Transactions of the Nigerian Association of Mathematical Physics
***Volume 8,* (*January, 2019*), pp1 – 4**
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HANKEL DETERMINANT FOR λ-PSEUDO-SPIRALIKE FUNCTIONS

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Abstract

For function $f(z) \in S_p(\rho, \lambda, \beta)$ ***of the form:*** $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ ***which are analytic and univalent in the open unit disk*** $E = \{z : |z| < 1\}$***. We determine sharp upper bounds for the functional*** $|a_2 a_4 - a_3^2|$ ***in the unit disk.***

Keywords: Analytic, univalent, Hankel determinant, pseudo-spiralike function.

1. Introduction

Let S denote the class of function $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$. For ρ real, $0 \le |\rho| < \pi/2$, a function f of the form (1.1) is said to be in $S_p(\rho)$, the class of ρ - spiralike functions if and only if

$$\Re\left\{e^{i\rho} \frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in E. \tag{1.2}$$

The likes of Silvia, Libera, Robertson [1-5] to mention but few, have widely studied the above class of functions and their investigation has led to many interesting results. In particular, the class $S_p(\rho)$ was introduced and shown to be a subfamily of S by Spacek [6]. Later, Zamonski [7] obtained sharp coefficient bounds for the class. For $\rho = 0$, $S_p(0) = S^*$ which is the well-known class of functions starlike with respect to the origin. For $\rho \ne 0$, it is known that $S_p(\rho)$ is not contained in S^*. Furthermore, Robertson [3, 4] showed that the radius of starlikeness $S_p(\rho)$ is $(\cos\rho + |\sin\rho|)^{-1}$. An approach of defining subclasses of $S_p(\rho)$ builds on the observation (1.2) that for $f \in S_p(\rho)$,

$$A = e^{i\rho} \frac{zf'(z)}{f(z)}, \qquad z \in D$$

is contained in the right half-plane, $\Re(z) > 0$. Subclasses can be obtained nicely by restriction to functions f for which A is contain in geometrically meaningful subsets of the right half-plane. Libera [1, 2] introduced the first formulation of subclasses of $S_p(\rho)$ by placing A in half-plane contained in $\Re(z) > 0$. The class of $\rho - spiralike$ functions of order β $(0 \le \beta < 1)$, denoted by $S_p(\rho, \beta)$, to be the set of function f of the form (1.1) that satisfy

$$\Re\left\{e^{i\rho} \frac{zf'(z)}{f(z)}\right\} > \beta\cos\rho, \quad z \in E.$$

Definition 1.1: Let $f \in S$, suppose $0 \le \beta < 1$, $0 \le \rho < \frac{\pi}{2}$ and λ is real. Then $f(z)$ belongs to the class $S_p(\rho, \lambda, \beta)$ *of* $\lambda - pesudo - spiralike$ functions of order β in the unit disk if and only if

$$\Re\left\{e^{i\rho} \frac{zf'(z)^{\lambda}}{f(z)}\right\} > \beta\cos\rho, \quad z \in E. \tag{1.3}$$

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The following remarks are noteworthy:

(i.) All power shall mean principal determination only throughout this work. Now, if $\rho = 0$ we have the class of $\lambda -$ *pseudo – starlike functions of order* β.

(ii.) If $\rho = 0,\ \lambda = 1$ we have the class of starlike functions of order β which in turn could be called 1- *pseudo– starlike* functions of order β.

(iii.) For $\lambda = 2$ we shall have the class of functions that satisfies

$$\Re\left\{e^{i\rho} f'(z)\frac{zf'(z)}{f(z)}\right\} > \beta\cos\rho, \quad z \in E$$

which is a product of combination of geometric expression for bounded turning and spiralike functions. More details on the above definition can be seen in Babalola [8].

Now, in 1976, Noonan and Thomas [9] stated the q^{th} Hankel determinant for $q \geq 1$ and $m \geq 1$ as

$$H_q(m) = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+q-1} \\ a_{m+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m+q-1} & \cdots & \cdots & a_{m+2q-2} \end{vmatrix}.$$

Several researchers have studied the above determinant extensively and several interesting results were obtained. For more details on the determinant $H_q(m)$ interested reader can refer to [9-15]. Obviously, one can observe that the Fekete and Szego functional is the one given by $H_2(1)$ that is, when $q = 2$ *and* $m = 1$. Fekete and Szego [16] further generalized the estimate $\left|a_3 - \mu a_2^2\right|$ where μ is real and $f \in S$. However, for our present discussion, we shall consider the Hankel determinant in which case $q = 2$ *and* $m = 2$, such that for function *f* of the form (1.1), we have

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a4 \end{vmatrix}.$$

Here, we seek sharp upper bound of the functional $\left|a_2 a_4 - a_3^2\right|$ for functions belonging to the analytic class $S_p(\rho, \lambda, \beta)$ defined in (1.3).

It is noted that the results proved by Mehrok and Singh [13] and Janteng et al. [17] follow as special cases in the present work. At this juncture, the following Lemmas shall be necessary.

Let P denotes the family of all functions p analytic in E for which $\Re\{p(z)\} > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^3 + p_3 z^3 + \ldots, z \in E. \tag{1.4}$$

Then the following Lemmas are immediate.

Lemma 1.1: [18]. If $p \in P$, then $|p_k| \leq 2$ for each k $(k = 1, 2, 3, \ldots)$.

Lemma 1.2: [17]. If $p \in P$, then

$$P_2 = \frac{1}{2}\left\{P_1^2 + \left(4 - P_1^2\right)x\right\}$$

$$p_3 = \frac{1}{4}\left\{p_1^3 + 2p_1(4 - p_1^2)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z\right\}$$

for some x and z satisfying

$|x| \leq 1,\ |z| \leq 1$ *and* $p_1 \in [0,2]$.

2. Main Results

Theorem 2.1: If $f \in S_p(\rho, \lambda, \beta)$, then

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{4(e^{i\rho} - \beta\cos\rho)^2}{(3\lambda - 1)^2}. \tag{2.1}$$

Proof: Since $f \in S_p(\rho, \lambda, \beta)$, then there exists $p \in P$ such that

$$e^{i\rho} z\left(f'(z)^{\lambda}\right) = f(z)\left[\beta\cos\rho + \left(e^{i\rho} - \beta\cos\rho\right)p(z)\right] \tag{2.2}$$

By comparing the coefficients in (2.2), we have that

$$a_2 = \frac{p_1(e^{i\rho} - \beta\cos\rho)}{(2\lambda e^{i\rho} - e^{i\rho})}, \tag{2.3}$$

$$a_3 = \frac{(e^{i\rho} - \beta\cos\rho)}{(3\lambda e^{i\rho} - e^{i\rho})}\left\{ p_2 - \frac{(e^{i\rho} - \beta\cos\rho)(2\lambda^2 e^{i\rho} - 4\lambda e^{i\rho} + e^{i\rho})}{(2\lambda e^{i\rho} - e^{i\rho})^2 (3\lambda e^{i\rho} - e^{i\rho})} p_1^2 \right\} \quad (2.4)$$

and

$$a_4 = \frac{(e^{i\rho} - \beta\cos\rho)}{(4\lambda e^{i\rho} - e^{i\rho})} p_3 + \frac{(e^{i\rho} - \beta\cos\rho)^2 (11\lambda e^{i\rho} - 6\lambda^2 e^{i\rho} + 2e^{i\rho})}{(2\lambda e^{i\rho} - e^{i\rho})(3\lambda e^{i\rho} - e^{i\rho})(4\lambda e^{i\rho} - e^{i\rho})} p_1 p_2$$

$$+ \frac{(e^{i\rho} - \beta\cos\rho)^3 (24\lambda^4 e^{i2\rho} - 80\lambda^3 e^{i2\rho} + 84\lambda^2 e^{i2\rho} - 28\lambda e^{i2\rho} + 3e^{i3\rho})}{3(2\lambda e^{i\rho} - e^{i\rho})^3 (3\lambda e^{i\rho} - e^{i\rho})(4\lambda e^{i\rho} - e^{i\rho})} p_1^3. \quad (2.5)$$

Now,

$$a_2 a_4 - a_3^2 = \frac{1}{H(\lambda)}\left\{ A p_1^4 + B p_1^2 p_2 + C p_1 p_3 - D p_2^2 \right\}$$

where

$$A = \left(e^{i\rho} - \beta\cos\rho\right)^4 \left(24\lambda^5 e^{i3\rho} - 60\lambda^4 e^{i3\rho} + 44\lambda^3 e^{i3\rho} + 12\lambda^2 e^{i3\rho} + \lambda e^{i3\rho}\right),$$

$$B = 3\left(e^{i\rho} - \beta\cos\rho\right)^3 \left(2\lambda e^{i\rho} - e^{i\rho}\right)^2 \left(-2\lambda^3 e^{i2\rho} - 3\lambda^2 e^{i\rho}\left(e^{i\rho} - 2\right) - \lambda e^{i2\rho} - 4e^{i2\rho}\right),$$

$$C = 3\left(e^{i\rho} - \beta\cos\rho\right)^2 \left(2\lambda e^{i\rho} - e^{i\rho}\right)^3 \left(3\lambda e^{i\rho} - e^{i\rho}\right)^2,$$

$$D = 3\left(e^{i\rho} - \beta\cos\rho\right)^2 \left(2\lambda e^{i\rho} - e^{i\rho}\right)^4 \left(4\lambda e^{i\rho} - e^{i\rho}\right)$$

and

$$H(\lambda) = 4.3\left(2\lambda e^{i\rho} - e^{i\rho}\right)^4 \left(3\lambda e^{i\rho} - e^{i\rho}\right)^2 \left(4\lambda e^{i\rho} - e^{i\rho}\right).$$

Applying Lemma (1.1) and Lemma (1.2), *we obtain*

$$\left|a_2 a_4 - a_3^2\right| = \frac{1}{4H(\lambda)}\left| \begin{array}{l} p_1^4(4A + 2B + C - D) - 2p_1^2(D - C - B)\left(4 - p_1^2\right)x \\ -\left(Cp_1^2 + D\left(4 + p_1^2\right)\right)\left(4 + p_1^2\right)x^2 + 2Cp_1\left(4 - p_1^2\right)\left(1 - |x|^2\right)z \end{array} \right|.$$

Assuming $p_1 = p \in [0,2]$ and using triangle inequality with $|z| \le 1$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{4H^*(\lambda)}\left\{ \begin{array}{l} p^4\left(4A^* + 2B^* + C^* - D^*\right) + 2p^2\left(D^* - C^* - B^*\right)\left(4 - P^2\right)|x| \\ +\left(C^* p^2 + D^*\left(4 - p^2\right)\right)\left(4 - p^2\right)|x|^2 + 2C^* p_1\left(4 - p^2\right)\left(1 - |x|^2\right) \end{array} \right\}$$

where

$$A^* = \left(e^{i\rho} - \beta\cos\right)^4 \left(24\lambda^5 - 60\lambda^4 + 44\lambda^3 - 12\lambda^2 + \lambda\right),$$

$$B^* = 3\left(e^{i\rho} - \beta\cos\rho\right)^3 (2\lambda - 1)^2 \left(-2\lambda^3 - 3\lambda^2\left(e^{i\rho} - 2\right) - \lambda - 4\right),$$

$$C^* = 3\left(e^{i\rho} - \beta\cos\rho\right)^3 (2\lambda - 1)^3 (3\lambda - 1)^2, D^* = 3\left(e^{i\rho} - \beta\cos\rho\right)^2 (2\lambda - 1)^4 (4\lambda - 1)$$

and

$$H^*(\lambda) = 4.3(2\lambda - 1)^4 (3\lambda - 1)^2 (4\lambda - 1).$$

If we set $\delta = |x| \le 1$, then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{4H^*(\lambda)}\left\{ \begin{array}{l} p^4\left(4A^* + 2B^* + C^* - D^*\right) + 2p^2\left(D^* - C^* - B^*\right)\left(4 - P^2\right)\delta \\ +\left(C^* p^2 + D^*\left(4 - p^2\right) - 2C^* p\right)\left(4 - p^2\right)\delta^2 + 2C^* p_1\left(4 - p^2\right) \end{array} \right\}$$

where

$$F(p,\delta) = p^4\left(4A^* + 2B^* + C^* - D^*\right) + 2p^2\left(D^* - C^* - B^*\right)\left(4 - P^2\right)\delta$$
$$+\left(C^* p^2 + D^*\left(4 - p^2\right) - 2C^* p\right)\left(4 - p^2\right)\delta^2 + 2C^* p_1\left(4 - p^2\right)$$

is an increasing function.

Now,

$$Max.F(p, \delta) = F(p, 1) = G(p).$$

Consequently,

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{4H^*(\lambda)} G(p) \quad (2.6)$$

Then,

$$G(p) = M(\lambda, \beta)p^4 - N(\lambda, \beta)p^2 + 16D^*$$

where,

$$M(\lambda, \beta) = 4A^* + 4B^* + 2C^* - 2D^*, \quad N(\lambda, \beta) = 8B^* + 4C^*$$

$A^*,\ B^*,\ C^*\ and\ \ D^*$ are as earlier defined.

So,

$G'(p)=4M(\lambda,\beta)p^3-2N(\lambda,\beta)p\ \ and\ \ G''(p)=12M(\lambda,\beta)p^2-2N(\lambda,\beta)$

such that if $G'(p)=0$, then

$2p[2M(\lambda,\beta)p^2-N(\lambda,\beta)]=0.$

Therefore $G(p)$ attains it maximum value at $p=0$ and max $G(p)=G(0)$. Hence, from (2.6), we obtain our result. The result is sharp for $p_1=0,\ p_2=-2\ \ and\ \ p_3=0$.

Corollary 2.1: If $f\in S_p(\rho,\lambda,0)$, then

$$\left|a_2a_4-a_3^2\right|\le\frac{4e^{i2\rho}}{(3\lambda-1)^2}. \qquad (2.7)$$

Corollary 2.2: If $f\in S_p(0,\lambda,\beta)$, then

$$\left|a_2a_4-a_3^2\right|\le\frac{4(1-\beta)^2}{(3\lambda-1)^2}. \qquad (2.8)$$

Corollary 2.3: If $f\in S_p(\rho,1,\beta)$, then

$$\left|a_2a_4-a_3^2\right|\le(e^{i\rho}-\beta\cos\rho)^2 \qquad (2.9)$$

Corollary 2.4: If $f\in S_p(0,1,\beta)$, then

$$\left|a_2a_4-a_3^2\right|\le(1-\beta)^2 \qquad (2.10)$$

Corollary 2.5: If $f\in S_p(0,1,0)$, then

$$\left|a_2a_4-a_3^2\right|\le 1. \qquad (2.11)$$

Incidentally, (2.11) coincide with the result obtained by Janteng et al. [17]. For recent work on Hankel determinant, interested reader can refer to [19-21] among others.

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