Transactions of the Nigerian Association of Mathematical Physics
***Volume 8,* (*January, 2019*), pp5 – 10**
© Trans. of NAMP

FIXED POINT THEOREMS OF ZAMFIRESCU'S TYPE IN COMPLEX VALUED G_b-METRIC SPACES

O. K. Adewale, J. O. Olaleru, and H. Akewe

University of Lagos, Akoka, Lagos

Abstract

In this paper, the proof of the existence of fixed point for contraction type and Zamfirescu's type operator are established and proved. The results generalized several known results in literature among others.

Key words and phrases: complex value G_b-metric spaces and fixed point.

1. Introduction and Preliminaries

The importance of fixed point theorems cannot be overemphasized. The study of metric fixed point theory has been researched broadly in the past decades since fixed point theory plays a vital role in mathematics and applied sciences such as optimization, mathematical models and economic theories. In light of this, many authors had been trying to generalized the usual metric space to a more general one. Gahler introduced 2-metric [1], Dhage in 1992 recommended the notion of a D-metric space in a bid to attain analogous results to those for metric spaces [2]. In 2006, Mustafa and Sims proved that these attempts were invalid [3]. They later introduced a new structure of generalised metric spaces called the G-metric spaces, the generalisation of the usual metric space (X; d) [4]. Bakhtin introduced b-metric space as a generalization of the usual metric space and proved analogue of Banach contraction principle in a b - metric space [5]. Akbar, Brian and Khan introduced the notion of complex valued metric space [6]. Sedghi, Shobe and Aliouche gave the notion of S-metric space and proved some fixed point theorems for a self-mapping on a complete S-metric space [7]. Aghajani, Abbas and Roshan presented a new type of metric which is called G_b - metric and studied some properties of this metric [8]. Recently Sedghi et al. defined S_b-metric spaces using the concept of S - metric spaces [9]. Adewale and Akinremi proved the analogous of the Zamfirescu's type fixed point theorem in generalized cone metric spaces [10]. In this paper, analogous of the Zamfirescu's type fixed point theorem in a complex valued G_b–metric space are proved. Some examples are included which shows that these generalizations are genuine.

Mustafa and Sims extended the notion of usual metric space from distance between two points to perimeter of a triangle as shown below:

Definition 1.1 [4]**.** Let X be a non-empty set and $G : X \times X \times X \to [0;\ 1)$ be a function satisfying the following properties:
(i) G(x; y; z) = 0 if and only if x = y = z
(ii) G(x; x; y) > 0, for all $x, y \in X$, with $x \neq y$
(iii) G(x; x; y) ≤ G(x; y; z), $x, y, z \in X$, with $z \neq y$
(iv) G(x; y; z) = G(x; z; y) = G(y; x; z) =… (symmetry).
(v) G(x; y; z) ≤ G(x; a; a) + G(a; y; z) $a, x, y, z \in X$ (rectangle inequality)
The function G is called a G-metric on X and (X, G), a G-metric space.

Example 1.2 [10]**.** Let $X = [0;\ 1)$, $T(x) = \frac{x}{4}$ and
$G\ (x, y, z) = \max\{/x - y/, /y - z/, /z - x/\}$
Then (X; G) is a G-metric space but not G-complete, since the sequence
$x_n = 1 - \frac{1}{n}$ is G-Cauchy in (X, G) and not G-convergent in (X; G), that is
$\lim\limits_{n\to\infty} 1 - \frac{1}{n} = 1 \notin [0;\ 1)$.
If X = [0; 1], then it is G-complete.

Definition 1.3 [11]. In a G-metric space X, G is said to be symmetric if G(x, y, y) = G(x, x, y) for all $x, y \in X$.

Bakhtin presented b - metric space as a generalization of the usual metric space by introducing Constant $s \geq 1$ in the triangle inequality as seen in the definition below:

Definition 1.4 [5]**.** Let X be a non-empty set and $d : X \times X \to [0;\ 1)$ be a function satisfying the following properties:
(i) d(x, y) = 0 if and only if x = y.

Correspondence Author: Adewale O.K., Email: Adewalekayode2@yahoo.com, Tel: +2348033833498

(ii) $d(x, y) \geq 0$, for all $x, y \in X$.
(iii) $d(x, y) = d(y, x)$ (symmetry).
(iv) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$ and $s \geq 1$ (triangle inequality)
The function d is called a b-metric on X and (X, d), a b-metric space. Examples had been given to show that a b-metric is not necessarily a metric.
Aghajani et al. extended this concept to G_b-metric spaces and defined the following:
Definition 1.5 [8]. Let X be a non-empty set and $G : X \times X \times X \to [0; 1)$ be a function satisfying the following properties:
(i) $G(x, y, z) = 0$ if and only if $x = y = z$
(ii) $G(x, x, y) > 0$, for all $x, y \in X$ with $x \neq y$
(iii) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$
(iv) $G(x, y, z) = G(y, z, x) = G(x, z, y) = \ldots$ (symmetry).
(v) There exists a real number $s \geq 1$ such that $G(x, y, z) \leq s[G(x, a, a)+G(a, y, z)]$ for all $a, x, y, z \in X$. Then (X, G_b) is a G_b-metric space.

In 2015, Ozgur Ege introduced the following definition as a generalization of both G-metric space and b-metric space.
Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Like other authors working in this area, we define a partial order $\preccurlyeq$ on $\mathbb{C}$ as follows:
$z_1 \preccurlyeq z_2$ if and only if $\mathrm{Re}(z_1) \leq R(z_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$. It follows that $z_1 \preccurlyeq z_2$ if one of the following properties is satisfied:
(i) $\mathrm{Re}(z_1) = R(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$
(ii) $\mathrm{Re}(z_1) < R(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$
(iii) $\mathrm{Re}(z_1) < R(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$
(iv) $\mathrm{Re}(z_1) = R(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$
Notably, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied.
Note that
$0 \preccurlyeq z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$
$z_1 \preccurlyeq z_2, z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$
Definition 1.6 [12]. Let X be a non-empty set, C, a set of complex numbers and $G_b : X \times X \times X \to \mathbb{C}$ be a function satisfying the following properties:
(i) $G_b(x, y, z) = 0$ if and only if $x = y = z$
(ii) $G_b(x, x, y) \succ 0$, for all $x, y \in X$ with $x \neq y$
(iii) $G_b(x, x, y) \preccurlyeq G_b(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$
(iv) $G_b(x, y, z) = G_b(y, z, x) = G_b(x, z, y) = \ldots$ (symmetry).
(v) There exists a real number $s \geq 1$ such that $G_b(x, y, z) \preccurlyeq s[G_b(x, a, a)+ G_b(a, y, z)]$ for all $a, x, y, z \in X$.
Then the function G_b is called a complex valued G_b-metric and (X, G_b) is the complex valued G_b-metric space. A complex valued G_b - metric space is complete if every Cauchy sequence in it is G_b - convergent in it.

2. Main Results

The following lemmas will be needed in our work.
Lemma 2.1. Let (X, G_b) be a complex valued G_b-metric space and $\{x_n\}$ a sequence in X. $\{x_n\}$ converges to $x \in X$ if and only if $|G_b(x_n, x, x)| \to 0\ j$ as $n \to \infty$.
Lemma 2.2. Let $(X; G_b)$ be a complex valued G_b-metric space and $\{x_n\}$, a sequence in X. $\{x_n\}$ is a Cauchy sequence if and only if $|G_b(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.
Example 2.3. Let $X = R$. If $x, y, z \in X$ with $k \in N$, then $G_b : X \times X \times X \to \mathbb{C}$ defined by:

$$G_b(x,y,z) = \begin{cases} 0 & \text{if } x = y = z; \\ k + \sqrt{-k^2} \text{ if } x,y,z \in R^+ \text{ and } x = y, & y = z \text{ or } x = z; \\ k + 2\sqrt{-k^2} & \text{if } x;\ y;\ z \in R^+ \text{ and } x \neq y \neq z. \end{cases}$$

is a complex valued Gb - metric on X but not Gb - metric on X.

Example 2.4. For a set of natural numbers N, let $X = \left\{\frac{1}{n}, n \in N\right\}$ and for all $z_1, z_2, z_3 \in X$ with
$G_b(z_1, z_2, z_3) = G_b(z_1, z_3, z_2) = G_b(z_2, z_1, z_3) = \cdots$
then $G_b: X^3 \to C$ defined by:
$G_b(z_1, z_2, z_3) = e^{z_3 i}|z_1 - z_2|$
where $z_1 = (x_1, y_1),\ z_2 = (x_2, y_2),\ z_3 = (x_3, y_3)$ is a complex valued G_b-metric on X.

We now prove the following theorems and proposition which are analogues of some results in a metric space, G-metric space and G_b-metric space.(see [12,13,14,15,16]).
Theorem 2.5 . Let X be a complete complex valued G_b – metric space and $T : X \to X$ a map for which there exists the real number k satisfying $0 \leq k < 1$ with

$\Phi(a) = \begin{cases} 0 \ if \ a = 0 \\ a \ if \ a \neq 0 \end{cases}$

and $s < \frac{1}{k^n+k}$ such that for each pair $x, y, z \in X$; $a \in N \cup \{0\}$,

$G_b(Tx, Ty, Tz) \preccurlyeq kG_b(x, y, z) + \Phi\big(G_b(Tx, y, y)\big)$

where $\Phi: [0, \infty) \to [0, \infty)$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = Tx_n$ for all $n \in N$, then we have

$G_b(x_n, x_{n+1}, x_{n+1}) \preccurlyeq kG_b(x_{n-1}, x_n, x_n) + \Phi\big(G_b(x_n, x_n, x_n)\big)$

$G_b(x_n, x_{n+1}, x_{n+1}) \preccurlyeq kG_b(x_{n-1}, x_n, x_n) + \Phi(0)$

$G_b(x_n, x_{n+1}, x_{n+1}) \preccurlyeq kG_b(x_{n-1}, x_n, x_n) \dots \dots \dots (1)$

From (1), we deduce that

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq kG_b(x_{n-1}, x_n, x_n) \\ &\preccurlyeq k^2G_b(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\preccurlyeq k^3G_b(x_{n-3}, x_{n-2}, x_{n-2}) \\ &\ \ \vdots \\ &\preccurlyeq k^nG_b(x_0, x_1, x_1) \dots \dots \dots (2) \end{aligned}$$

By repeated use of rectangle inequality with $m > n$, we have

$$\begin{aligned} G_b(x_n, x_m, x_m) &\preccurlyeq sG_b(x_n, x_{n+1}, x_{n+1}) + s^2G_b(x_{n+1}, x_{n+2}, x_{n+2}) + s^3G_b(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + \dots + s^nG_b(x_{m-1}, x_m, x_m) \dots \dots \dots (3) \end{aligned}$$

From (2) and (3), we have

$$\begin{aligned} G_b(x_n, x_m, x_m) &\preccurlyeq sG_b(x_n, x_{n+1}, x_{n+1}) + s^2G_b(x_{n+1}, x_{n+2}, x_{n+2}) + \dots + s^nG_b(x_{m-1}, x_m, x_m) \\ &\preccurlyeq sk^nG_b(x_0, x_1, x_1) + s^2k^{n+1}G_b(x_0, x_1, x_1) + \dots + s^nk^{m-1}G_b(x_0, x_1, x_1) \\ &\preccurlyeq [sk^n + s^2k^{n+1} + \dots + s^nk^{m-1}]G_b(x_0, x_1, x_1) \\ &\preccurlyeq sk^n[1 + sk + \dots + s^{n-1}k^{m-n-1}]G_b(x_0, x_1, x_1) \\ &\preccurlyeq sk^n[1 - sk]^{-1}G_b(x_0, x_1, x_1) \end{aligned}$$

Taking the limit of $G_b(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$\lim_{n\to\infty} |G_b(x_n, x_m, x_m)| = \lim_{n\to\infty} |sk^n[1 - sk]^{-1}G_b(x_0, x_1, x_1)| = 0$

For $n, m, l \in N$

$G_b(x_n, x_m, x_l) \preccurlyeq sG_b(x_n, x_m, x_m) + sG_b(x_m, x_l, x_l)$

Taking the limit of $G_b(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$\lim_{n\to\infty} |G_b(x_n, x_m, x_l)| = \lim_{n\to\infty} |sG_b(x_n, x_m, x_m) + sG_b(x_m, x_l, x_l)| = 0$

So, x_n is a G_b-Cauchy sequence.

By completeness of $(X; G_b)$, there exist $u \in X$ such that x_n is G_b-convergent to u.

Suppose $Tu \neq u$,

$G_b(x_n, Tu, Tu) \preccurlyeq kG_b(x_{n-1}, u, u) + \Phi\big(G_b(x_n, u, u)\big)$

Taking the limit as $n \to \infty$ and using the fact that function is G_b-continuous in its variables, we get:

$G_b(u, Tu, Tu) \preccurlyeq kG_b(u, u, u) + \Phi\big(G_b(u, u, u)\big) \preccurlyeq 0$

A contradiction. So, $Tu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$G_b(Tu, Tv, Tv) \preccurlyeq kG_b(u, v, v) + \Phi\big(G_b(Tu, v, v)\big)$

Since $Tu = u$ and $Tv = v$, we have

$G_b(u, v, v) \preccurlyeq kG_b(u, v, v) + \Phi\big(G_b(u, v, v)\big)$

which implies that $v = u$.

If $\Phi\big(G_b(Tx, y, y)\big) = \Phi(0)$ in Theorem 2.5, we have Theorem 3.7 in [12] and setting $d(x, y) = G_b(Tx, y, y)$, the theorem reduces to Banach contraction principle [13]. We can also obtain generalized Banach contraction principle in G-metric spaces if $s = 1$.

Corollary 2.6. Let X be a complete complex valued G_b-metric space and $T: X \to X$ a map for which there exists the real number k satisfying $0 \leq k < 1$ such that for each pair $x, y, z \in X$,

$G_b(Tx, Ty, Tz) \preccurlyeq kG_b(x, y, z)$.

Then T has a unique fixed point.

Proposition 2.7. Let X be a complete complex valued G_b - metric space and $T: X \to X$ a map for which there exists the real numbers a, b, c satisfying $0 \leq a < \frac{3}{2s+3}$, $b \leq \min\left\{\frac{1}{2}, \frac{3}{2s+3}\right\}$, and $c \leq \min\left\{\frac{1}{2}, \frac{3}{2s+3}\right\}$ such that for each pair $x, y, z \in X$ at least one of the following is true.

(GBZ_1) $G_b(Tx, Ty, Tz) \preccurlyeq aG_b(x, y, z)$;

$(\mathrm{GBZ}_2) G_b(Tx, Ty, Tz) \preccurlyeq b[G_b(x, Tx, Tx) + G_b(y, Ty, Ty)]$;
$(\mathrm{GBZ}_3) G_b(Tx, Ty, Tz) \preccurlyeq c[G_b(x, Ty, Ty) + G_b(y, Tx, Tx)]$.
Then T has a unique fixed point.
Proof. Adding (GBZ_1), (GBZ_2) and (GBZ_3),we have

$$G_b(Tx, Ty, Ty) \preccurlyeq qG_b(x, y, y) + qG_b(x, Tx, Tx) + qG_b(y, Ty, Ty) + qG_b(x, Ty, Ty) + qG_b(y, Tx, Tx)$$

where $q = max\left\{\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right\}$.
Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = Tx_n$ for all $n \in N$, then we have

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq q[2G_b(x_{n-1}, x_n, x_n) + G_b(x_{n-1}, x_{n+1}, x_{n+1}) + G_b(x_n, x_{n+1}, x_{n+1})] \\ &\preccurlyeq q[2G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1}) + sG_b(x_{n-1}, x_n, x_n) \\ &\quad + sG_b(x_n, x_{n+1}, x_{n+1})] \\ &\preccurlyeq \frac{2q + sq}{1 - q - sq} G_b(x_{n-1}, x_n, x_n) \end{aligned}$$

Let $p = \frac{2q+sq}{1-q-sq}, p \in [0,1)$,
We deduce that

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq pG_b(x_{n-1}, x_n, x_n) \\ &\preccurlyeq p^2 G_b(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\preccurlyeq p^3 G_b(x_{n-3}, x_{n-2}, x_{n-2}) \\ &\quad \vdots \\ &\preccurlyeq p^n G_b(x_0, x_1, x_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (4) \end{aligned}$$

By repeated use of rectangle inequality with $m > n$, we have

$$\begin{aligned} G_b(x_n, x_m, x_m) &\preccurlyeq sG_b(x_n, x_{n+1}, x_{n+1}) + s^2 G_b(x_{n+1}, x_{n+2}, x_{n+2}) + s^3 G_b(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + \cdots + s^n G_b(x_{m-1}, x_m, x_m) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (5) \end{aligned}$$

From (4), we have

$$\begin{aligned} G_b(x_n, x_m, x_m) &\preccurlyeq sG_b(x_n, x_{n+1}, x_{n+1}) + s^2 G_b(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + s^n G_b(x_{m-1}, x_m, x_m) \\ &\preccurlyeq sp^n G_b(x_0, x_1, x_1) + s^2 p^{n+1} G_b(x_0, x_1, x_1) + \cdots + s^n p^{m-1} G_b(x_0, x_1, x_1) \\ &\preccurlyeq [sp^n + s^2 p^{n+1} + \cdots + s^n p^{m-1}] G_b(x_0, x_1, x_1) \\ &\preccurlyeq skp^n [1 + sp + \cdots + s^{n-1} p^{m-n-1}] G_b(x_0, x_1, x_1) \\ &\preccurlyeq sp^n [1 - sp]^{-1} G_b(x_0, x_1, x_1) \end{aligned}$$

Taking the limit of $G_b(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$$\lim_{n\to\infty} |G_b(x_n, x_m, x_m)| = \lim_{n\to\infty} |sp^n [1 - sp]^{-1} G_b(x_0, x_1, x_1)| = 0$$

For $n, m, l \in N$

$$G_b(x_n, x_m, x_l) \preccurlyeq sG_b(x_n, x_m, x_m) + sG_b(x_m, x_l, x_l)$$

Taking the limit of $G_b(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n\to\infty} |G_b(x_n, x_m, x_l)| = \lim_{n\to\infty} |sG_b(x_n, x_m, x_m) + sG_b(x_m, x_l, x_l)| = 0$$

So, x_n is a G_b-Cauchy Sequence.
By completeness of $(X; G_b)$, there exist $u \in X$ such that x_n is G_b-convergent to u.
Suppose $Tu \neq u$,

$$G_b(x_n, Tu, Tu) \preccurlyeq qG_b(x_{n-1}, u, u) + qG_b(x_{n-1}, x_n, x_n) + qG_b(u, Tu, Tu) + qG_b(x_n, Tu, Tu) + qG_b(u, x_n, x_n)$$

Taking the limit as $n \to \infty$ and using the fact that function is G_b-continuous in its variables, we get

$$\begin{aligned} G_b(u, Tu, Tu) &\preccurlyeq qG_b(u, u, u) + qG_b(u, u, u) + qG_b(u, Tu, Tu) + qG_b(u, Tu, Tu) \\ &\quad + qG_b(u, u, u) \\ &\preccurlyeq 0 \end{aligned}$$

A contradiction. So, $Tu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$G_b(Tu, Tv, Tv) \preccurlyeq qG_b(u, v, v) + qG_b(u, Tu, Tu) + qG_b(v, Tv, Tv) + qG_b(Tu, Tv, Tv) + qG_b(v, Tu, Tu)$$

Since $Tu = u$ and $Tv = v$, we have

$$G_b(u, v, v) \preccurlyeq 0$$

which implies that $u = v$.

Corollary 2.8. Let X be a complete complex-valued G_b - metric space and $T : X \to X$ a map for which there exist the real numbers a satisfying $0 \leq a < \frac{3}{2s+3}$ such that for each pair $x, y, z \in X$,

$G_b(Tx, Ty, Tz) \preccurlyeq aG_b(x, y, z)$;
Then T has a unique fixed point.

Corollary 2.9. Let X be a complete complex-valued G_b - metric space and $T : X \to X$ a map for which there exist the real numbers b satisfying $b \le \min\left\{\frac{1}{2}, \frac{3}{2s+3}\right\}$ such that for each pair $x, y, z \in X$,
$G_b(Tx, Ty, Tz) \preccurlyeq b[G_b(x, Tx, Tx) + G_b(y, Ty, Ty)]$;
Then T has a unique fixed point.

Corollary 2.10. Let X be a complete complex-valued G_b – metric space and $T : X \to X$ a map for which there exist the real numbers c satisfying $c \le \min\left\{\frac{1}{2}, \frac{3}{2s+3}\right\}$ such that for each pair $x, y, z \in X$,
$G_b(Tx, Ty, Tz) \preccurlyeq c[G_b(x, Ty, Ty) + G_b(y, Tx, Tx)]$.
Then T has a unique fixed point.

In view of Corollary 2.6 and Proposition 2.7, we have the following:
Theorem 2.11. Let X be a complete complex valued G_b - metric space and $T : X \to X$ a map for which there exist the real numbers a, b, c satisfying $0 \le a < 1$, $b \le \min\left\{\frac{1}{2}, \frac{3}{2s+3}\right\}$, and $c \le \min\left\{\frac{1}{2}, \frac{3}{2s+3}\right\}$ such that for each pair $x, y, z \in X$ at least one of the following is true.
$(\mathrm{GBZ}_1)\, G_b(Tx, Ty, Tz) \preccurlyeq aG_b(x, y, z)$;
$(\mathrm{GBZ}_2)\, G_b(Tx, Ty, Tz) \preccurlyeq b[G_b(x, Tx, Tx) + G_b(y, Ty, Ty)]$;
$(\mathrm{GBZ}_3)\, G_b(Tx, Ty, Tz) \preccurlyeq c[G_b(x, Ty, Ty) + G_b(y, Tx, Tx)]$.
Then T has a unique fixed point.

Proof: It follows from Proposition 2.7 and Corollary 2.6.
Remark 2.12. If $s = 1$ in Theorem 2.11 and we set $d(x, y) = G_b(x, y, y)$, it reduces to Zamfirescu's fixed point theorem [16].
Theorem 2.13.
Let X be a complete complex-valued G_b - metric space and $T : X \to X$ a map for which there exists the real numbers a, b, c satisfying $0 \le a + b + c < 1$ with

$$\Phi(t) = \begin{cases} 0 \ if \ t = 0 \\ a \ if \ t \ne 0 \end{cases}$$

and $s < \frac{a^n + a(1-b-c)^{n-1}}{(1-b-c)^n}$ such that for each pair $x, y, z \in X$, $t \in N \cup \{0\}$,
$G_b(Tx, Ty, Tz) \preccurlyeq a\Phi\big(G_b(x, Tx, Tx)\big) + b\Phi\big(G_b(y, Ty, Ty)\big) + b\Phi\big(G_b(z, Tz, Tz)\big)$
where $\Phi: [0, \infty) \to [0, \infty)$. Then T has a unique fixed point.

Proof:
Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = Tx_n$ for all $n \in N$, then we have

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq a\Phi\big(G_b(x_{n-1}, x_n, x_n)\big) + b\Phi\big(G_b(x_n, x_{n+1}, x_{n+1})\big) + b\Phi\big(G_b(x_n, x_{n+1}, x_{n+1})\big) \\ &\preccurlyeq \frac{a}{1-b-c} G_b(x_{n-1}, x_n, x_n) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6) \end{aligned}$$

Let $r = \frac{a}{1-b-c} < 1$. From (6), we deduce

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq rG_b(x_{n-1}, x_n, x_n) \\ &\preccurlyeq r^2 G_b(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\preccurlyeq r^3 G_b(x_{n-3}, x_{n-2}, x_{n-2}) \\ &\quad \vdots \\ &\preccurlyeq r^n G_b(x_0, x_1, x_1) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (7) \end{aligned}$$

By repeated use of rectangle inequality with $m > n$, we have

$$\begin{aligned} G_b(x_n, x_m, x_m) &\preccurlyeq sG_b(x_n, x_{n+1}, x_{n+1}) + s^2 G_b(x_{n+1}, x_{n+2}, x_{n+2}) + s^3 G_b(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + \cdots + s^n G_b(x_{m-1}, x_m, x_m) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . . (8) \end{aligned}$$

From (7) and (8), we have

$$\begin{aligned} G_b(x_n, x_m, x_m) &\preccurlyeq sG_b(x_n, x_{n+1}, x_{n+1}) + s^2 G_b(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + s^n G_b(x_{m-1}, x_m, x_m) \\ &\preccurlyeq sr^n G_b(x_0, x_1, x_1) + s^2 r^{n+1} G_b(x_0, x_1, x_1) + \cdots + s^n r^{m-1} G_b(x_0, x_1, x_1) \\ &\preccurlyeq [sr^n + s^2 r^{n+1} + \cdots + s^n r^{m-1}] G_b(x_0, x_1, x_1) \\ &\preccurlyeq sr^n [1 + sr + \cdots + s^{n-1} r^{m-n-1}] G_b(x_0, x_1, x_1) \\ &\preccurlyeq sr^n [1 - sr]^{-1} G_b(x_0, x_1, x_1) \end{aligned}$$

Taking the limit of $G_b(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$$\lim_{n\to\infty} |G_b(x_n, x_m, x_m)| = \lim_{n\to\infty} |sr^n [1 - sr]^{-1} G_b(x_0, x_1, x_1)| = 0$$

For $n, m, l \in N$
$G_b(\mathrm{x_n}, \mathrm{x_m}, \mathrm{x_l}) \preccurlyeq sG_b(\mathrm{x_n}, \mathrm{x_m}, \mathrm{x_m}) + sG_b(\mathrm{x_m}, \mathrm{x_l}, \mathrm{x_l})$
Taking the limit of $G_b(\mathrm{x_n}, \mathrm{x_m}, \mathrm{x_l})$ as $n, m, l \to \infty$, we have
$\lim_{n\to\infty} |G_b(\mathrm{x_n}, \mathrm{x_m}, \mathrm{x_l})| = \lim_{n\to\infty} |sG_b(\mathrm{x_n}, \mathrm{x_m}, \mathrm{x_m}) + sG_b(\mathrm{x_m}, \mathrm{x_l}, \mathrm{x_l})| = 0$
So, $\mathrm{x_n}$ is a G_b-Cauchy Sequence.

By completeness of (X; G_b), there exist u ∈ X such that $\mathrm{x_n}$ is G_b-convergent to u.
Suppose $T\mathrm{u} \neq \mathrm{u}$,
$G_b(\mathrm{x_n}, Tu, Tu) \preccurlyeq a\Phi\big(G_b(\mathrm{x_{n-1}}, \mathrm{x_n}, \mathrm{x_n})\big) + b\Phi\big(G_b(\mathrm{u}, Tu, Tu)\big) + c\Phi\big(G_b(\mathrm{u}, Tu, Tu)\big)$
Taking the limit as $n \to \infty$ and using the fact that function is G_b-continuous in its variables, we get:
$G_b(\mathrm{u}, Tu, Tu) \preccurlyeq a\Phi\big(G_b(\mathrm{u}, \mathrm{u}, \mathrm{u})\big) + b\Phi\big(G_b(\mathrm{u}, Tu, Tu)\big) + c\Phi\big(G_b(\mathrm{u}, Tu, Tu)\big) \preccurlyeq 0$
A contradiction. So, $T\mathrm{u} = \mathrm{u}$.
To show the uniqueness, suppose $v \neq \mathrm{u}$ is such that $T\mathrm{v} = \mathrm{v}$, then
$G_b(\mathrm{Tu}, Tv, Tv) \preccurlyeq a\Phi\big(G_b(\mathrm{u}, T\mathrm{u}, T\mathrm{u})\big) + b\Phi\big(G_b(\mathrm{v}, Tv, Tv)\big) + c\Phi\big(G_b(\mathrm{v}, Tv, Tv)\big)$
Since $T\mathrm{u} = \mathrm{u}$ and $T\mathrm{v} = \mathrm{v}$, we have
$G_b(\mathrm{u}, v, v) \preccurlyeq 0$
which implies that $v = \mathrm{u}$.

Remark 2.14. If $\Phi\big(G_b(\mathrm{x}, y, y)\big) = \Phi(d(x, y))$ in Theorem 2.13, we have the result in [14].

References

[1] Gahler, S., 2-Metrische Raume und ihre topologische Struktur, Mathematishe Nachrichten, 26, (1963), 115-148.
[2] Dhage, B. C., Generalized metric space and mapping with fixed points, Bulletin of the Calcutta Mathematical Society, 84(12), (1992), 329-336.
[3] Mustafa, Z. and Sims, B., Some remarks concerning D-metric spaces, International conference on Fixed Point Theory and Applications, Yokohama, Japan, (2004).
[4] Mustafa, Z. and Sims, B., A new approach to generalized metric spaces, J. Nonlinear Convex Analysis, 7, (2006), 289-297.
[5] Bakhtin A., The contraction mapping principle in quasimetric spaces, Funct. Anal., Unianowsk Gos. Ped. Inst., 30(1), (1989).
[6] Akbar, A., Brian, F. and Khan, M., Common fixed point theorem in complex valued metric spaces, Numerical Functional Analysis and Optimizaton, 32, (2011), 245-255.
[7] Sedghi, S., Gholidahneh, A., Dosenovic, T., Esfahani, J. and Randenovic, S., Common fixed point of four maps in S_b -metric spaces, J. Linear Topol. Algebra, 5, (2016), 93-104.
[8] Aghajani, A., Abbas, M. and Roshan, JR., Common fixed point of generalized weak contractive mappings in partially ordered Gb - metric spaces, Filomat, 28, (2014), 1087-1101.
[9] Sedghi, S., Shobe, N. and Aliouche, A., A generalization of fixed point theorem in S-metric spaces, Mat. Vesn., 64, (2012), 258-266.
[10] Adewale O. K. and Akinremi B. V., Fixed point theorem of Zamfirescus type in generalized cone metric spaces, International Journal of Advanced Mathematical Sciences, 3, (2013), 104-108.
[11] Mustafa,Z. and Sims, B., Fixed point theorems for contractive mappings in complete G-metric spaces. doi: 10.1155/2009/917175.
[12] Ozgur E., Complex Valued Gb-Metric Spaces, J. Computational Analysis and Applications, 21(2), (2015).
[13] Banach, S., Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fundamenta Mathematicae, (1922), 133-181.
[14] Emmanuele G., Fixed Point Theorems in Complete Metric Spaces, Nonlinear Analysis, Theory, Melhods and Applrrurrons. 5, (1980), 287-292.
[15] Mishra, S.N., Sharma, B.K. and Iseki, K., Contractive Type Mapping on 2-Metric Space, Math. Japonica, 21, (1976), 67-70.
[16] Olaleru, J. O. and Samet, B., Some fixed point theorems in cone rectangular metric spaces, J. Nigeria Mathematics Society. 33, (2014), 145-158.
[16] Zamfirescu, T., Fixed point theorems in metric spaces, Archive for Mathematical Logic (Basel), (23) (1972), 292-298.