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A PURSUIT DIFFERENTIAL GAME WITH DIFFERENT CONSTRAINTS ON CONTROLS OF THE PLAYERS

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Abstract

We study a pursuit differential game problem with finite number pursuers and one evader in the space l_2. Pursuers' motions are described by first order differential equations and that of the evader by second order differential equation. Control functions of the pursuer and evader are subject to integral and geometric constraints respectively. Duration of the game is denoted by the positive number θ. Pursuit is said to be completed if the geometric positions of a pursuer and evader coincide. We prove theorems, each of which provides a condition for completion of pursuit. Consequently, strategies of the pursuers that ensure completion of pursuit are constructed.

Keywords: Pursuit, integral constraint, geometric constraint, Hilbert space.
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1. INTRODUCTION

There are numerous papers on differential game problems by many researchers. The papers [1 - 19] and some references therein, are examples.

In many studies of pursuit and evasion differential game problems, motions of the two players (i.e. pursuer and evader) are explicitly stated and are considered to be differential equations of the same order. For example, in the papers [2], [4-6], [10], and [12-13], motion of each of the player is considered to obey first order differential equation. In other studies such as [1], [3], [9] and [14], players' motions are described by second order differential equations. Whereas in [19] motion of all the two players obey higher order differential equation.

Specifically, there are various studies of pursuit differential game problems in which motions of the layers are described by the first order differential equations of the form

$$\dot{x} = u(t),\ x(0) = x_0. \tag{1.1}$$

The studies in the references [2], [4-6], [10] and [12] are examples. On the other hand, in some studies such as [1], [3] and [9] motion of each player is described by the following second order differential equation:

$$\ddot{x} = u(t),\ x(0) = x_0 \tag{1.2}$$

In view of this, it will be appropriate to study a differential game in which motion of one player described by the equation of the form (1.1) and that of another player be described by the equation (1.2).

In this piece of research, we study pursuit differential game problem in a Hilbert space l_2, where motions of the pursuers and evader described by first and second order differential equations respectively. Control functions of the pursuers are subject to integral constrains. Whereas, geometric constraint is imposed on the control function of the evader.

2. STATEMENT OF THE PROBLEM

Consider the pursuit differential game problem where motions of pursuers P_i and one evader are described by the equations

$$\begin{cases} P_i : \dot{x} = u_i(t), x_i(0) = x_{0i}, i = 1, 2, ...m, \\ E : \ddot{y} = v(t), \dot{y}(0) = y^1, y(0) = y^0, \end{cases} \tag{2.1}$$

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where x_i, x_{i0}, u_i, y, y^0, y^1, $v \in l_2$, $u_i = (u_{i1}, u_{i2}, \dots)$ is a control parameter of the pursuer P_i and $v = (v_1, v_2, \dots)$ is that of the evader E. In what follows (unless stated otherwise) we consider that $x_{0i} \neq y_0$, for all $i \in I = \{1, 2, 3, \dots, m\}$.

Definition 2.1. *A function* $u(t) = (u_1(t), u_2(t), u_3(t)\dots)$ with measurable coordinates such that

$$\int_0^\infty \| \mathrm{v}(\mathrm{t}) \|^2 \, ds \leq \rho_i^2, \mathrm{t} \geq 0, \tag{2.2}$$

is called admissible control of the pursuer.

Definition 2.2. *A function* $\mathrm{v}(t) = (\mathrm{v}_1(\mathrm{t}), \mathrm{v}_2(\mathrm{t}), \mathrm{v}_3(\mathrm{t}), \dots)$ measurable coordinates such that

$$\| \mathrm{v}(\mathrm{t}) \| \leq \sigma, \mathrm{t} \geq 0, \tag{2.3}$$

is called admissible control of the evader.

Definition 2.3. *A function* $U_i(t, x, y, v), U_i : [0, \infty) \times l_2 \times l_2 \times l_2 \to l_2$, *such that the system*

$$\begin{cases} \dot{x} = U_i(t, x, y, v(t)), x_i(0) = x_{0i}, \\ \ddot{y} = v(t), \dot{y}(0) = y^1, y(0) = y^0, \end{cases}$$

has a unique solution $(x(\cdot), y(\cdot))$ with continuous components $x(\cdot), y(\cdot)$ in l_2 for an arbitrary admissible control v= v(t), $0 \leq t \leq \theta$, of the evader E is called strategy of the pursuer P_i if along this solution

$$\int_0^\infty \| U_i(s, x(s), y(s), v(s)) \|^2 \, ds \leq \rho_i^2.$$

Definition 2.4 . Pursuit is said to be completed in the game (2.1) – (2.3) if there exists pursuer P_i with strategy U_i that ensure the quality $x(\theta) = y(\theta)$ for some $\theta > 0$

The problem is to find sufficient conditions for completion of pursuit in this game problem.

3. RESULTS

It is well known that (see, for example [1], [3] and [9]) the state $y(\theta)$ of the second system in (2.1) at a specified time θ can be found from the following equation

$$\dot{y} = (\theta - t)v(t), y(0) = y_0 = y^1\theta.$$

Therefore, to show that $x_i(\theta) = y(\theta)$ at a the time θ in the game (2.1)-(2.3) we consider the system

$$\begin{cases} P_I : \dot{x}_i = u_i(t), x_i(0) = x_{0i}, i \in I, \\ E : \dot{y} = (\theta - t)v(t), y(0) = y^0. \end{cases} \tag{3.1}$$

If the pursuer P_i and evader E use admissible controls $u_i(t) = (u_{i1}(t), u_{i2}(t), u_{i3}(t), \dots)$ and $v(t) = (v_1(t), v_2(t), v_3(t), \dots)$ respectively, then by (3.1) their corresponding motions is given by

$$x_i(t) = x_{i1}(t), x_{i2}(t), x_{i3}(t), \dots), \quad y(t) = (y_1(t), y_2(t), y_3(t), \dots),$$

where

$$x_{ik}(t) = x_{i0k} + \int_0^t u_{ik}(s) ds$$

$$y_k(t) = y_{0k} + \int_0^t (\theta - s) v_k(s) ds.$$

We denote that $\beta = \dfrac{\sigma}{\sqrt{3}}; \gamma = \max_{i \in I} \| y_0 - x_{0i} \|$ and

$$\Omega = \bigcup_{i \in I} \left\{ z \grave{o} l_2 : 2\langle z, y_0, -x_{0i} \rangle \leq \left(\rho_i^2 - \sigma^2 \frac{\theta^3}{3} \right) \theta + \| y_0 \|^2 - \| x_{0i} \|^2 \right\}$$

Each of the theorems below gives a sufficient condition for completion of pursuit in the game described by (2.1)-(2.3).

Theorem 3.1 If there exists $i \in I$ such that $x_{0i} = y_0$ for $i \in I$, then pursuit can be completed in the game (2.1)-(2.3) at time $\theta = \left(\dfrac{\rho_i}{\beta} \right)^{\frac{2}{5}}$.

Proof: We construct the strategy of the pursuer P_i follows:

$$U_i(t) = \begin{cases} (\theta - t)v(t), 0 \le t \le \theta, \\ 0, \qquad t > \theta. \end{cases} \tag{3.2}$$

The admissibility of this strategy can be shown as follows:

$$\int_0^\infty ||\theta - s)v(s)||^2 ds = \int_0^\theta ||\theta - s)^2 ||v(s)||^2 ds$$

$$= \sigma^2 \int_0^\theta (\theta - s)^2 ds$$

$$= \frac{1}{3}\sigma^2\theta^3 = \frac{1}{3}\sigma^2\left(\frac{\rho_i}{\beta}\right)^2$$

$$= \frac{1}{3}\sigma^2\left(\frac{\sqrt{3\rho_i}}{\sigma}\right)^2 \rho_i^2.$$

We now show that the pursuer's strategy (3.2) ensures completion of pursuit in the time θ.

$$x_i(\theta) = x_{0i} + \int_0^\theta (\theta - s)v(s)ds = y_0 + \int_0^\theta \theta - s)v(s)ds = y(\theta).\cdot$$

Theorem 3.2. If there exists $i \in I$ such that $\rho_i \ge 1 + \gamma\beta^{\frac{1}{3}}$, then pursuit can be completed in the game (2.1)-(2.3) at the time $\theta = \beta^{-\frac{2}{3}}$.

Proof: let the strategy of the *ith* pursuer be

$$U_i(t) = \begin{cases} \frac{y_0 - x_{0i}}{\theta} + (\theta - t)v(t), & 0 \le t \le \theta, \\ 0, & t > \theta. \end{cases}$$

The admissibility of this strategy can be shown as follows:

$$\left(\int_0^\infty \|\frac{y_0 - x_{0i}}{\theta} + (\theta - s)v(s)\|^2 ds\right)^{\frac{1}{2}} = \left(\int_0^\theta \|\frac{y_0 - x_{0i}}{\theta} + (\theta - s)v(s)\|^2 ds,\right)^{\frac{1}{2}}$$

$$\le \left(\int_0^\theta \frac{\| y_0 - x_{0i}\|^2}{\theta^2} ds\right)^{\frac{1}{2}} + \left(\int_0^\theta (\theta - s)^2 \|v(s)\|^2 ds\right)^{\frac{1}{2}}$$

$$\le \frac{\| y_0 - x_{0i}\|}{\theta}\sqrt{\theta} + \sigma\left(\int_0^\theta (\theta - s)^2 ds\right)^{\frac{1}{2}}$$

$$= \frac{\| y_0 - x_{0i}\|}{\sqrt{\theta}} + \frac{\sigma}{\sqrt{3}}\theta^{\frac{3}{2}}$$

$$\le \gamma\beta^{\frac{1}{3}} + \frac{\sigma}{\sqrt{3}}\beta^{-1}$$

$$\le \gamma\beta^{\frac{1}{3}} + 1 \le \rho_{i.}$$

We now show that if the pursuer p_i uses the strategy (3.2), then pursuit can be completed indeed,

$$x_i(\theta) = x_{0i} + \int_0^\theta \left(\frac{y_0 - x_{0i}}{\theta} + (\theta - t)v(s)\right)ds = y_0 + \int_0^\theta (\theta - s)v(s)ds = y(\theta).$$

Theorem 3.3. If there exists $i \in I$, such that $\rho_i > \gamma$ and $\sigma \le \frac{8}{3}(\rho_i - \gamma)$, the pursuit can be completed in the game (2.1)-(2.3) at the time $\theta = \frac{3}{4}$.

Proof: It is not difficult to see that

$$\left\langle y_0 - x_{0i}, \int_0^\theta (\theta - s)v(s)ds \right\rangle \le \frac{\theta^2}{2}\gamma\sigma \tag{3.4}$$

Let the pursuer P_i, uses the strategy (3.3). It is obvious that pursuit can be completed in the game (2.1)-(2.3). It remains to show the admissibility of this strategy. Indeed, using inequality (3.4) we have

$$\int_0^\theta \| \frac{y_0 - x_{0i}}{\theta} + (\theta - s)v(s) \|^2 \, ds = \int_0^\theta \frac{y_0 - x_{0i}}{\theta} + (\theta - s)v(s) \|^2 \, ds$$
$$= \int_0^\theta \| \frac{y_0 - x_{0i}}{\theta} \|^2 \, ds + \frac{2}{\theta}\left\langle y_0 - x_{0i}, \int_0^\theta (\theta - s)v(s)ds \right\rangle$$
$$+ \int_0^\theta \| (\theta - s)v(s) \|^2 \, ds$$
$$\le \| y_0 - x_{0i} \|^2 + \frac{2}{\theta}\left\langle y_0 - x_{0i}, \int_0^\theta (\theta - s)v(s)ds \right\rangle + \sigma^2 \frac{\theta^3}{3}$$
$$\le \gamma^2 + \frac{2}{\theta}\left(\frac{\theta^2}{2}\gamma\sigma\right) + \sigma^2 \frac{\theta^3}{3}$$
$$\le \gamma^2 + \theta\gamma \frac{8}{3}(\rho_i - \gamma) + \left(\frac{8}{3}(\rho_i - \gamma)\right)^2 \frac{\theta^3}{3}$$
$$= \gamma^2 + \frac{3}{4}\gamma \frac{8}{3}(\rho_i - \gamma) + \left(\frac{8}{3}(\rho_i - \gamma)\right)^2 \frac{3^2}{4^3}$$
$$= \gamma^2 + 2\gamma(\rho_i - \gamma) + (\rho_i - \gamma)^2 \, \rho_i^2.$$

Theorem 3.4. If $y(\theta) \in \Omega$, then pursuit can be completed in the game (2.1)-(2.3).

Proof: Let the pursuer P_i uses the strategy (3.3). From the prove of theorem 3.2, we have seen that this strategy ensures completion of pursuit. This means that for the proof of this theorem it remains to show the admissibility of this strategy using the condition $y(\theta) \in \Omega$ of the theorem.

It is clear to see that the inclusion $y(\theta) \in \Omega$, implies that there exists $i \in I$ such that the following inequality holds.

$$2\left\langle y_0 - x_{0i}, \int_0^\theta (\theta - s)v(s)ds \right\rangle \le \left(\rho_i^2 - \sigma^2 \frac{\theta^3}{3}\right)\theta - \| y_0 - x_{0i} \|^2. \qquad \textbf{(3.5)}$$

By using the inequality (3.5), we can show the admissibility of strategy (3.2) as follows:

$$\int_0^\infty \| \frac{y_0 - x_{0i}}{\theta} + (\theta - s)v(s) \|^2 \, ds = \int_0^\theta \| \frac{y_0 - x_{0i}}{\theta} \|^2 \, ds + \frac{2}{\theta}\left\langle y_{0i} - x_0, \int_0^\theta (\theta - s)v(s)ds \right\rangle$$
$$+ \int_0^\theta (\theta - s)v(s) \|^2 \, ds$$
$$\le \frac{\| y_0 - x_{0i} \|^2}{\theta}\left(\rho_i^2 - \sigma^2 \frac{\theta^3}{3}\right) - \frac{\| y_0 - x_{0i} \|^2}{\theta} + \sigma^2 \frac{\theta^3}{3}$$
$$= \rho_i^2.$$

4. CONCLUSION

We have studied a pursuit problem in which the motions of the pursuers and evader are described by first and second order differential equations. That is a game problem where the two players are considered to have different behavior in their motions. This is in contrast to what is obtainable in the vast related literature.

In each of the four theorems stated and proved, we provide the time for completion of pursuit. It is important to observe that we provide three different sufficient conditions, each for completion of pursuit, when the pursuer uses the same strategy in the game problem. For further research, value of the game and optimality of the pursuit times can also be investigated.

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