\cup}\,(\widehat{H},C)=(\hat{F},A)\,\widetilde{\cup}\,((\hat{G},B)\,\widetilde{\cup}\,(\widehat{H},C))$.

Theorem 5.3. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\widehat{H},C)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A)\,\widetilde{\cap}_E\,\emptyset_E=\emptyset_E$.

(ii) $(\hat{F},A)\,\widetilde{\cap}_E\,(\hat{F},A)=(\hat{F},A)$

(iii) $(\hat{F},A)\,\tilde{\cap}_E\,(\hat{G},B) = (\hat{G},B)\,\tilde{\cap}_E\,(\hat{F},A)$

(iv) $\left((\hat{F},A)\,\tilde{\cap}_E\,(\hat{G},B)\right)\tilde{\cap}_E\,(\hat{H},C) = (\hat{F},A)\,\tilde{\cap}_E\left((\hat{G},B)\,\tilde{\cap}_E\,(\hat{H},C)\right)$.

Proof: The proof is straight forward.

Theorem 5.4. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\hat{H},C)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A)\,\tilde{\cup}_R\,\emptyset_E = (\hat{F},A)$.

(ii) $(\hat{F},A)\,\tilde{\cup}_R\,(\hat{F},A) = (\hat{F},A)$

(iii) $(\hat{F},A)\,\tilde{\cup}_R\,(\hat{G},B) = (\hat{G},B)\,\tilde{\cup}_R\,(\hat{F},A)$

(iv) $\left((\hat{F},A)\,\tilde{\cup}_R\,(\hat{G},B)\right)\tilde{\cup}_R\,(\hat{H},C) = (\hat{F},A)\,\tilde{\cup}_R\left((\hat{G},B)\,\tilde{\cup}_R\,(\hat{H},C)\right)$.

Proof: The proof is straight forward.

Theorem 5.5. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\hat{H},C)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A) \Cap (\hat{G},B) = (\hat{G},B) \Cap (\hat{F},A)$.

(ii) $\left((\hat{F},A) \Cap (\hat{G},B)\right) \Cap (\hat{H},C) = (\hat{F},A) \Cap \left((\hat{G},B) \Cap (\hat{H},C)\right)$.

Proof: The proof is straight forward.

The following theorem shows that, the absorption law with respect to the operations union ($\tilde{\cup}$) and restricted intersection ($\Cap$) holds.

Theorem 5.6. Let $(\hat{F},A)$ and $(\hat{G},B)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $\left((\hat{F},A)\,\tilde{\cup}\,(\hat{G},B)\right) \Cap (\hat{F},A) = (\hat{F},A)$

(ii) $\left((\hat{F},A) \Cap (\hat{G},B)\right)\tilde{\cup}\,(\hat{F},A) = (\hat{F},A)$.

Proof: (i) $\left((\hat{F},A)\,\tilde{\cup}\,(\hat{G},B)\right) \Cap (\hat{F},A) = (\hat{H},(A\cup B)\cap A)$. For any $e \in A$, we consider the following cases.

Case 1: $e \in B$. Then $\hat{H}(e) = \left(\hat{F}(e)\cup G(e)\right)\cap \hat{F}(e) = \hat{F}(e)$.

Case 2: $e \notin B$. Then $\hat{H}(e) = \hat{F}(e)\cap \hat{F}(e) = \hat{F}(e)$.

Therefore, $\hat{F}$ and $\hat{H}$ are the same operators and so

$\left((\hat{F},A)\,\tilde{\cup}\,(\hat{G},B)\right) \Cap (\hat{F},A) = (\hat{F},A)$.

(ii) The proof of (ii) follows from (i).

The following theorem shows that, the absorption law with respect to the operations extended intersection ($\tilde{\cap}_E$) and restricted union ($\tilde{\cup}_R$) holds.

Theorem 5.7. Let $(\hat{F},A)$ and $(\hat{G},B)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $\left((\hat{F},A)\,\tilde{\cap}_E\,(\hat{G},B)\right)\tilde{\cup}_R\,(\hat{F},A) = (\hat{F},A)$

(ii) $\left((\hat{F},A)\,\tilde{\cup}_R\,(\hat{G},B)\right)\tilde{\cap}_E\,(\hat{F},A) = (\hat{F},A)$.

Proof. The proof is similar to the proof of theorem 5.6 and hence omitted.

The absorption laws with respect to the operation $\tilde{\cap}_E$ and $\tilde{\cup}$, $\Cap$ and $\tilde{\cup}_R$ may not hold in general as shown in the following example.

Example 5.1. Let U be a universe, $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\}$ and $B = \{e_1, e_3\}$. Let $(\hat{F},A)$ and $(\hat{G},B)$ be any intuitionistic fuzzy soft set over U. Suppose that

$\left((\hat{F},A)\,\tilde{\cap}_E\,(\hat{G},B)\right)\tilde{\cup}\,(\hat{F},A) = (\hat{H}, A\cup B)$ and $\left((\hat{F},A) \Cap (\hat{G},B)\right)\tilde{\cup}_R\,(\hat{F},A) = (\hat{I}, A\cap B)$.

Since, $A \subset E = A\cup B$ and $A\cap B = \{e_1\} \subset A$, we have

$\left((\hat{F},A)\,\tilde{\cap}_E\,(\hat{G},B)\right)\tilde{\cup}\,(\hat{F},A) \neq (\hat{F},A)$ and

$\left((\hat{F},A) \Cap (\hat{G},B)\right)\tilde{\cup}_R\,(\hat{F},A) \neq (\hat{F},A)$.

The following theorem shows that, the distributive law with respect to the operations restricted intersection ($\Cap$) and union ($\tilde{\cup}$) holds.

Theorem 5.8. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\hat{H},C)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A) \Cap \left((\hat{G},B)\,\tilde{\cup}\,(\hat{H},C)\right) = \left((\hat{F},A) \Cap (\hat{G},B)\right)\tilde{\cup}\left((\hat{F},A) \Cap (\hat{H},C)\right)$.

(ii) $(\hat{F},A)\,\tilde{\cup}\left((\hat{G},B) \Cap (\hat{H},C)\right) = \left((\hat{F},A)\,\tilde{\cup}\,(\hat{G},B)\right) \Cap \left((\hat{F},A)\,\tilde{\cup}\,(\hat{H},C)\right)$

Proof:

(i) Let $(\hat{F},A) \Cap \left((\hat{G},B) \tilde{\cup} (\hat{H},C)\right) = (\hat{I}, A \cap (B \cup C))$,

$\left((\hat{F},A) \Cap (\hat{G},B)\right) \tilde{\cup} \left((\hat{F},A) \Cap (\hat{H},C)\right) = (\hat{J}, (A \cap B) \cup (A \cap B))$,

$= (\hat{J}, A \cap (B \cup C))$.

Now for any $e \in A \cap (B \cup C)$, it follows that $e \in A$ and $e \in B \cup C$.

We consider the following cases.

Case 1: $e \in A$, $e \notin B$ and $e \in C$. Then $\hat{I}(e) = \hat{F}(e) \cap \hat{H}(e) = \hat{J}(e)$.

Case 2: $e \in A$, $e \in B$ and $e \notin C$. Then $\hat{I}(e) = \hat{F}(e) \cap \hat{G}(e) = \hat{J}(e)$.

Case 3: $e \in A$, $e \in B$ and $e \in C$. Then $\hat{I}(e) = \hat{F}(e) \cap \left(\hat{G}(e) \cup \hat{H}(e)\right) =$

$\left(\hat{F}(e) \cap \hat{G}(e)\right) \cup \left(\hat{F}(e) \cap \hat{H}(e)\right) = \hat{J}(e)$.

Therefore, $\hat{I}$ and $\hat{J}$ are the same operators and so

$(\hat{F},A) \Cap \left((\hat{G},B) \tilde{\cup} (\hat{H},C)\right) = \left((\hat{F},A) \Cap (\hat{G},B)\right) \tilde{\cup} \left((\hat{F},A) \Cap (\hat{H},C)\right)$.

(ii) The proof of (ii) follows from (i).

The following theorem shows that, the distributive law with respect to operations extended intersection ($\tilde{\cap}_E$) and restricted ($\tilde{\cup}_R$) holds.

Theorem 5.9. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\hat{H},C)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A) \tilde{\cup}_R \left((\hat{G},B) \tilde{\cap}_E (\hat{H},C)\right) = \left((\hat{F},A) \tilde{\cup}_R (\hat{G},B)\right) \tilde{\cap}_E \left((\hat{F},A) \tilde{\cup}_R (\hat{H},C)\right)$.

(ii) $(\hat{F},A) \tilde{\cap}_E \left((\hat{G},B) \tilde{\cup}_R (\hat{H},C)\right) = \left((\hat{F},A) \tilde{\cap}_E (\hat{G},B)\right) \tilde{\cup}_R \left((\hat{F},A) \tilde{\cap}_E (\hat{H},C)\right)$.

Proof. The proof is similar to the proof of theorem 5.8.

The following theorem shows that, the distributive law with respect to operations extended intersection ($\Cap$) and restricted ($\tilde{\cup}_R$) holds.

Theorem 5.10. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\hat{H},C)$ be intuitionistic fuzzy soft sets over U. Then,

(i) $(\hat{F},A) \tilde{\cup}_R \left((\hat{G},B) \Cap (\hat{H},C)\right) = \left((\hat{F},A) \tilde{\cup}_R (\hat{G},B)\right) \Cap \left((\hat{F},A) \tilde{\cup}_R (\hat{H},C)\right)$.

(ii) $(\hat{F},A) \Cap \left((\hat{G},B) \tilde{\cup}_R (\hat{H},C)\right) = \left((\hat{F},A) \Cap (\hat{G},B)\right) \tilde{\cup}_R \left((\hat{F},A) \Cap (\hat{H},C)\right)$.

Proof. The proof follows from Theorem 5.8.

Theorem 5.11. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\hat{H},C)$ be intuitionistic fuzzy soft sets over U. Then, $(\hat{F},A) \tilde{\subseteq} (\hat{G},B) \Rightarrow (\hat{F},A) \tilde{\cup} (\hat{H},C) \tilde{\subseteq} (\hat{G},B) \tilde{\cup} (\hat{H},C)$ and $(\hat{F},A) \Cap (\hat{H},C) \tilde{\subseteq} (\hat{G},B) \Cap (\hat{H},C)$

Proof: Let $(\hat{F},A) \tilde{\cup} (\hat{H},C) = (\hat{I}, A \cup C)$ and $(\hat{G},B) \tilde{\cup} (\hat{H},C) = (\hat{J}, B \cup C)$.

From $(\hat{F},A) \tilde{\subseteq} (\hat{G},B)$, we have $A \subseteq B$ and $\hat{F}(e) \subseteq \hat{G}(e)$ for any $e \in A$.

Now for any $e \in A \cup C$, we consider the following cases:

Case 1: $e \in A - C$. Then $e \in B - C$. Hence, $\hat{I}(e) = \hat{F}(e) \subseteq \hat{G}(e) = \hat{J}(e)$.

Case 2: $e \in (B \cap C) - A$. Then, $\hat{I}(e) = \hat{H}(e) \subseteq \hat{F}(e) \cup \hat{H}(e) = \hat{J}(e)$.

Case 3: $e \in C - B$. Then, $e \in C - A$. Hence $\hat{I}(e) = \hat{H}(e) = \hat{J}(e)$.

Case 4: $e \in (A \cap C)$. Then, $e \in (B \cap C)$. Hence $\hat{I}(e) = \hat{F}(e) \cup \hat{H}(e) \subseteq \hat{G}(e) \cup \hat{H}(e) = \hat{J}(e)$.

The following theorem shows that, the associative law with respect to operations extended intersection ($\tilde{\cap}_E$) and union ($\tilde{\cup}$) holds under certain condition.

Theorem 5.12. Let $(\hat{F},A)$, $(\hat{G},B)$ and $(\hat{H},C)$ be intuitionistic fuzzy soft sets over U. such that $(\hat{H},C) \tilde{\subseteq} (\hat{F},A)$. Then,

(i) $(\hat{F},A) \tilde{\cap}_E \left((\hat{G},B) \tilde{\cup} (\hat{H},C)\right) \tilde{\subseteq} \left((\hat{F},A) \tilde{\cap}_E (\hat{G},B)\right) \tilde{\cup} (\hat{H},C)$.

(ii) $(\hat{F},A) \tilde{\cap}_E \left((\hat{G},B) \tilde{\cup} (\hat{H},C)\right) = \left((\hat{F},A) \tilde{\cap}_E (\hat{G},B)\right) \tilde{\cup} (\hat{H},C)$, if $A \subseteq B$.

Proof.

(i) Suppose that $(\hat{F},A) \tilde{\cap}_E \left((\hat{G},B) \tilde{\cup} (\hat{H},C)\right) = \left(\hat{I}, A \cup (B \cup C)\right)$,

$\left((\hat{F},A) \tilde{\cap}_E (\hat{G},B)\right) \tilde{\cup} (\hat{H},C) = \left(\hat{J}, (A \cup B) \cup C\right)$.

Now, for any $e \in A \cup (B \cup C)$, we consider the following cases.

Case 1: $e \in A, e \in B$ and $e \notin C$. Then $\hat{I}(e) = \hat{F}(e) \cap \hat{G}(e) = \hat{J}(e)$.

Case 2: $e \in A, e \notin B$ and $e \in C$. Then,

$\hat{I}(e) = \hat{F}(e) \cap \hat{H}(e) = \hat{H}(e) \subseteq \hat{F}(e) = \hat{F}(e) \cap \hat{H}(e) = \hat{J}(e)$.
Case 3: $e \in A, e \notin B$ and $e \notin C$. Then $\hat{I}(e) = \hat{F}(e) = \hat{J}(e)$.
Case 4: $e \in A, e \in B$ and $e \in C$. Then,
$\hat{I}(e) = \hat{F}(e) \cap \left(\hat{G}(e) \cup \hat{H}(e)\right) = \left(\hat{F}(e) \cap \hat{G}(e)\right) \cup \left(\hat{F}(e) \cap \hat{H}(e)\right) = \hat{J}(e)$.
Case 5: $e \notin A, e \in B$ and $e \notin C$. Then $\hat{I}(e) = \hat{G}(e) = \hat{J}(e)$.
Therefore, $(\hat{F}, A) \tilde{\cap}_E \left((\hat{G}, B) \tilde{\cup} (\hat{H}, C)\right) \tilde{\subseteq} \left((\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)\right) \tilde{\cup} (\hat{H}, C)$.
If $A \subseteq B$, then cases 2 and 3 do not hold. It follows that
$(\hat{F}, A) \tilde{\cap}_E \left((\hat{G}, B) \tilde{\cup} (\hat{H}, C)\right) = \left((\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)\right) \tilde{\cup} (\hat{H}, C)$.
It is worth noting that $(\hat{F}, A) \tilde{\cap}_E \left((\hat{G}, B) \tilde{\cup} (\hat{H}, C)\right) = \left((\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)\right) \tilde{\cup} (\hat{H}, C)$ may not be true if $A \nsubseteq B$ as shown in the following example. Hence, the associative law with respect to the operations $\tilde{\cap}_E$ and $\tilde{\cup}$ does not hold in general.
Example 5.2. Let $U = \{x, y\}, A = E = \{e_1, e_2, e_3\}, B = \{e_1, e_2\}$ and $C = \{e_3\}$. Define intuitionistic fuzzy soft sets $(\hat{F}, A)$, $(\hat{G}, B)$ and $(\hat{H}, C)$ over U as follows:
$\hat{F}(e_1) = \{\langle x, 0.4, 0.4\rangle, \langle y, 0.3, 0.6\rangle\}$,
$\hat{F}(e_2) = \{\langle x, 0.5, 0.4\rangle, \langle y, 0.4, 0.5\rangle\}$,
$\hat{F}(e_3) = \{\langle x, 0.5, 0.4\rangle, \langle y, 0.6, 0.3\rangle\}$,
$\hat{G}(e_1) = \{\langle x, 0.4, 0.4\rangle, \langle y, 0.4, 0.6\rangle\}$.
$\hat{G}(e_2) = \{\langle x, 0.5, 0.3\rangle, \langle y, 0.4, 0.5\rangle\}$,
$\hat{H}(e_1) = \{\langle x, 0.3, 0.6\rangle, \langle y, 0.4, 0.5\rangle\}$.
Suppose that
$(\hat{F}, A) \tilde{\cap}_E \left((\hat{G}, B) \tilde{\cup} (\hat{H}, C)\right) = (\hat{I}, A \cup (B \cup C))$.
$((\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)) \tilde{\cup} (\hat{H}, C) = (\hat{J}, (A \cup B) \cup C)$.
Now, we have
$\mu_{\hat{I}(e_3)}(x) = min\{\mu_{\hat{F}(e_3)}(x), \mu_{\hat{H}(e_3)}(x)\} = min\{0.5, 0.3\} = 0.3$ and
$\mu_{\hat{J}(e_3)}(x) = max\{\mu_{\hat{F}(e_3)}(x), \mu_{\hat{H}(e_3)}(x)\} = max\{0.5, 0.3\} = 0.5$.
It follows that, $\mu_{\hat{I}(e_3)}(x) < \mu_{\hat{J}(e_3)}(x)$.
Therefore, $(\hat{F}, A) \tilde{\cap}_E \left((\hat{G}, B) \tilde{\cup} (\hat{H}, C)\right) \neq ((\hat{F}, A) \tilde{\cap}_E (\hat{G}, B)) \tilde{\cup} (\hat{H}, C)$.

6. Intuitionistic Fuzzy Soft Set Based Decision Making

Definition 6.1. Let $\Sigma = \langle \hat{F}, A \rangle$ be an intuitionistic fuzzy soft set over U, where $A \subseteq E$ and E is a set of parameters. For $s, t \in [0, 1]$, the $(\boldsymbol{s}, \boldsymbol{t}) - \boldsymbol{level}$ soft set of Σ is a crisp soft set $L(\Sigma; s, t) = \langle \hat{F}_{(s,t)}, A \rangle$ defined by $\hat{F}_{(s,t)}(a) = L(\hat{F}(a); s, t) = \{x \in U: \mu_{\hat{F}(a)}(x) \geq s \text{ } and \text{ } \lambda_{\hat{F}(a)}(x) \leq t\}$ for all $a \in A$.
This definition is clearly an extension of level soft sets of fuzzy soft sets. That is, $s \in [0, 1]$ can be viewed as a given **least threshold** on membership values and $t \in [0, 1]$ can be seen as a given **greatest threshold** on non-membership values. In a real-life application of intuitionistic fuzzy sets based decision making, normally the thresholds are chosen in advance by the decision makers and represent their requirements on membership levels and non-membership levels, respectively.
Definition 6.2. Let $\Sigma = \langle \hat{F}, A \rangle$ be an intuitionistic fuzzy soft set over U, where $A \subseteq E$ and E is a set of parameters. Let $\eta: A \rightarrow [0, 1] \times [0, 1]$ be an intuitionistic fuzzy set in A which is called a **Threshold intuitionistic fuzzy set.** The level soft set of Σ with respect to η is a crisp soft set $L(\Sigma, \eta) = \langle \hat{F}_\eta, A \rangle$ defined by
$\hat{F}_\eta(a) = L\left(\hat{F}(a); \eta(a)\right) = \{x \in U: \mu_{\hat{F}(a)}(x) \geq \mu_\eta(a) \text{ } and \text{ } \lambda_{\hat{F}(a)}(x) \leq \lambda_\eta(a)\}$ for all $a \in A$. Clearly, the level soft sets of intuitionistic fuzzy soft sets with respect to an intuitionistic fuzzy set are extensions of the level soft set.
Definition 6.3. (**The mid-level soft set** of an intuitionistic fuzzy soft set). Let $\Sigma = \langle \hat{F}, A \rangle$ be an intuitionistic fuzzy soft set over U, where $A \subseteq E$ and E is a set of parameters. Based on the intuitionistic fuzzy soft set $\Sigma = \langle \hat{F}, A \rangle$**,** we can define an intuitionistic fuzzy set
$mid_\Sigma: A \rightarrow [0, 1] \times [0, 1]$ by $\mu_{mid\Sigma}(a) = \frac{1}{|U|} \sum_{x \in U} \mu_{\hat{F}(a)}(x)$ and $\lambda_{mid\Sigma}(a) = \frac{1}{|U|} \sum_{x \in U} \lambda_{\hat{F}(a)}(x)$ for all $a \in A$. The intuitionistic fuzzy set mid_Σ is called the **mid-threshold** of the intuitionistic fuzzy soft set Σ**.** In addition, the level soft set of Σ with respect to the mid-threshold intuitionistic fuzzy set mid_Σ, namely $L(\Sigma, mid_\Sigma)$ is called the mid-level soft set of Σ and is represented simply by $L(\Sigma; mid)$. In what follows the mid-level decision rule will mean using the mid-threshold and considering the mid-level soft set in intuitionistic fuzzy soft sets based decision making.

Definition 6.4. (The **Top-Bottom-level soft set**, **Top-Top-level soft set** and **Bottom-Bottom-level soft set** of an intuitionistic fuzzy soft set). Let $\mathcal{E} = \langle \hat{F}, A \rangle$ be an intuitionistic fuzzy soft set over U, where $A \subseteq E$ and E is a set of parameters. Based on the intuitionistic fuzzy soft set
$\mathcal{E} = \langle \hat{F}, A \rangle$**,** we can define an intuitionistic fuzzy set $\boldsymbol{topbottom}_{\mathcal{E}}: A \to [0,1] \times [0,1]$ by
$\mu_{topbottom\ \mathcal{E}}(a) = \max_{x \in U} \mu_{\hat{F}(a)}(x)$ and $\lambda_{topbottom\ \mathcal{E}}(a) = \min_{x \in U} \lambda_{\hat{F}(a)}$ for all $a \in A$.
Intuitionistic fuzzy set $\boldsymbol{toptop}_{\mathcal{E}}: A \to [0,1] \times [0,1]$ is defined by
$\mu_{toptop\ \mathcal{E}}(a) = \max_{x \in U} \mu_{\hat{F}(a)}(x)$ and $\lambda_{toptop\ \mathcal{E}}(a) = \max_{x \in U} \lambda_{\hat{F}(a)}(x)$ for all $a \in A$.
Also, intuitionistic fuzzy set $\boldsymbol{bottombottom}_{\mathcal{E}}: A \to [0,1] \times [0,1]$ is define by
$\mu_{bottombottom\ \mathcal{E}}(a) = \min_{x \in U} \mu_{\hat{F}(a)}(x)$ and $\lambda_{bottombottom\ \mathcal{E}}(a) = \min_{x \in U} \lambda_{\hat{F}(a)}$ for all $a \in A$.
To illustrate the above definitions, we shall consider the following Example 6.1.

Example 6.1. Let us consider an intuitionistic fuzzy soft set $\mathcal{E} = \langle \hat{F}, A \rangle$ which describes the conditions of some states in a country that an investor Mr. X with enough budgets is considering to locate his manufacturing industry.
Suppose that there are six states in the initial universe $U = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ under consideration and that $A = \{a_1, a_2, a_3, a_4, a_5\}$ is a set of decision parameters or attribute related to U. The $a_i (i = 1,2,3,4,5)$ stand for the parameters "peaceful", "power supply", "accessible", "densely populated" and "good weather", respectively. Suppose that
$\hat{F}(a_1) = \{\langle S_1, 0.8, 0.2\rangle, \langle S_2, 0.6, 0.2\rangle, \langle S_3, 0.7, 0.2\rangle, \langle S_4, 0.4, 0.3\rangle, \langle S_5, 0.9, 0.1\rangle, \langle S_6, 0.6, 0.4\rangle\}$,
$\hat{F}(a_2) = \{\langle S_1, 0.7, 0.2\rangle, \langle S_2, 0.8, 0.1\rangle, \langle S_3, 0.5, 0.3\rangle, \langle S_4, 0.6, 0.4\rangle, \langle S_5, 0.8, 0.2\rangle, \langle S_6, 0.4, 0.3\rangle\}$,
$\hat{F}(a_3) = \{\langle S_1, 0.7, 0.1\rangle, \langle S_2, 0.6, 0.2\rangle, \langle S_3, 0.3, 0.4\rangle, \langle S_4, 0.8, 0.1\rangle, \langle S_5, 0.9, 0.1\rangle, \langle S_6, 0.6, 0.3\rangle\}$,
$\hat{F}(a_4) = \{\langle S_1, 0.6, 0.3\rangle, \langle S_2, 0.8, 0.1\rangle, \langle S_3, 0.9, 0.1\rangle, \langle S_4, 0.8, 0.2\rangle, \langle S_5, 0.7, 0.1\rangle, \langle S_6, 0.8, 0.2\rangle\}$,
$\hat{F}(a_5) = \{\langle S_1, 0.8, 0.2\rangle, \langle S_2, 0.4, 0.5\rangle, \langle S_3, 0.9, 0.1\rangle, \langle S_4, 0.8, 0.1\rangle, \langle S_5, 0.5, 0.4\rangle, \langle S_6, 0.8, 0.2\rangle\}$.
The intuitionistic fuzzy soft set $\mathcal{E} = \langle \hat{F}, A \rangle$ is a parameterized family $\{\hat{F}(a_i), i = 1, 2,3,4,5\}$ of fuzzy sets on U and $\langle \hat{F}, A \rangle =$

$$\left\{\begin{array}{c} Peaceful\ states = \left\{\begin{array}{c}\langle S_1, 0.8, 0.2\rangle, \langle S_2, 0.6, 0.2\rangle, \langle S_3, 0.7, 0.2\rangle, \langle S_4, 0.4, 0.3\rangle, \langle S_5, 0.9, 0.1\rangle, \\ \langle S_6, 0.6, 0.4\rangle\end{array}\right\}, \\ power\ supply\ states = \left\{\begin{array}{c}\langle S_1, 0.7, 0.2\rangle, \langle S_2, 0.8, 0.1\rangle, \langle S_3, 0.5, 0.3\rangle, \langle S_4, 0.6, 0.4\rangle, \\ \langle S_5, 0.8, 0.2\rangle, \langle S_6, 0.4, 0.3\rangle\end{array}\right\}, \\ accessible\ states = \left\{\begin{array}{c}\langle S_1, 0.7, 0.1\rangle, \langle S_2, 0.6, 0.2\rangle, \langle S_3, 0.3, 0.4\rangle, \langle S_4, 0.8, 0.1\rangle, \\ \langle S_5, 0.9, 0.1\rangle, \langle S_6, 0.6, 0.3\rangle\end{array}\right\}, \\ densely\ populated\ states = \left\{\begin{array}{c}\langle S_1, 0.6, 0.3\rangle, \langle S_2, 0.8, 0.1\rangle, \langle S_3, 0.9, 0.1\rangle, \langle S_4, 0.8, 0.2\rangle, \\ \langle S_5, 0.7, 0.1\rangle, \langle S_6, 0.8, 0.2\rangle\end{array}\right\} \\ states\ with\ good\ weather = \left\{\begin{array}{c}\langle S_1, 0.8, 0.2\rangle, \langle S_2, 0.4, 0.5\rangle, \langle S_3, 0.9, 0.1\rangle, \langle S_4, 0.8, 0.1\rangle, \\ \langle S_5, 0.5, 0.4\rangle, \langle S_6, 0.8, 0.2\rangle\end{array}\right\} \end{array}\right\}$$

Algorithm

(I) Input the intuitionistic fuzzy soft set $\mathcal{E} = \langle \hat{F}, A \rangle$**.**

(II) Input a threshold intuitionistic fuzzy set $\eta: A \to [0,1] \times [0,1]$ (or give a threshold value pair $(s,t) \in [0,1] \times [0,1]$; or choose a mid-level decision rule; or choose the top-bottom-level decision rule; or choose the top-top-level decision rule; or choose the bottom-bottom-level decision rule) for decision making.

(III) Compute the level soft set $L(\mathcal{E}; \eta)$ with respect to the threshold intuitionistic fuzzy set η (or the (s,t) −level soft set $L(\mathcal{E}; s, t)$; or the mid-level soft set $L(\mathcal{E}; mid)$; or the top-bottom-level soft set $L(\mathcal{E}; topbottom)$; or the top-top-level soft set $L(\mathcal{E}; toptop)$ or the bottom-bottom-level soft set $L(\mathcal{E}; bottombottom)$).

(IV) Present the level soft set $L(\mathcal{E}; \eta)$ (or $L(\mathcal{E}; s, t)$; $L(\mathcal{E}; mid)$; $or\ L(\mathcal{E}; topbottom)$; $orL(\mathcal{E}; toptop)$; $or\ L(\mathcal{E}; bottombottom)$) in tabular form and compute the choice value c_i of o_i, for all i.

(v) The optimal decision is to select o_k if $c_k = max_i c_i$.

(vi) If k has more than one value then any one of o_k may be chosen.

Table 1 gives the tabular representation of the intuitionistic fuzzy soft set $\mathcal{E} = \langle \hat{F}, A \rangle$**.**

Table 1: Tabular representation of intuitionistic fuzzy soft set $\mathcal{E} = \langle \hat{F}, A \rangle$

U/A	a_1	a_2	a_3	a_4	a_5
S_1	$\langle 0.8, 0.2\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.8, 0.2\rangle$
S_2	$\langle 0.6, 0.2\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.4, 0.5\rangle$
S_3	$\langle 0.7, 0.2\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.9, 0.1\rangle$
S_4	$\langle 0.4, 0.3\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.8, 0.1\rangle$
S_5	$\langle 0.9, 0.1\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.5, 0.4\rangle$
S_6	$\langle 0.6, 0.4\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.8, 0.2\rangle$

Now, let us take $S = 0.7$ and $t = 0.3$, then we have the following:
$L(\hat{F}(a_1); 0.7, 0.3) = \{S_1, S_3, S_5\}$,
$L(\hat{F}(a_2); 0.7, 0.3) = \{S_1, S_2, S_5\}$,
$L(\hat{F}(a_3); 0.7, 0.3) = \{S_1, S_4, S_5\}$,
$L(\hat{F}(a_4); 0.7, 0.3) = \{S_2, S_3, S_4, S_5, S_6\}$,
$L(\hat{F}(a_5); 0.7, 0.3) = \{S_1, S_3, S_4, S_6\}$.
Hence, the $(0.7, 0.3)$ −level soft set $\Sigma = \langle \hat{F}, A\rangle$ is a soft set $L(\Sigma; 0.7, 0.3) = \langle \hat{F}_{(0.7,0.3)}, A\rangle$, where the set- valued mapping $\hat{F}_{(0.7,0.3)}: A \to P(U)$ is defined by
$\hat{F}_{(0.7,0.3)}(a_i) = L(\hat{F}(a_i); 0.7,0.3)$ for $i = 1,2,3,4,5$. Table 2 gives the tabular representation of the $(0.7,0.3) - level$ soft set $L(\Sigma; 0.7, 0.3)$ with choice value.

Table 2: Tabular representation of the level soft set $L(\Sigma; \mathbf{0.7}, \mathbf{0.3})$ with choice value

U/A	a_1	a_2	a_3	a_4	a_5	$Choice\ Value$
S_1	1	1	1	0	1	**4**.
S_2	0	1	0	1	0	2.
S_3	1	0	0	1	1	3.
S_4	0	0	1	1	1	3.
S_5	1	1	1	1	0	**4**.
S_6	0	0	0	1	1	2.

From the Table **2,** it follows that, the maximum choice value is $c_1 = c_5 = \mathbf{4}$ and therefore the optimal decision is to select either state $\boldsymbol{S_1}$ or state $\boldsymbol{S_5}$.
For mid-level soft sets, let us consider Example **6.1** and $\Sigma = \langle \hat{F}, A\rangle$ with tabular representation in Table **3.**
it is clear that, the mid-threshold of $\langle \hat{F}, A\rangle$ is an intuitionistic fuzzy set
$mid_{\langle \hat{F},A\rangle} = \{\langle a_1, 0.67, 0.23\rangle, \langle a_2, 0.63, 0.25\rangle, \langle a_3, 0.65, 0.2\rangle, \langle a_4, 0.77, 0.16\rangle, \langle a_5, 0.7, 0.25\rangle\}$, and the mid-level soft set of $\langle \hat{F}, A\rangle$ is a soft set $L(\langle \hat{F}, A\rangle; mid)$ with its tabular representation given by Table 3**.**

Table 3: Tabular representation of the mid-level soft set $L(\langle \hat{F}, A\rangle; mid)$ with choice value

U/A	a_1	a_2	a_3	a_4	a_5	$Choice\ Value$
S_1	1	1	1	0	1	**4**.
S_2	0	1	0	1	0	2.
S_3	1	0	0	1	1	3.
S_4	0	0	1	0	1	2.
S_5	1	1	1	0	0	3.
S_6	0	0	0	0	1	1.

From the Table 3**,** the maximum choice value is $c_1 = 4$ and so the optimal decision is to select state $\boldsymbol{S_1}$**.**
For illustrative example of top-bottom-level soft sets, top-top-level soft sets and bottom-bottom-level soft sets, let us again consider the intuitionistic fuzzy soft set $\Sigma = \langle \hat{F}, A\rangle$ with its tabular representation given by Table 1.
It is obvious that the top-bottom-threshold of $\langle \hat{F}, A\rangle$ is an intuitionistic fuzzy set
$topbottom_{\langle \hat{F},A\rangle} = \{\langle a_1, 0.9, 0.1\rangle, \langle a_2, 0.8, 0.1\rangle, \langle a_3, 0.9, 0.1\rangle, \langle a_4, 0.9, 0.1\rangle, \langle a_5, 0.9, 0.1\rangle\}$.
and the top-bottom-level soft set of $\langle \hat{F}, A\rangle$ is a soft set $L(\langle \hat{F}, A\rangle; topbottom)$ with its tabular representation given by Table 4.
it is also clear that the top-top-threshold of $\langle \hat{F}, A\rangle$ is an intuitionistic fuzzy set
$toptop_{\langle \hat{F},A\rangle} = \{\langle a_1, 0.9, 0.4\rangle, \langle a_2, 0.8, 0.4\rangle, \langle a_3, 0.9, 0.4\rangle, \langle a_4, 0.9, 0.3\rangle, \langle a_5, 0.9, 0.5\rangle\}$.
and the top-top-level soft set of $\langle \hat{F}, A\rangle$ is a soft set $L(\langle \hat{F}, A\rangle; toptop)$ with its tabular representation given by Table 5.
It is obvious that the bottom-bottom-threshold of $\langle \hat{F}, A\rangle$ is an intuitionistic fuzzy set
$bottombottom_{\langle \hat{F},A\rangle} = \{\langle a_1, 0.4, 0.1\rangle, \langle a_2, 0.4, 0.1\rangle, \langle a_3, 0.3, 0.1\rangle, \langle a_4, 0.6, 0.1\rangle, \langle a_5, 0.4, 0.1\rangle\}$.
and the bottom-bottom-level soft set of $\langle \hat{F}, A\rangle$ is a soft set $L(\langle \hat{F}, A\rangle; bottombottom)$ with its tabular representation given by Table 6.

Table 4: Tabular representation of the top-bottom-level soft set $L(\Sigma; topbottom)$ with choice value

U/A	a_1	a_2	a_3	a_4	a_5	$Choice\ Value$
S_1	0	0	0	0	0	0.
S_2	0	1	0	0	0	1.
S_3	0	0	0	1	1	**2**.
S_4	0	0	0	0	0	0.
S_5	1	0	1	0	0	**2**.
S_6	0	0	0	0	0	0.

From Table 4, it follows that the maximum choice value is $c_3 = c_5 = \mathbf{2.}$ The optimum decision is to select either state S_3 or state S_5.

Table 5: Tabular representation of the top-top-level soft set $L(\Sigma; \boldsymbol{toptop})$ with choice value

U/A	a_1	a_2	a_3	a_4	a_5	$Choice\ Value$
S_1	0	0	0	0	0	0.
S_2	0	1	0	0	0	1.
S_3	0	0	0	1	1	**2**.
S_4	0	0	0	0	0	0.
S_5	1	1	1	0	0	**3**.
S_6	0	0	0	0	0	0.

From Table 5, it follows that the maximum choice value is $c_5 = \mathbf{3.}$ The optimum decision is to select state S_5.

Table 6: Tabular representation of the bottom-bottom-level soft set $L(\Sigma; \boldsymbol{bottombottom})$ with choice value

U/A	a_1	a_2	a_3	a_4	a_5	$Choice\ Value$
S_1	0	0	1	0	0	1.
S_2	0	1	0	1	0	2.
S_3	0	0	0	1	1	2.
S_4	0	0	1	0	1	2.
S_5	1	0	1	1	0	**3**.
S_6	0	0	0	0	0	0.

From Table 6, it follows that the maximum choice value is $c_5 = \mathbf{3.}$ The optimum decision is to select state S_5.

Remark: It has been observed that when we constructed the Table of **Bottom-top-**level soft set of an intuitionistic fuzzy soft set in Example 6.1, we noticed that all the possible choice options have the same maximum choice value and hence no unique choice can be made.

Also, after employing all decision rules on Example 6.1, we observed that the choice of the states differs and this is due to the decision maker's preference (or the choice of the decision rules).

7. Conclusion

In this paper, we investigated some basic results on intuitionistic fuzzy soft set. We stated De Morgan's laws and proved them in details. We also presented some detailed results on restricted union, union, restricted intersection, extended intersection, AND and OR products and absorption laws with respect to the various operations. Finally, an adjustable approach to decision making problem using level soft set of an intuitionistic fuzzy soft set was presented with illustrative example.

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