Transactions of the Nigerian Association of Mathematical Physics
Volume 8, (January, 2019), p53 – 60

NATURE OF EQUILIBRIUM SOLUTIONS, RESONANCE BEHAVIOR AND STABILITY OF A FORCED DAMPED DRIVEN MORSE OSCILLATOR

Usman A. Marte[1], Usman A. Umar[2] and Abdulahi N. Njah [3]

[1]Department of Mathematical sciences, University of Maiduguri, Maiduguri, Nigeria.
[2]Department of Physics, University of Maiduguri, Maiduguri, Nigeria
[3]Department of Physics, University of Lagos, Lagos, Nigeria;

Abstract

The classically damped driven Morse oscillator is considered. The method of multiple scales MMS is used to obtain the equilibrium solutions, then the nature of the equilibrium solutions are analyzed from the eigenvalues of the autonomous set of equations obtained from the MMS. The frequency response curve for detuning parameter value ranging from [-4, 9] is also shown, showing the behavior of principal resonance of the system. The stability region for the system is explored through characteristic properties eigenvalues of the perturbed linear system and results were compared with the nature of the equilibrium solutions found. The peculiarity of the forced damped driven oscillator turn out to be for a particular energy level the stability region is approximately simple harmonic(SH) for small values of the oscillating frequency. As the oscillating frequency approaches the dissociation limit the system loses stability till the oscillating frequency reaches the stability region of the next energy level where it again becomes approximately SH.

1.0 Introduction

The Morse oscillator is frequently used in the description of the motion of diatomic molecules in an external electromagnetic field. A lot of investigation on the Morse oscillator has been done with the classical, semi classical and quantum mechanical methods for the description of the diatomic molecule [1-11]. Some areas where Morse oscillator is used include the modeling of multi photon excitation of diatomic molecule in a dense medium or in a gaseous cell under a high pressure and the modeling of the pumping of a local mode of a poly atomic molecule by an infra red laser where the energy flow out of the molecule decay with a constant rate. Not much has been done on the Morse oscillator in the area of nonlinear dynamics. The dynamics of the damped driven Morse oscillator has not received much attention in comparison with the Duffing and the van-der Pol oscillators with has been looked into by many. The bifurcation structure of the classically damped driven oscillator has been looked into where the driving frequency was considered as the bifurcation parameter [12, 13]. Where the authors considered the driving frequency in the range of 0.0 to 3.0 and presented their results, a very rich dynamical behavior was presented; in particular they showed from a bifurcation diagram that a chaotic orbit is found for a driving frequency value of 0.5. The dynamical behavior of the damped driven Morse oscillator with varying forcing amplitude for the driving frequency fixed at 0.5 was looked into in [15] where different types of dynamical behavior were obtained. This paper tries to look into the dynamics and the bifurcations structures of the damped driven Morse oscillator using the method of multiple scales MMS perturbation theory. The equation of motion under consideration here is given by

$$\ddot{x}+\alpha\dot{x}+\tfrac{dV}{dx}= f_0 cos(\omega t) \qquad (1)$$

Where α, v, f_0 and ω are the damping, the Morse potential function, the forcing amplitude and the driving frequency respectively for Morse oscillator given by (1).
The Morse potential is given by

$$V(x)=(1-e^{-\beta x})^2 \qquad (2)$$

The potential function given in (2) is expanded using the Taylors series to obtain

$$\tfrac{dV}{dx}=x-\tfrac{3}{2}x^2+\tfrac{7}{6}x^3-\tfrac{5}{8}x^4+\cdots \qquad (3)$$

To facilitate the use of MMS

Correspondence Author: Usman A.M., Email: auamarte@yahoo.com, Tel: +2348023576675, +2348038863149

2.0 Approximate Solution

The approximate solution of equation (1) can be found by employing MMS as given in [16, 17]. For a small but finite value of x the power series solution of the form can be assumed for (1)

$$x(t,\in) = x_0(T_0,T_1) + \in x_1(T_0,T)\cdots \quad (4)$$

Where T_0 = t is a fast time scale and T_1 = εt is a slow time scale. The slow time scale T_1 characterizes the modulation in the amplitude and phase caused by the nonlinearity and damping, while the fast scale T_0 is associated with the relatively fast changes in the response of the system.

The first and the second derivatives with respect to the time t are given by

$$\tfrac{d}{dt} = D_0 + \varepsilon D_1 + \cdots, \tfrac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \quad (5)$$

Where $D_n = \partial/\partial t$. Substituting equations (4) and (5) in (1) and equating coefficients of equal powers of ε. Results in

$O(\varepsilon^0)$: $$(D_0^2 + \omega_0^2)x_0 = 0 \quad (6)$$

$O(\varepsilon^1)$: $$(D_0^2 + \omega_0^2)x_1 = -2D_0D_1x_0 - \alpha D_0 x_0 + \tfrac{3}{2}\omega_0^2 x_0^2 - \tfrac{7}{6}\omega_0^2 x_0^3 + f_0\cos(\omega t) \quad (7)$$

Where $\omega_0^2 = \beta$. The general solution of equation (6) can be written in the form

$$x_0 = A(T_1)\cos(\omega_0 T_0 + \varphi(T_1)) \quad (8)$$

Where $A(T_1)$ and $\varphi(T_1)$ are the amplitude and phase of the response which are determined by imposing the solvability condition in the next level of approximation. Substituting equation (8) into (7) results in

$$(D_0^2 + \omega_0^2)x_1 = 2\omega_0(A'\sin\theta + A\cos\theta\varphi') + \alpha\omega_0 A\sin\theta x_0 + \tfrac{3}{4}\omega_0^2(1+\cos 2\theta)$$
$$-\tfrac{7}{24}\omega_0^2 A^3(3\cos\theta + \cos 3\theta) + f_0\cos(\omega T_0) \quad (9)$$

Where $\theta = \omega_0 T_0 + \varphi$

The solvability condition demands that

$$2\omega_0 A' + \alpha\omega_0 A = -f_0\sin\psi$$
$$\omega_0 A\phi' - \tfrac{7}{8}\omega_0^2 A^3 = -f_0\cos\psi \quad (10)$$

Where $\psi = \varphi - \sigma T_1$ which can be rewritten as

$$A' = -\tfrac{\alpha}{2}A - \tfrac{f_0}{2\omega_0}\sin\psi$$
$$\psi' = \tfrac{7}{8}\omega_0 A^2 + \sigma - \tfrac{f_0}{\omega_0 A}\cos\psi \quad (11)$$

Where σ is the detuning parameter

2.1. Systems equilibrium solutions and their nature

The equilibrium solutions can be obtained from (11) by setting ψ' = A' = 0, which results in the equation

$$\alpha\omega_0 A = -f_0\sin\psi$$
$$\omega_0\sigma A - \tfrac{7}{8}\omega_0^2 A^3 = -f_0\cos\psi \quad (12)$$

The nature of the fixed points can analyzed from the eigenvalues of the Jacobian matrix of the system given by equation (11) which can be written in the form

$$J = \begin{pmatrix} -\tfrac{\alpha}{2} & -\tfrac{f_0}{2\omega_0}\cos\psi \\ \tfrac{14}{8}\omega_0 A + \tfrac{f_0}{\omega_0 A^2}\cos\psi & \tfrac{f_0}{\omega_0 A}\sin\psi \end{pmatrix} \quad (13)$$

2.2 Principal Resonance

The principal resonance solution can be found by setting ω = ω₀ + εσ where σ is a detuning parameter. From (12) by squarering and adding it results in

$$(\alpha\omega_0 A)^2 + (\omega_0\sigma A - \tfrac{7}{8}\omega_0^2 A^3)^2 = f_0^2 \quad (14)$$

Resulting in the so called the frequency response curve FRC

2.3 Systems regions of stability

The stability of the equilibrium solution can be studied by introducing a small perturbation to the solution obtained in (10) and allow the system to run for some time and see what happens to the small perturbation. For stable solution the perturbation dies down with time while for unstable solution the perturbation grows with time.

The small perturbation can be introduced in the form :

$$A = A_0 + A_1$$
$$\psi = \psi_0 + \psi_1 \quad (15)$$

Where A_0 and ψ_0 satisfy equation (11). Substituting (15) into (11) and keeping only linear terms in A_1 and ψ_1 results in

$$(A_0+A_1)' = -\tfrac{\alpha}{2}(A_0+A_1) - \tfrac{f_0}{2\omega_0}\sin(\psi_0+\psi_1)$$
$$(\psi_0+\psi_1)' = \tfrac{1}{\omega_0(A_0+A_1)}[7(A_0+A_1)^3 - f_0\cos(\psi_0+\psi_1] - \sigma \qquad (16)$$

Simplifying (16) and keeping only the linear terms A_1 and ψ_1 results in

$$A_1' = -\tfrac{\alpha}{2}A_1 - (\tfrac{f_0}{2\omega_0}\cos\psi_0)\psi_1$$
$$\psi_1' = \tfrac{1}{A_0}[(21A_0 - 7A_0^2 + \tfrac{f_0}{A_0}\cos\psi_0)A_1 + (f_0\sin\psi_0)\psi_1] \qquad (17)$$

Whose Jacobian matrix is given by

$$J = \begin{pmatrix} -\frac{\alpha}{2} & -\frac{f_0}{2\omega_0}\cos\psi_0 \\ \frac{1}{A_0}(21A_0 - 7A_0^2 + \frac{f_0}{A_0}\cos\psi_0) & f_0\sin\psi_0 \end{pmatrix} \qquad (18)$$

The eigenvalues of this Jacobian can be obtained from

$$\begin{vmatrix} -\frac{\alpha}{2}-\lambda & -\frac{1}{2\omega_0}(7A_0^3 - \sigma\omega_0 A_0) \\ \frac{1}{A_0}(21A_0 - \sigma\omega_0) & \alpha\omega_0 A_0 - \lambda \end{vmatrix} = 0 \qquad (19)$$

Expanding this determinant gives the characteristic equation

$$\lambda^2 + \alpha(\tfrac{1}{2} + \omega_0 A_0)\lambda + \tfrac{1}{2}\alpha^2\omega_0 A_0 + \tfrac{1}{2\omega_0}(7A_0^2 - \sigma\omega_0)(21A_0 - \sigma\omega_0) = 0 \qquad (20)$$

Leading to the stability condition for the equilibrium solution given by

$$\tfrac{1}{2}\alpha^2\omega_0 A_0 + \tfrac{1}{2\omega_0}(7A_0^2 - \sigma\omega_0)(21A_0 - \sigma\omega_0) > 0 \qquad (21)$$

3.0 Results and discussion.

3.1 Equilibrium solutions

The equilibrium solutions for the system is obtained from equation (12) the forcing amplitude value of f_0 = 4.0, damping coefficient α= 0.8 and natural frequency $\omega_0 = \sqrt{8}$, for detuning parameter values σ ranging from -4 to 9.0 and results are presented in Table 1

Table 1:

σ	**Equilibrium solutions**
-4	A→-0.326043,ψ→2.95609 A→-0.203985-1.30805 ™,ψ→0.09275 +0.687651 ™ A→-0.203985+1.30805 ™,ψ→0.09275 -0.687651 ™ A→0.203985 -1.30805 ™,ψ→-3.04884-0.687651 ™ A→0.203985 +1.30805 ™,ψ→-3.04884+0.687651 ™ **A→0.326043,ψ→-0.1855**
-3	A→-0.404363,ψ→2.91081 A→-0.244153-1.16342 ™,ψ→0.115393 +0.621705 ™ A→-0.244153+1.16342 ™,ψ→0.115393 -0.621705 ™ A→0.244153 -1.16342 ™,ψ→-3.0262-0.621705 ™ A→0.244153 +1.16342 ™,ψ→-3.0262+0.621705 ™ **A→0.404363,ψ→-0.230785**
-2	A→-0.511385,ψ→2.84811 A→-0.298776-1.01398 ™,ψ→0.146739 +0.551446 ™ A→-0.298776+1.01398 ™,ψ→0.146739 -0.551446 ™ A→0.298776 -1.01398 ™,ψ→-2.99485-0.551446 ™ A→0.298776 +1.01398 ™,ψ→-2.99485+0.551446 ™ **A→0.511385,ψ→-0.293478**
-1	A→-0.646727,ψ→2.76705 A→-0.367682-0.865089 ™,ψ→0.18727 +0.479491 ™ A→-0.367682+0.865089 ™,ψ→0.18727 -0.479491 ™

	A→0.367682 -0.865089 ™,ψ→-2.95432-0.479491 ™ A→0.367682 +0.865089 ™,ψ→-2.95432+0.479491 ™ **A→0.646727,ψ→-0.37454**
0	A→-0.79883,ψ→2.67271 A→-0.445122-0.719165 ™,ψ→0.23444 +0.406938 ™ A→-0.445122+0.719165 ™,ψ→0.23444 -0.406938 ™ A→0.445122 -0.719165 ™,ψ→-2.90715-0.406938 ™ A→0.445122 +0.719165 ™,ψ→-2.90715+0.406938 ™ **A→0.79883,ψ→-0.468879**
1	A→-0.953328,ψ→2.57201 A→-0.523951-0.569982 ™,ψ→0.284793 +0.329944 ™ A→-0.523951+0.569982 ™,ψ→0.284793 -0.329944 ™ A→0.523951 -0.569982 ™,ψ→-2.8568-0.329944 ™ A→0.523951 +0.569982 ™,ψ→-2.8568+0.329944 ™ **A→0.953328,ψ→-0.569586**
2	A→-1.1017,ψ→2.46874 A→-0.599968-0.398391 ™,ψ→0.336425 +0.236536 ™ A→-0.599968+0.398391 ™,ψ→0.336425 -0.236536 ™ A→0.599968 -0.398391 ™,ψ→-2.80517-0.236536 ™ A→0.599968 +0.398391 ™,ψ→-2.80517+0.236536 ™ **A→1.1017,ψ→-0.672851**
3	A→-1.24101,ψ→2.36336 A→-0.671784-0.0957135 ™,ψ→0.389115 +0.0584849 ™ A→-0.671784+0.0957135 ™,ψ→0.389115 -0.0584849 ™ A→0.671784 -0.0957135 ™,ψ→-2.75248-0.0584849 ™ A→0.671784 +0.0957135 ™,ψ→-2.75248+0.0584849 ™ **A→1.24101,ψ→-0.77823**
4	A→-1.37091,ψ→2.25408 A→-1.09974,ψ→0.671432 A→-0.37902,ψ→0.216084 **A→0.37902,ψ→-2.92551** **A→1.09974,ψ→-2.47016** **A→1.37091,ψ→-0.887515**
5	A→-1.4919,ψ→2.13699 A→-1.31531,ψ→0.839116 A→-0.291201,ψ→0.165483 **A→0.291201,ψ→-2.97611** **A→1.31531,ψ→-2.30248** **A→1.4919,ψ→-1.0046**
6	A→-1.60435,ψ→2.00416 A→-1.48916,ψ→1.00171 A→-0.239179,ψ→0.135716 **A→0.239179,ψ→-3.00588** **A→1.48916,ψ→-2.13988** **A→1.60435,ψ→-1.13743**

7	A→-1.70738,ψ→1.83292
	A→-1.64323,ψ→1.1932
	A→-0.203672,ψ→0.115471
	A→0.203672,ψ→-3.02612
	A→1.64323,ψ→-1.94839
	A→1.70738,ψ→-1.30867
8	A→-1.79359-0.0159167 ™,ψ→1.52047 +0.178058 ™
	A→-1.79359+0.0159167 ™,ψ→1.52047 -0.178058 ™
	A→-0.177616,ψ→0.100644
	A→0.177616,ψ→-3.04095
	A→1.79359 -0.0159167 ™,ψ→-1.62112-0.178058 ™
	A→1.79359 +0.0159167 ™,ψ→-1.62112+0.178058 ™
9	A→-1.90398-0.0317803 ™,ψ→1.52617 +0.392762 ™
	A→-1.90398+0.0317803 ™,ψ→1.52617 -0.392762 ™
	A→-0.157585,ψ→0.0892622
	A→0.157585,ψ→-3.05233
	A→1.90398 -0.0317803 ™,ψ→-1.61543-0.392762 ™
	A→1.90398 +0.0317803 ™,ψ→-1.61543+0.392762 ™

The equilibrium solutions of the dynamical system (1) as approximated to the first order using equation (4), for detuning parameter range [-4, 9] for the forcing amplitude $f_0 = 4.0$. Where A is the response amplitude and ψ is the phase of the response.

It is seen that the equilibrium solutions obtained from equation (12) presented in table 1 has some of the solutions with no physical meaning. The response amplitude must be real and positive, as a result the physically realizable solutions are separated in bold form and is underlined in the table; which are selected and subjected for further investigation as given in Table 2.

Table 2

σ	Equilibrium points EP	Eigenvalues
-4	A→0.326043,ψ→-0.1855	-0.6+3.07303 ™,-0.6-3.07303 ™
-3	A→0.404363,ψ→-0.230785	-0.6+2.49859 ™,-0.6-2.49859 ™
-2	A→0.511385,ψ→-0.293478	-0.6+2.01733 ™,-0.6-2.01733 ™
-1	A→0.646727,ψ→-0.37454	-0.6+1.66604 ™,-0.6-1.66604 ™
0	A→0.79883,ψ→-0.468879	-0.6+1.4453 ™,-0.6-1.4453 ™
1	A→0.953328,ψ→-0.569586	-0.6+1.31672 ™,-0.6-1.31672 ™
2	A→1.1017,ψ→-0.672851	-0.600002+1.23695 ™,-0.600002-1.23695 ™
3	A→1.24101,ψ→-0.77823	-0.599999+1.17599 ™,-0.599999-1.17599 ™
3.4	A→0.481457,ψ→ -2.86575	-0.600007+1.83869 ™,-0.600007-1.83869 ™
	A→0.917159,ψ→ -2.59612	-0.976059,-0.223939
	A→1.29408,ψ→ -0.821312	-0.599999+1.15242 ™,-0.599999-1.15242 ™
3.5	A→0.45832,ψ→ -2.87933	-0.600002+1.96287 ™,-0.600002-1.96287 ™
	A→0.953851,ψ→ -2.57166	-1.10463,-0.0953647
	A→1.30711,ψ→ -0.832199	-0.6+1.14641 ™,-0.6-1.14641 ™
3.6	A→0.438536,ψ→ -2.8909	-0.600003+2.07705 ™,-0.600003-2.07705 ™
	A→0.98711,ψ→ -2.54914	-1.1891,-0.0109094
	A→1.32005,ψ→ -0.843139	-0.600001+1.14032 ™,-0.600001-1.14032 ™

4	A→0.37902,ψ→-2.92551	-0.599998+2.4785 ™,-0.599998-2.4785 ™
	A→1.09974,ψ→-2.47016	-1.37371,0.173707
	A→1.37091,ψ→-0.887515	-0.600001+1.11494 ™,-0.600001-1.11494 ™
5	A→0.291201,ψ→-2.97611	-0.600001+3.32826 ™,-0.600001-3.32826 ™
	A→1.31531,ψ→-2.30248	-1.53243,0.33243
	A→1.4919,ψ→-1.0046	-0.600001+1.03904 ™,-0.600001-1.03904 ™
6	A→0.239179,ψ→-3.00588	-0.599989+4.10211 ™,-0.599989-4.10211 ™
	A→1.48916,ψ→-2.13988	-1.54976,0.349759
	A→1.60435,ψ→-1.13743	-0.6+0.92854 ™,-0.6-0.92854 ™
7	A→0.203672,ψ→-3.02612	-0.600007+4.8473 ™,-0.600007-4.8473 ™
	A→1.64323,ψ→-1.94839	-1.45985,0.259848
	A→1.70738,ψ→-1.30867	-0.6+0.728374 ™,-0.6-0.728374 ™
8	A→0.177616,ψ→-3.04095	-0.599992+5.5785 ™,-0.599992-5.5785 ™
9	A→0.157585,ψ→-3.05233	-0.600003+6.30196 ™,-0.600003-6.30196 ™

The equilibrium solutions of the dynamical system (1) as approximated to the first order using equation (4), for detuning parameter range [-4, 9] for the forcing amplitude $f_0 = 4.0$. For A and ψ physically realizable solutions.

From the eigenvalues of the equilibrium solutions found in the Table 2 it is seen that the equilibrium points are two types. For detuning parameter σ from [0-3] the eigenvalues are complex conjugate pairs with a negative real part indicating that the fixed points are stable in the form of incoming spirals. For σ between 3.4 and 3.8 the EQS split into three with two incoming spirals and a stable node while for σ between 4 and 7 (i) The stable node losses its stability through a pitchfork bifurcation into a saddle EQS, 2 real eigenvalues with opposite signs and the two incoming spirals continue to remain stable. For σ > 7.0 the saddle EQS disappear through a reverse pitchfork bifurcation and only one EQS continues whose eigenvalues are complex conjugate pairs with a negative real part i.e. stable incoming spirals continue to exist.

3.2 Principal Resonance

Equation (13) is used to plot the FRC for the detuning parameter σ values ranging [-4, 9] for three forcing amplitude values f_0 = 3.2, 4.0 and 6.0

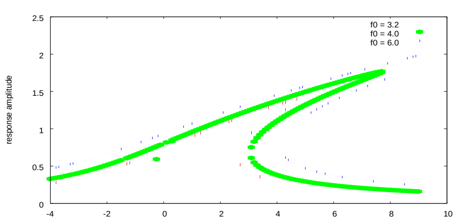

Fig 1 FRC for the detuning parameter σ values ranging [-5, 9] for three forcing amplitude values

Figure 1 is showing a clear nonlinearity for large values of the detuning parameter, from the figure it can also be seen that for detuning parameter values less than 2.25 only one equilibrium solution are found for all the 3 forcing amplitudes considered, for detuning parameter σ values in the range [3.0-7.0] three equilibrium solutions are found for the case of $f_0 = 4.0$, in agreement with the EQS found in Table 2, while for detuning parameter σ > 7.0 another region of one equilibrium reappears as also found in Table 2.

3.3 Stability analysis

From equations (13) and (21) superposed on the same frame to show the regions where the equilibrium solutions are stable and where they are unstable.

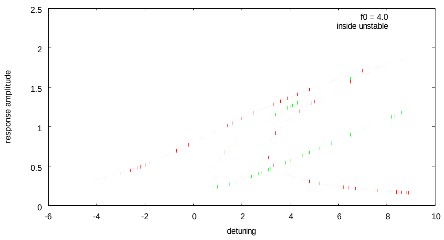

Fig 2.0 The FRC for f_0 = 4.0 and the stability boundary curve.

4.0 Concluding Remarks

In this paper the dynamical behavior of the damped driven Morse oscillator was considered. The method of multiple scales was used to obtain the approximate analytic solution, showing regions of incoming spiral solutions which are stable for small detuning parameter values up to 3.0 for the forcing amplitude of 4.0 at the detuning parameter value of [3.4 -3.8] a pitchfork bifurcation occurs, solutions splitting into three with two stable spirals and a stable node in the middle and detuning parameter value of 4.0 the stable node loses its stability and turns into a saddle equilibrium point up to a detuning parameter value of 7.0. For detuning parameter values greater than 7.0 a reverse pitchfork bifurcation occur where the response amplitude suddenly drops down to again another stable spiral, these results show that the dynamical behavior of the system subjected to the Morse potential type has only some allowed values of position and energy levels where system response amplitudes are stable.

Indicating that for the forced damped driven oscillator at every energy level the stability region is approximately simple harmonic (SH) for small values of the oscillating frequency. As the oscillating frequency approaches the dissociation limit the system loses stability till the oscillating frequency reaches the stability region of the next energy level where it again becomes approximately SH. Showing alternate switching of regions of stability and instability.

5.0 Acknowledgment

UAM immensely acknowledges the useful comments and suggestions offered by Prof. John O. A. Idiodi of the Department of Physics, University of Benin. Nigeria

6.0 References

[1] Christoffel K.M. and Bowman J.M. J. Phys. Chem. **85** 2159(1981)

[2] Gray S.K. Chem. Phys. 75, 67 (1983)

[3] Noid D.W. and Stine J.R. Chem. Phys. Lett. **65**, 153 (1979)

[4] Brown R.C. and Wyatt R.E. Phys. Rev Lett. **57**. 1 (1986)

[5] Dardi P.S. and Gray S.K. J. Chem. Phys. **77**, 1345 (1982)

[6] Goggin M.E. and Milloni, Phys. Rev. A. **37**,796 (1988)

[7] Gray S.K. Chem. Phys. 83 125 (1984)

[8] Walker R.B. and Presston R.K. J. Chem Phys. **67**, 2017 (1977)

[9] Tung M. Eschennazi E. and Yuan J.M. Chem. Phys. Lett. 115, 405 (1985)

[10] Tung M. and Yuan J.M. Phys. Rev A **36**, 4463 (1987)

[11] Lie G.C. and Yuan J.M. J. Chem. Phys.**84** 5486 (1986)

[12] Knop W. and Lauterborn W J. Chem. Phys. **93**, 3950 (1990)

[13] Lakshmanan M. and Murali K. Chaos in Nonlinear Oscillators: Controlling and Synchronization. World Scientific, Singapore, 1996.

[14] Wei X, Ruihong L and Shuang L Nonlinear Dynamics **46** 211-221(2006)

[15] Marte, U.A., Umar, U.A. and Hassan, M. Transactions of the Nigerian Association of Mathematical Physics **7**(2018) pp 23 - 26

[16] Nayfeh, A.H. *Perturbation Methods*; Wiley and Sons: New York, NY, USA, 1973.

[17] Nayfeh, A.H.; Mook, D.T. *Nonlinear Oscillations*; Wiley and Sons: New York, NY, USA, 1979.