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ISOMETRIC LINES OF $\mathfrak{Z}_d \times \mathfrak{Z}_d$

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Abstract

Lines of $\mathfrak{Z}_d \times \mathfrak{Z}_d$ were discussed. A comparison was made between each lines in $\mathfrak{G}_d$, as a result, for d a prime, lines with $\mathfrak{L}$ $(a\mu, a\lambda)$ is isomorphic to $\mathfrak{L}$ (μ, λ) for μ, λ a coprime.

Keywords: Isometric lines, finite geometries

I. INTRODUCTION

Over the years, Finite quantum systems with variables in $\mathfrak{Z}_d$ had received a lot of attention with particular interest on mutually unbiased bases in Finite Hilbert space [1-7]. In the past, a connection between finite geometry and finite dimensional Hilbert spaces has been established. Recently, attention is focused on finite geometry especially on near-linear finite geometry [8-13]. For instance, it was discussed in [7-8] that there exists a duality between lines non-near linear finite geometry and weak mutually unbiased bases in finite quantum systems.

In this work, our attention is focused on one thing; lines of near-linear finite geometry. Our finding is that, there are many lines in a near-linear finite geometry out which there exists a bijection between points of these lines and as a result leads to partition of these lines distinctly.

The whole work is partitioned into six sections thus: preliminary of this work is discussed in section II. Here, we define the notation we used in the discourse. The concept of finite geometry along with the review of previous work is discussed in Section III. In section IV, we discuss isometric lines of $\mathfrak{Z}_d \times \mathfrak{Z}_d$.

Symplectic on $\mathfrak{Z}_d \times \mathfrak{Z}_d$with numerical example is discussed in section V. The conclusion of our work is in section VI.

Definition: Let X and Y be two sets, a map $\pi: X \to Y$ is an isometry if and only if there exists a bijection between X and Y. in this article, X and Y are finite sets of integer modulo in d. It is denoted by $\mathfrak{Z}_d$.

II. PRELIMINARIES

(i) A field of integer modulo d is denoted by $\mathfrak{Z}_d$.
(ii) The number of invertible element is represented by $\varphi(d)$ where

$$\varphi(d) = d \prod \left(1 - \frac{1}{d}\right); \ d = \text{prime} \tag{1}$$

(iii) Let $\psi(d)$ represents the Dedekind psi function, it is defined as;

$$\psi(d) = d \prod \left(1 + \frac{1}{d}\right); \ d = \text{prime} \tag{2}$$

(iv) GCD(μ, λ) denotes the greatest common divisor of two elements μ and λ in $\mathfrak{Z}_d$.
(v) The notation a ≅b means a maps b.

III. FINITE GEOMETRY $\mathfrak{G}_d = \mathfrak{Z}_d \times \mathfrak{Z}_d$

A finite geometry is defined as the pair

$$\mathfrak{G}_d = (\text{Pd; Ld}) \tag{3}$$

where Pd and Ld denote the set of points and lines in G_d respectively

$$\text{Pd} = \{(m, n) \mid m, n \epsilon\, \mathfrak{Z}_d\} \tag{4}$$

and a line through the origin is defined as

$$\mathfrak{L}\,(\mu, \lambda) = \{(k\mu; k\lambda) \mid \mu, \lambda \,\epsilon\, \mathfrak{Z}_d\}, k \,\epsilon\, \mathfrak{Z}_d \tag{5}$$

denotes set of lines.

In this work we study Near-linear finite geometry. In it two lines for example intersect only at the origin. This is linked to the fact that $\mathfrak{Z}_d$ is a field of integer modulo d and all the lines in this work are through the origin. We confirm the following proposition in our previous work of [6].

Proposition III.1.

(i) $\mathfrak{L}\,(\mu, \lambda) = \mathfrak{L}\,(p\mu, p\lambda), \forall\, p \in \mathfrak{Z}_d^*$ (6)

and

$\mathfrak{L}\,(p\mu, p\lambda) \prec \mathfrak{L}\,(\mu, \lambda), \forall\, p \in \mathfrak{Z}_d - \mathfrak{Z}_d^*$ (7)

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A line $\mathfrak{L}\,(p\mu, p\lambda)$ is a maximal line in $\mathfrak{G}_d$ if GCD$(\mu,\lambda) \in \mathfrak{Z}_d^*$, and $\mathfrak{L}\,(\mu,\lambda)$ is a subline in $\mathfrak{G}_d$ if GCD$(\mu,\lambda) \in \mathfrak{Z}_d - \mathfrak{Z}_d^*$
(ii) There are $\psi(d)$ maximal lines in finite geometry $\mathfrak{G}_d$ with exactly d points each.

IV. ISOMETRIC LINES OF $\mathfrak{Z}_d \times \mathfrak{Z}_d$

Suppose we define a finite geometry as expressed earlier in equation (3); If $p \in \mathfrak{Z}_d^*$ then
$\mathfrak{L}\,(\mu,\lambda) \cong \mathfrak{L}\,(p\mu, p\lambda)$ (8)
Hence $\mathfrak{L}\,(\mu,\lambda)$ and $\mathfrak{L}\,(p\mu, p\lambda)$ are isometric to each other.
Proposition IV.1. If $\mathfrak{L}\,(\mu,\lambda)$ is a line of near-linear finite geometry $\mathfrak{G}_d$ where d is a prime integer.
then $\mathfrak{L}\,(\mu,\lambda)$ is isometric to $\mathfrak{L}\,(q\mu, q\lambda)\ \forall q \in \mathfrak{Z}_d^*$ (9)
Proof. Let $a, b \in \mathfrak{Z}_d$ where d is a prime integer. The proof is self evident since $\mathfrak{Z}_d$ is a field, it means every non-zero integer has an inverse and p in equation (9) is the additive generator.
Hence complete the proof.

A. Example

The concept of isometry in $\mathfrak{G}_d = \mathfrak{Z}_d \times \mathfrak{Z}_d$ is shown below as illustration for d = 5: using equation (3).

$\mathfrak{L}$ (0;1) ≅ $\mathfrak{L}$ (0; 2) ≅ $\mathfrak{L}$ (0; 3) ≅ $\mathfrak{L}$ (0; 4) (10)
$\mathfrak{L}$ (1; 0) ≅ $\mathfrak{L}$ (2; 0) ≅ $\mathfrak{L}$ (3;0) ≅ $\mathfrak{L}$ (4; 0) (11)
$\mathfrak{L}$ (1; 1) ≅ $\mathfrak{L}$ (2; 2) ≅ $\mathfrak{L}$ (3;3) ≅ $\mathfrak{L}$ (4; 4) (12)
$\mathfrak{L}$ (1; 2) ≅ $\mathfrak{L}$ (2; 4) ≅ $\mathfrak{L}$ (3;1) ≅ $\mathfrak{L}$ (4; 3) (13)
$\mathfrak{L}$ (1; 3) ≅ $\mathfrak{L}$ (2; 1) ≅ $\mathfrak{L}$ (3; 4) ≅ $\mathfrak{L}$ (4; 2) (14)
$\mathfrak{L}$ (1; 4) ≅ $\mathfrak{L}$ (2; 3) ≅ $\mathfrak{L}$ (3; 2) ≅ $\mathfrak{L}$ (4; 1) (15)

where from equation (5) a line $\mathfrak{L}$ (1; 2) is defined as set of points
$\mathfrak{L}$ (1; 2) = {(0, 0); (1, 2), (2, 4), (3,1), (4,3)} (16)
since 2 ∈ $\mathfrak{Z}_5^*$; then $\mathfrak{L}$ (2; 4) is
$\mathfrak{L}$ (2; 4) = {(0,0),(2,4),(4,3),(1,2),(3,1)} (17)

V. SYMPLECTIC GROUP ON $\mathfrak{G}_d$

Let
$\mathfrak{M}\,(\sigma,\tau \mid \lambda,\mu)$ (18)
represents a unitary transformation where
$\mathfrak{M}\,(\sigma,\tau \mid \lambda,\mu) \equiv \begin{pmatrix} \sigma & \tau \\ \lambda & \mu \end{pmatrix}$
$|\mathfrak{M}| = |\sigma\mu - \tau\lambda| = 1 \ (mod\ d)$, where $\mu,\sigma,\tau,\ \lambda \in \mathfrak{Z}_d$, $\mathfrak{M}$ form a group called Symplectic group Sp(2, $\mathfrak{Z}_d$) group.
Acting $\mathfrak{M}$ on all points of line $\mathfrak{L}$ (x, y) in $\mathfrak{Z}_d \times \mathfrak{Z}_d$ generates all the points of the line $\mathfrak{L}$ (σx+ τ y, λx+ μ y).
This is expressed in this work as $\mathfrak{M}\,(\sigma,\tau \mid \lambda,\mu)$ $\mathfrak{L}$ (x; y).
Example: Suppose d = 5 and we substitute 2,1,1, and 1 for σ,τ,λ,μ respectively into $\mathfrak{M}$ and act $\mathfrak{M}$ on $\mathfrak{L}$ (x, y) where x = 0; y∈ $\mathfrak{Z}_5$ it yields $\mathfrak{L}$ (1,1) in equation (12). Suppose d is a prime, acting $\mathfrak{M}$ (0, 1$\mid -1$;ℵ) on the line $\mathfrak{L}$ (0, 1), we obtain all the lines through the origin. For
ℵ = 0,1,…, d-1→ ■(ℵ)=$\mathfrak{M}$ (0, 1$\mid -1$,ℵ) $\mathfrak{L}$ (0,1) = $\mathfrak{L}$ (1, ℵ), (19)
In this work, we fix a rule that if ℵ =-1, $\mathfrak{M}$ (0, 1$\mid -1$;ℵ) is replaced by $\mathfrak{M}$ (I, 0$\mid$0,I).
If we substitute the value of ℵ = -1,… d, we obtain all the lines in $\mathfrak{G}_d$ we obtain all the points in the line as shown in equations(10) - (15) for a = 0; b ∈ $\mathfrak{Z}_d$.

VI. CONCLUSION

This article focuses on near-linear finite geometry. Lines of $\mathfrak{Z}_d \times \mathfrak{Z}_d$ was carefully examined. It was found that although lines may have different naming system but looking at their internal structure they can have identical points. For d a prime, $\mathfrak{Z}_d$ is a field of integer modulo d and all lines of $\mathfrak{Z}_d \times \mathfrak{Z}_d$ partitioned into $\psi(d)$ distinct lines. We discover that on each partition, there exists $\varphi(d)$ lines. Having critically examined each partition, we confirm a one-to-one correspondence within each representation of the lines and other lines in the partition. Furthermore, an action of the Symplectic matrix on each lines of $\mathfrak{Z}_d \times \mathfrak{Z}_d$ also produced other lines in the same representation. Hence they form complete isometric lines in the geometry.

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