Transactions of the Nigerian Association of Mathematical Physics
Volume 7, (March, 2018), pp229 – 232

BROWNIAN MOTION PROCESS: AN APPLICATION TO INVESTMENT FINANCE

Tsetimi J.[1], Disu A.B.[2] and Mamadu E.J.[3]

[1]Department of Mathematics, Delta State University, Abraka, Nigeria.
[2] Department of Mathematics, National Open University of Nigeria, Abuja.
[3] Department of Mathematics, University of Ilorin, P.M.B 1515, Ilorin, Nigeria.

Abstract

In this paper, a numerical stimulation of one-dimensional Brownian motion process to financial investment is considered. For this purpose, we treat the one-dimensional Brownian motion process in reference to constant and time-dependent drift and volatility parameters in a given time-steps in yearly returns for an investor to invest to maximize some utility functions such as wealth or cash flow. The Euler-Marayama scheme was applied for discretizing the Brownian motion process. Results are presented graphically, and showed that for constant drift and volatility 50 to 100 time-steps is a risky investment which is at least profitable than safe investment for drift<volatility, and vice versa. However, for time dependent drift and volatility, the results showed that an investor making risky or safe investment fluctuates. All computational frameworks of this research are implemented using the software Maple 18.

***Keywords*:** Stochastic differential equation, drift volatility, Brownian motion, Ito process.

1.0 Introduction

Brownian motion process is a stochastic process X_t, governed by the stochastic differential equation (SDE)

$$X_t = \alpha(X_t)dt + \beta(X_t)dW_t, \quad (1)$$

where $\alpha(X_t)$ is the drift, $\beta(X_t)$ is the volatility and W_t is the standard Wiener process $\{W_t: t \geq 0\}$. In many real life applications equation (1) results from internally or externally random fluctuation in the dynamical analysis of a system. A typical example is the molecular bombardment of dust on the surface of water, which results in Browian motion [1].

Considering the random fluctuations of share prices on the stock exchange market, it is imperative to model price dynamics or investment finance as stochastic differential as equations [2]. The idea of applying SDE to investment finance was first conceived in [3]. The method in [3] enabled the investor to decide between two classes of investments, one safe and the other risky. Merton [3] also argued that an investor will maximize some class of utility functions such as wealth or cash flow by implementing some investment strategy to avoid possible hiccups such as bankruptcy. A price dynamics model can be formulated from the exponential growth of an ordinary differential equation as follows [3];

$$\frac{dp_s}{dt} = \alpha p_s, \ \alpha > 0, \quad (2)$$

where p_s is the price of safe investment. We also observe that noisy fluctuation in stock exchange market is proportional to the share price. Thus, *Equation (2)* is remodeled as;

$$\frac{dp_s}{dt} = ap_s + ap_s\varepsilon_t, \quad (3)$$

where ε_t is the Gaussian white noise. *Equation (3)* can be interpreted as an Ito stochastic differential equation

$$dp_t^s = ap_t^s dt + bp_t^s dW_t, \quad (4)$$

where W_t is the standard Wiener process $\{W_t: t \geq 0\}$ and $a, b > 0$.

Several researchers over the years have studied the random fluctuations of share prices in real and ambiguous investment processes. For instance, the effect of neo-additive noise capacities on ambiguous investment in a Choquest-Brownian motion setting was studied in [4]. In like manner, the stimulation of stock price trajectories to checkmate the effect of the actual stock price reversal was explored in [5]. Crichowsky et. al. [6] considered the characteristics of shadow prices under a fractional Brownian motion process for portfolio optimization under transaction costs. The effect of optimal investment under uncertainty in a competitive environment was considered in [7].

Corresponding Author: Tsetimi J., Email: tsetimi@yahoo.com, Tel: +2348068561884

In this paper, we aim at exploring the numerical stimulation of one-dimensional Brownian motion process to investment finance with constant and time-dependent drift and volatility in the share prices in an algebraic form in a given time-steps (an aspect Merton's model [1] did not explore explicitly). The analysis and results presented here will enable investors in implementing strategy in a specified time-steps in yearly returns to avoid possible hiccups such as bankruptcy. The time-steps we mean the correspond to the time frame for an investor to maximize utility functions such as wealth or cash flow in a constant or time-dependent drift and volatility share prices in the stock exchange.

2.0 Definition of Parameters

Useful definition of parameters can be seen in [2], and are presented in the table below.

Table 1: Prevalent Parameters for application of Brownian Motion to investment finance.

x_0	Initial value (real constant)
$\alpha = \alpha(X_t)$	Drift parameter
$\beta = \beta(X_t)$	Volatility parameter
X_0	Vector (initial value)
Ω	Vector (drift parameter)
T	Time parameter
$\sum$	Covariance matrix

3.0 Brownian Motion

The Brownian motion (x_0, α, β) and (x_0, α, β, t) provides a new one-dimensional Brownian motion process. This Brownian motion process is a stochastic process governed by the SDE in *Equation (1)* [2]

Remark 1:

I. If α is a real constant, then α must be a real number.

II. If α is a time-dependent drift, then α can be written as an algebraic expression.

III. β can either be constant or time dependent.

IV. The Brownian motion $(X_0, \Omega, \sum)$ gives an n-dimensional Brownian motion with Ω and $\sum$ .It is defined uniquely by the SDE [9]

$$dX_t = Mdt + NdW_t, \qquad (5)$$

where M is a n-dimensional vector, N is a $n \times m$ - matrix and $NN^T = \sum$. W_t is a Wiener process $\{W_t : t \geq 0\}$ of m-dimensional.

V. If Ω and $\sum$ are time-dependent parameters, then Ω must be presented as a vector, and $\sum$ must be a symmetric matrix.

4.0 Euler – Marayuma Scheme

In this section, the Euler scheme is implemented for executing the Brownian motion process for the investment with the aid of Maple 18 software.

Let an Ito process $\{X_t : t \in [t_0, T]\}$ satisfying the scalar SDE be given as [2]

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t,\ t \in [t_0, T], \qquad (6)$$

with the initial value

$$X_{t_0} = X_0. \qquad (7)$$

Discretizing *Equation (6)* in the interval $t \in [t_0, T]$, as $t_0 = \gamma_1 < \gamma_2 < \gamma_3 < \cdots < \gamma_n < \cdots \gamma_N = T$, then the Euler or Euler-Marayuma approximation is a time-dependent stochastic process $S = \{S(t) : t \in [t_0, T]\}$ satisfying the scheme

$$S_{n+1} = S_n + (\gamma_n, S_n)(\gamma_{n+1} - \gamma_n) + \beta(\gamma_n, S_n)(W_{\gamma_{n+1}} - W_{\gamma_n}),\ S_0 = X_0. \qquad (8)$$

5.0 Numerical Stimulations

In this section, we implement some numerical stimulations for Brownian motion process using the Euler-Marayuma scheme with constant and time-dependent drift and volatility parameters with the aid of Maple 18 software.

a. Maple code for Brownian motion process with constant drift and volatility.

```
> restart :
> with(Finance) :
> T := 2
> X := BrownianMotion(0.0, 0.02, 0.3) :
> S_n := SamplePath(X(t), t = 0 ..T, timesteps = 100, replications = 10)
```

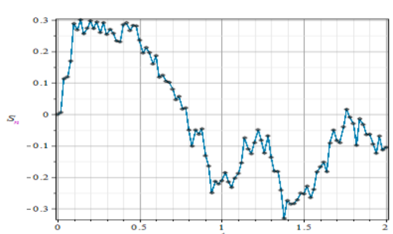

Figure 1. $Pathplot\ for\ X(t)\ with\ timesteps = 100.$

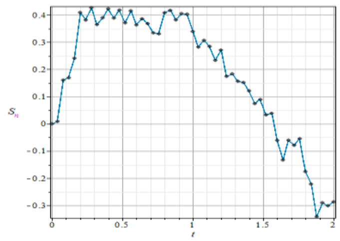

Figure 2. $Pathplot\ for\ X(t)\ with\ timesteps = 50.$

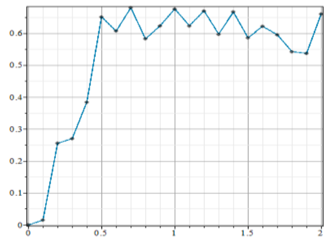

Figure 3. $Pathplot\ for\ X(t)\ with\ timesteps = 20.$

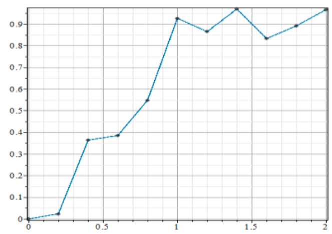

Figure 4. $Pathplot\ for\ X(t)\ with\ timesteps = 10.$

b. One-dimensional Brownian motion with time-dependent parameters given in algebraic form.

> $restart:$

> $\alpha := piecewise\left(t < 1, 0.02, \frac{0.02}{t}\right)$

$$\alpha := \begin{cases} 0.02 & t < 1 \\ \frac{0.02}{t} & otherwise \end{cases}$$

> $\beta := evalf_3(CurveFitting\text{:-}Spline([[0, 0.5], [1, 0.3], [1.5, 1.5], [2, 1.0]], t, degree = 3))$

$$\beta := \begin{cases} 0.500 - 1.40\,t + 1.20\,t^3 & t < 1. \\ -1.90 + 2.20\,t + 3.60\,(-1. + t)^2 - 6.40\,(-1. + t)^3 & t < 1.5 \\ 1.\,t - 6.\,(t - 1.5)^2 + 4.\,(t - 1.5)^3 & otherwise \end{cases}$$

> $X_t := BrownianMotion(0.0, \alpha, \beta, t):$

> $S := SamplePath(X(t), t = 0..T, replications = 10^6):$

6. Conclusion

We have treated in this paper the one-dimensional Brownian motion process in reference to constant and time-dependent drift and volatility parameters in a given time-steps in yearly returns for an investor to invest to maximize some utility functions such as wealth or cash flow. Results were obtained using Maple 18 software and are presented graphically showing that for constant drift and volatility 50 to 100 time-steps is a risky investment which is at least profitable than safe investment for drift<volatility, and vice versa. However, for time dependent drift and volatility, the results showed that an investor making risky or safe investment fluctuates.

References

[1] Brown, B.M. and Hewitt, J.T. (1972). Asymptotic likelihood theorem for diffusion processes. J. Appl. Prob. **12, 228-238.**

[2] Kloeden, P.E. (1992). Numerical solution of stochastic differential equations. Springer-Verlag, New York

[3] Merton, R.C. (1973). Theory of rational option pricing. Bell. J. Econs. Manag. Sc. **4, 141-183.**

[4] Kongsgaard , N.S. (2013). Ambiguous investment under Choquet-Brownian motion with neo-additive capacities, Master's Thesis, Department of Economics, Copenhagen Business School.

[5] Reddy, K. and Clinton, Y. (2016). Simulating stock prices using geometric Brownian motion: Evidence from Australian companies, Australasian Accounting, Business and Finance Journal, 10(3), 2016, 23-47. doi:10.14453/aabfj.v10i3.3

[6] Czichowsk, C., Peyre, R., Schachermayer, W., Yang, J. (2016). Shadow prices, fractional Brownian motion, and portfolio optimisation under transaction costs, arXiv:1608.01415v1 [q-fin.MF] 4 Aug 2016.

[7] Novy-Marx, R. (2004). An equilibrium model of investment under uncertainty, graduate school of business, University of Chicago.

[8] Kloeden, P.E. and Pearson, R.A. (1977). The numerical solution of stochastic differential equations. J. Austral. Math. Soc., Series B, **20, 8-12.**

[9] Karandikar, R.L. (1981). Pathwise solution of stochastic differential equations. Sankhya **43A, 121-132.**