Transactions of the Nigerian Association of Mathematical Physics
Volume 8, (January, 2019), p11 – 16
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ON THE CONSTRUCTION OF MAXIMA AND MINIMA OF CONVEX FUNCTIONS IN INFINITE DIMENSIONAL SPACES

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Abstract

In this paper, the maxima and minima of convex functions in infinite dimensional spaces were considered. The necessary and sufficient conditions for maxima and minima were stated. Some examples were used to illustrate when a function attains its extreme values. Then the relevant theorems were reviewed and proofs of the results given, which extends some results in literature.

***Keywords*:** Maxima, Minima, Convex functions, infinite dimensional Spaces

MSC: C-30C80, 74G65, 58Exx

1. Introduction

Consider the minimization or maximization of a convex function
$\underset{v\in V}{Max} f(v) = f(v)$ or $\underset{x\in K}{Min} f(x) = f(\bar{x})$
Where V is a normed linear space and $f : V \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued function $f : X \to \mathbb{R} \cup \{+\infty\}$.
Many researchers have worked on the constructions of maximizations and minimizations of convex functions in infinite dimensional spaces with resounding results see for instance,[1-8].

Taha[**1**] discussed that an extreme point of a function $f(x)$ defines either a maximum or a minimum of the function.He said that mathematically a point $x_0 = (x_1, \ldots, x_j, \ldots, x_n)$ *is a maximum if*
$f(x_0 + h) \le f(x_0)$ for all
h = $(h_1, \ldots, h_j, \ldots, h_n)$ such that $|h_j|$ is sufficiently small for all j. In other words x_0 is a maximum if the value of f at every point in the neighborhood of x_0 does not exceed $f(x_0)$ .In a similar manner x_0 is a minimum if for h as defined
$f(x_0 + h) \ge f(x_0)$
When a point is maximum of all the maximum points of a function, it is called a global maximum while others are local maximum and verse versa in minimum values.

Theorem 1.1 The necessary condition for x_0 to be an extreme point of $f(x)$ is that
$\nabla f(x_0) = 0$
This means that at extreme points, $\nabla f(x_0) = 0$ must vanish or that the gradient vector must be null.

Theorem 1.2 A sufficient condition for a stationary point x_0 to be extremum is that the Hessian matrix H evaluated at x_0 is
(i) Positive definite when x_0 is a minimum point.
(ii) Negative definite when x_0 is a maximum point.

Luenberger [**2**] stated the three classic results concerning minimization or maximization of convex functions.

Theorem 1.3 Let f be a convex function defined on the convex set Ω. Then the set Γ where f achieves its minimum is convex and any relative minimum is a global minimum.

Theorem 1.4 (Eberlien Smu'lyan) states that " A Banach Space E is reflexive if and only if every norm bounded sequence in E has a subsequence which converges weakly to an element of E".

Awunganyi [**3**] states that, let x be a vector in a Hilbert space H and let M be a closed convex subset of H. Then there is a unique vector $m_0 \in M$ such that

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$\| x - m_0 \| \leq \| x - m \|$ for all $m \in M$.
Furthermore a necessary and sufficient condition that m_0 to be a unique mini mizing vector is that $(x - m_0, m - m_0) \geq 0$,for all $m \in M$.
Chidume [**4**] stated the Weierstrass theorem as: Let $D \subseteq \mathbb{R}^n$ be a compact set (closed and bounded) and $f : D \to \mathbb{R}$ be a continuous function. Then f attains a global maximum or a global minimum on D i.e $\exists\ x_1$ and x_2 such that
$f(x_1) \geq f(x) \geq f(x_2)\ \forall\ x \in D$
Observe that from these theorems, the function achieves its maximum and minimum on the given domain. To solve an optimization problem is to find a global minimizer of f in a normed space.
Luenberger [**5**], Studied the extended familiar technique of minimizing a function of a single variable by ordinary calculus to a similar technique based on more general differentials. In this way we obtained analogues of the classical necessary condition for local extremas and in a later section, the lagrange technique for constrained extrema. Let f be a real-valued functional defined on a subset Ω of a normed space X. A point $x_0 \in \Omega$ is said to be a relative minimum of f on Ω if there is an open sphere N containing x_0 such that $f(x_0) \leq f(x)$ for all $x \in \Omega \cap N$. The point x_0 is said to be strict relative minimum of f on Ω if $f(x_0) < f(x)$ for all $x \neq x_0$, $x \in \Omega \cap N$. Relative maxima are defined similarly.
Peypouquet [**6**] discussed minimizers of convex functions and said that an extended real valued function $f: X \to \mathbb{R} \cup \{+\infty\}$ defined on a vector space X is convex if
$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ for each $x, y \in$ dom(f) and $\lambda \in (0,1)$. The inequality above holds trivially if $\lambda \in (0,1)$ or if either x *or* y are not in dom(f) .If the inequality is strict whenever $x \neq y$ and $\lambda \in (0,1)$,we say $f$is strictly convex.
Blume [**7**] Firms maximize profit and consumers maximize preferences. Also, firms minimize losses and risks which is the core of micro economics. Since the study of maxima and minima form the bases of optimization, when extended to convex functions in infinite dimensional spaces, the solution will lead to a long term solution to the cost incurred by business enterprises during the course of their production to the time of consumption. The mathematical solution of risks and losses will be the lowest minimum and that of profit will be the highest. Therefore, the maxima and minima of convex functions in infinite dimensional spaces will be beneficial to firms whose aim is to make profit and minimize cost.
The study of minimization of convex functions in infinite dimensional spaces is significant to a scientist for instance, a pharmacist whose mixture of drugs gave the best and long term solution to certain ailments. To a structural engineer whose mixture of materials brought about solid buildings, roads, bridges ,etc will be to his credit. In all aspect of life, maxima and minima of convex functions in infinite dimensional spaces will give a long term solution to problems of life.
This paper is motivated by studying [**4, 6, 7**] and the references therein. To study optimization we need to maximize or minimize functions. This is easy in finite dimensional spaces, since in finite dimensional spaces convex continuous functions defined on a compact domain attain their maximum or minimum on the given domain. The prove of this relies on the properties of the domain i.e the set is compact. But attempt to move this to infinite dimensional space proved abortive because compact sets are rare to be found in infinite dimensional spaces, 4,[8-10]. Also the topology of infinite dimensional spaces is "too big " to give us compactness. Therefore to obtain some form of compactness, we need to cut down a number of open sets under consideration i.e to reduce the size of the topology of infinite dimensional spaces E. This leads us to weak compactness.

2. PRELIMINARIES

Definition 1.1(Convex set) Let X be a real linear space and $C \subset X$. The set C is called convex if for each $x_1, x_2 \in C$ and for each $t \in [0,1]$.we have
$tx_1 + (1-t)x_2 \in C$.
Definition1.2 Let D be a subset of real vector space and $f : D \to \mathbb{R}\ \boldsymbol{U}\ \{+\infty\}$**;** then f is said to be convex if (a) D is convex and (b) for each $t \in [0.1]$ and for each $x_1, x_2 \in D$ we have
$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t) f(x_2)$.
Definition 1.3 Let $f : X \to \mathbb{R}\ \boldsymbol{U}\ \{+\infty\}$ be a map. The effective domain of f is the set defined by
$D(f) := \{x \in X : f(x) < +\infty\}$
Definition 1.4 A map $f : X \to \mathbb{R}\ \boldsymbol{U}\ \{+\infty\}$ is called proper if $D(f) \neq \emptyset$.
Definition 1.5 (Epigraph) The epigraph of f is the set defined by
$\text{epi}(f) := \{(x, \alpha) \in X \times R : x \in D(f) \text{ and } f(x) \leq \alpha\}$
Definition1.6 (Section of f), Let $\propto \in \mathbb{R}$, we have the following definition;
$Sf_{\propto} := \{x \in X : f(x) \leq \propto\} = \{x \in D(f) : f(x) \leq \propto\}$
Proposition 1.7 A mapping $f : X \to \mathbb{R}\ \boldsymbol{U}\ \{+\infty\}$ is convex if and only if the epi(f) is convex.
Definition 1,8 A function $f : X \to \mathbb{R}\boldsymbol{U}\{+\infty\}$ is called lower semi –continuous at $\bar{x}$ if $\liminf f(x) \geq f(\bar{x})\ x \to \bar{x}$

Definition 1.9 Let E be a normed linear space and let J be the canonical embedding of E into E^{**} i.e double dual space. If J is onto, then E is called reflexive. Thus a reflexive Banach space is one in which the canonical embedding is onto.

Definition 2.0 Let X be a reflexive space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$.

3. RESULTS

Theorem1.5: Let X be a reflexive Banach space and let K be a closed, convex, bounded and non-empty subset of X. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous and convex. Then there exist $\bar{x} \in K$ such that $f(\bar{x}) \leq f(x)$ $\forall x \in K$ i.e $f(\bar{x}) = \inf f(x) = \min f(x)$

Proof: f is lower semi continuous and convex implies f is weakly lower semi continuous.

Let b $= \inf_{x \in K} f(x)$

First suppose b $= -\infty$. Then for $n \subset \mathbb{N}, \exists\, x_m \in K$

Such that $f(x_m) < -n \;\forall n \in \mathbb{N}$

Since K is bounded it implies $\{x_m\}$ is bounded and $x_m \in K$, since X is a reflexive Banach space, by theorem[1.4] implies there exist $\{x_{mk}\}$, such that x_{mk}converges weakly to $x \in X$.But K is convex and closed implies K is weakly closed. Hence $x \in K$. By weak lower semi-continuity of f, we have,

$f(x) \leq \lim_{k\to\infty} \inf f(x_{mk})$

$f(x) \leq \lim_{k\to\infty} \inf f(x_{mk}) < -\infty$

This is a contradiction since $f(x) \in \mathbb{R} \cup \{+\infty\}$.Hence $m \in \mathbb{R}$.

Let $m \in \mathbb{N}$ and take $\varepsilon_m = \frac{1}{m}$, then there exist $x_m \in K$ such that

$m \leq f(x_m) < b + \frac{1}{m}$.

The sequence $\{x_m\}$ in K implies $\{x_m\}$ is bounded and so

$\exists\, \{x_{mk}\}_{k \in \mathbb{N}}$ subsequence of $\{x_m\}$and $\bar{x} \in K$ such that $x_{mk} \to \bar{x}$. Since f is weakly lower semi-continuous we have

$f(\bar{x}) \leq \lim\inf_{k\to\infty} f(x_{mk}) \leq \liminf_{k\to\infty} (b + \frac{1}{mk})$

$= \lim_{k\to\infty} (b + \frac{1}{mk}) = b$

But $f(\bar{x}) \leq b = \inf_{x \in k} f(x)$ and b $\leq f(\bar{x})$ $since\ b = \inf_{x \in k} f(x)$

Therefore,

$f(\bar{x}) = b = \inf_{x \in k} f(x)$.

Also, we present the second result of this study.

Suppose K lost boundedness and f is proper, lower semi-continuous,convex and coercive function, we prove that there exist $\bar{x} \in X$.

Theorem 1.6

Let X be a reflexive real Banach space and $f: X \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper, lower semi-continuous function. Suppose

$\lim_{\|x\| \to \infty} f(x) = +\infty$.

Then, there exit $\bar{x} \in X$ a minimizer of f such that

$f(\bar{x}) \leq f(x), x \in X$ i.e

$f(\bar{x}) = \inf_{x \in X} f(x)$

Proof : Since f is proper, then there exist $x^0 \in X$ such that $f(x^0) \neq +\infty$

i.e $f(x^0) \in \mathbb{R}$. Now we construct a set K and show that it is non-empty closed convex and bounded subset of X and apply theorem 2.1.

Consider $f: X \to \mathbb{R} \cup \{+\infty\}$ with $K = \{x \in X: f(x) \leq f(x^0)\}$

To show that, we know that K is a section with $\alpha = f(x^0)$

Since f is convex and lower semi-continuous, by proposition 4.1 and 4.2, K is convex and closed. Then we assume K is bounded.

Suppose K is not bounded, then there exist a sequence $x_m \in K$ such that

$\| X_m \| > m \;\forall m \in \mathbb{N}$

Since $X_m \in K$ we obtain

$f(X_m) \leq f(x^0)\, \|x_m\| > m \Rightarrow \lim_{m\to\infty} \|X_m\| = +\infty$
By hypothesis $\lim_{m\to\infty} f(x_m) = +\infty$
Contradicting the inequality above, and so K is bounded.
By theorem 2.1. there exist $\bar{x} \in K \subset X$ such that
$\forall x \in K, \quad f(\bar{x}) \leq f(x)$.
Suppose $x \in X \setminus K$ *ie* k^c i.e complement of k, then
$f(x) > f(x^0)\ \forall x^0 \in K$
Since $\bar{x} \in K$. $f(\bar{x}) \leq f(x^0)$
Thus $f(\bar{x}) \leq f(x)\ \forall x \in X$
$\Rightarrow f(\bar{x}) = \inf_{x\in k} f(x)$

Theorem1.7 Suppose $K \subseteq H$ is a weakly sequentially closed and bounded set. Suppose $f: K \to \mathbb{R}$ is weakly sequentially lower semi-continuous.Then f is bounded from below and has a minimizer on K.

Proof: First we show that f is bounded from below.Suppose to the contrary that f is not bounded from below. Then there exist a sequence $\{X_n\} \in K$ such that $f(X_n) < -n$ for all n. Now since K is bounded $\{X_n\}$ has a weakly convergent subsequence $\{X_{nk}\}$,
$X_{nk} \to x^*$.Moreover, K is weakly sequentially lower semi-continuous. We have $f(x^*) \leq \lim inf f(X_{nk}) = -\infty$, which a contradiction. Hence, f is bounded from below. Next. we show the existence of a minimizer. Let $\{X_n\} \in K$ be a minimizing sequence for f, that is $f(X_n) \to inf f(x)$. Let $\propto = \inf f(x.)$ Since K is bounded and K is weakly sequentially closed, it follows that $\{X_n\}$ has a weakly convergent subsequence $X_{nk} \to x^* \in K$.Since f is weakly sequentially lower semi-continuous, we have
$\propto \leq f(x^*) \leq \lim inf f(x_{nk}) = lim f(x_{nk}) = \propto$. Hence $f(x^*) = \propto$.

Theorem1.8 Let K be a convex, strongly closed and bounded subset of H.Suppose$f : K \to \mathbb{R}$ is a strongly lower semi-continuous and convex function. Then f is bounded from below and attains a minimizer on K.

Proof :K is strongly closed and convex and by lemma 4.1 is also weakly sequentially closed. But, since f is strongly lower semi-continuous, and convex, it is also weakly lower semi-continuous by corollary 4.2.Then, $f : K \to \mathbb{R}$ a weakly lower semi-continuous and K a weakly closed and bounded set in H which allows us to apply the generalized Weierstrass theorem to conclude that f is bounded from below and attain a minimizer on K. If f is strictly convex, the minimizer will be unique i.e if we have two distinct minimizers $u_1$and$u_2$in $K. f(u_1) = f(u_2) = inf f(u)$. Then by strict convexity of f we have
$f(u_1 + u_2)/2 \leq f(u_1)$ which is a contradiction[8]

Corollary 1.9 Let $f : H \to \mathbb{R}$ be a strongly lower semi-continuous,convex and coercive function. Then, f is bounded from below and attains a minimizer.

Proof:

Under the assumptions of the corollary, it is straight forward to note that f is bounded from below. Next, fix a $\delta > 0$, since$f$ is coercive, there exists $M \in \mathbb{R}$. such that
$f(x) \geq \inf f(y) + \delta$ for all $x \in \{ x: \|x\| > M$. }Then consider $f: C \to \mathbb{R}$ with
$C = \{ x : \| x \| \leq M\}$ and apply the previous theorem.

4. APPLICATION

Let us take a look at a real life problem of a situation where a man wants to divide his savings between three mutual funds with expected returns so as to minimize risk of the return on the investment. How should he divide his savings between three mutual funds with expected returns 10%,10% and 15% respectively, so as to minimize risk while achieving an expected return of 12%. We measure risk as the variance of the return on the investment. What fraction of x of the savings is invested in fund 1,y in fund 2 and z in fund 3 where $x + y + z = 1$, the variance of the return has been calculated to be $400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$.

The problem is modeled as
Min $f(x, y, z) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$.
Subject to
$x + y + 1.5z = 1.2$
$x + y + z = 1$
Using Lagrange multiplier,
Min $f(x, y, z) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$. (1)
Subject to $x + y + 1.5z = 1.2$ (2)
$x + y + z = 1$ (3)

From (2) $x + y + 1.5z - 1.2 = 0$
Also from (3) $x + y + z - 1 = 0$
$L(x_i, \lambda_i) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz + \lambda_1(x + y + 1.5z - 1.2) + \lambda_2(x + y + z - 1)$
The necessary conditions for $L(x_i, \lambda_i)$ to have an extreme point can be written as
$\frac{\partial L}{\partial x} = 800x + 200y + \lambda_1 + \lambda_2 = 0$ (4)
$\frac{\partial L}{\partial y} = 1600y + 200x + 400z + \lambda_1 + \lambda_2 = 0$ (5)
$\frac{\partial L}{\partial z} = 3200z + 400y + 1.5\lambda_1 + \lambda_2 = 0$ (6)
$\frac{\partial L}{\partial \lambda_1} = x + y + 1.5z - 1.2 = 0$ (7)
$\frac{\partial L}{\partial \lambda_2} = x + y + z - 1 = 0$ (8)
Since $x + y + z = 1$ from (8)
$z = 1 - x - y$ (9)
Subtract equation (4) from equation (5)
$1600y + 200x + 400z + \lambda_1 + \lambda_2 - 800x - 200y - \lambda_1 - \lambda_2$
$= -600x + 1400y + 400z = 0$ (10)
Recall that $z = 1 - x - y$
Substituting z in equation (10) we have
$-600x + 1400y + 400(1 - x - y) = -1000x + 1000y + 400 = 0$
Therefore, $y = -0.4 + x$ (11)
Substituting for y into equation (7) we have,
$x - 0.4 + x + 1.5(1 - x - (-0.4 + x) - 1.2 = 0 \quad = 2x - 3x + 0.5 = 0$
$= -x + 0.5 = 0$
$\therefore \quad x = 0.5$
Substitute for x into (11) we have $y = -0.4 + 0.5 = 0.1 => y = 0.1$.
Substitute for $x = 0.5$ and $y = 0.1$ $in\ eqs$ (9) we have
$z = 1 - 0.5 - 0.1 = 1 - 0.6 = 0.4,$ $\therefore z = 0.4.$
Hence, $f^*_{min} = 400(0.5)^2 + 800(0.1)^2 + 200(0.5)(0.1) + 1600(0.4)^2 + 400(0.1)(0.4) = 100 + 8 + 10 + 256 + 16 = 390$
Substitute for x, y and z in (5)and (6) we have that
$1600y + 200x + 400z + \lambda_1 + \lambda_2 = 0$ (5)
$3200z + 400y + 1.5\lambda_1 + \lambda_2 = 0$ (6)
$= 1600(0.1) + 200(0.5) + 400(0.4) + \lambda_1 + \lambda_2 = 0$
$= 160 + 100 + 160 + \lambda_1 + \lambda_2 = 0$
$= 420 + \lambda_1 + \lambda_2 = 0$
$3200(0.4) + 400(0.1) + 1.5\lambda_1 + \lambda_2 = 0$
$1280 + 40 + 1.5\lambda_1 + \lambda_2 = 0$
$1320 + 1.5\lambda_1 + \lambda_2 = 0$ (2)
$420 + \lambda_1 + \lambda_2 = 0$ (1)
Rearrange to $1.5\lambda_1 + \lambda_2 = -1320$ (1)
$\lambda_1 + \lambda_2 = -420$ (2)

$\frac{0.5\lambda_1}{0.5} = \frac{900}{0.5}$
$\lambda_1 = 1800$
$\therefore 1800 + \lambda_2 = -420$
$\lambda_2 = -420 - 1800 = -1320 \text{ or } -2220$

Reflexive Space: A space X is said to be reflexive if the canonical embedding J of X $into$ X^{**} $i.e$ $double\ dual\ space$ is onto i,e surjective. This means that every element of X is uniquely represented in X^{**}.The function above is not reflexive because reflexivity occurs in spaces but not in functions.
Convexity: Let K be a non-empty convex set. A function $f : K \to \mathbb{R}$ is said to be convex if the following inequality holds
$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in K$ and $\lambda \in (0,1)$

The expression $\lambda x + (1-\lambda)y$ is a convex combination and it is a special linear combination where the coefficients are non-negative and sum to one. Since our λ_1 and $\lambda_2 \in \mathbb{R}$ does not fall within this domain, we cannot have a convex combination and therefore the function cannot be convex as stated by the inequality above.

Continuity: Let $f : D(f) \subset \mathbb{R} \to \mathbb{R}$, f is said to be continuous at $x_0 \in D(f)$if and only if $\forall\, \varepsilon > 0$ there is a $\delta = \delta(\varepsilon, x_0) > 0$ such that $|x - x_0| < \delta \;=> |f(x) - f(x_0)| < \varepsilon$.

Theorem: If a linear functional on a normed linear space X is continuous at a single point, then it is continuous throughout X.

To prove that the function $f(x,y,z) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$
is continuous at $x = 0.5, y = 0.1, z = 0.4$
Let $\varepsilon > 0$ be given, we want to find a $\delta = \delta(\varepsilon, x_0) > 0$ such that if $|\, x - 0.5 \,| < \delta$ then
$|f(x) - f(0.5)| < \varepsilon$.If it is continuous at $x = 0.5$, then it is continuous at $y = 0.1, z = 0.4$
$|f(x) - f(0.5)| = |400x^2 + 200xy| = |400(x^2 - 0.5^2) + 200(xy - (0.5)(0.1)|$
$= |400(x - 0.5)(x + 0.5) + 200(x - 0.5)(y - 0.1)| = 400\delta\,|\,x + 0.5\,|\; + 200\delta\,|\,y - 0.1\,|$
$= 200(2|x + 0.5| + |y - 0.1|\delta$. $|x - 0.5| < \delta$ we have
$|x| = |x - 0.5 + 0.5| \le |\,x - 0.5\,| + 0.5 < \delta + 0.5$.
Here we can assume $\delta < 1$, so that $|x| < 1 + 0.5 = 1.5$
Therefore, x is continuous at $x = 0.5$, and if it is continuous there it is continuous throughout X i.e at $y = 0.1, z = 0.4$.

Coercivity: Let X be a reflexive real Banach space.A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be coercive if
$\lim_{\|x\| \to \infty} f(x) = +\infty$

But these values $x, y, z = 0.5, 0.1, 0.4 \in \mathbb{R}$ and $\neq +\infty$, therefore their functions cannot be coercive.

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