Transactions of the Nigerian Association of Mathematical Physics
Volume 7, (*March, 2018*), **pp221 – 224**
© *Trans. of NAMP*

QUANTUM THEORY OF SIMPLE LINEAR HARMONIC OSCILLATOR UNDER RIEMMANIAN GEOMETRY

L. W. Lumbi[1], I. I. Ewa[1],S. D. Yusuf[1], A. Z. Loko[1], I.K. Joseph[2],and E. N. Chifu[3]

[1]Physics Department, Nasarawa State University, Keffi, P.M.B.1022, Keffi, Nigeria.
[2]Mathematics Department, Nasarawa State University, Keffi, P.M.B.1022, Keffi, Nigeria.
[3]Physics Department, Federal University Dutse, P.M.B. 7156 Dutse, Nigeria.

Abstract

In this article the Riemannian geometry were applied to Schrodinger equation of simple linear harmonic oscillator to obtain generalized Eigen energies and Eigen function. The results obtained contain additional correction terms which are not found in the well-known Schrodinger's Eigen energies which were derived based upon the Euclidean geometry of space.
These additional correction term scan be used to explain the vibration spectra of diatomic molecules such as Hydrogen Fluoride (HF), Hydrogen gas (H_2), Nitrogen gas (N_2)and Chlorine gas (Cl_2) as well as to Schrodinger's mechanical wave equation for finite, infinite and rectangular potential well.

***Keywords*:** Riemannian geometry, Schrodinger Equation, Linear Harmonic Oscillator, Eigen Energies and Euclidean geometry.

1.0 Introduction

In the year 1686 Isaac Newton published his book entitled The Mathematical Philosophy of Natural Philosophy to lay the foundation of his dynamical Mathematical theories of classical mechanics in particular and Physics in general based upon Euclidean coordinate geometry of space [1].
In the year 1860 James Clark Maxwell formulated his dynamical Mathematical theory of classical electrodynamics based upon Euclidean coordinate geometry of space [1].
In the year 1926 Erwin Schrodinger introduced his dynamical Mathematical theory of quantum mechanics based upon Euclidean coordinate geometry of space. These theories have continued to be studied and developed and applied till today [1].
In the year 1854 Georg Friedrich Bernhard Riemann (1826 - 1866) published his theory of geometry and corresponding tensorial classical mechanics in the gravitational field. It is most interesting and instructive to note that F. B. Riemann did not discover any metric tensor(s) for any gravitational field in nature because they require the use of the speed of light in vacuum as the universal scaling constant [1, 2].
In 2013 S. X. K. Howusu in his book titled Riemannian Revolution in Physics and Mathematics discovered a unique metric tensor for all gravitational fields in nature that is necessary and sufficient for the formulation of theoretical Physics based upon Riemannian geometry [1, 2].
In this article the Riemannian Laplacian operator is applied to Schrodinger equation of simple linear harmonic oscillator in two- dimensionsto obtain generalized Eigen energies.

2.0 Theoretical Analysis

The unique metric tensor for all gravitational fields in nature is given explicitly by [1, 2]

$$g_{00} = \left(1 + \frac{2}{c^2} f\right) \qquad (1)$$

$$g_{11} = \left(1 + \frac{2}{c^2} f\right)^{-1} \qquad (2)$$

$$g_{22} = r^2 \qquad (3)$$

Corresponding Author: Lumbi L.W., Email: williamslucas44@gmail.com, Tel: +2348030849994, +2348096130493

$$g_{33} = r^2 \sin^2 \theta \tag{4}$$

$$g_{\mu\nu} = 0 \text{; otherwise} \tag{5}$$

In our previous paper [4], these metric tensors were used to formulate Riemannian Laplacian operator in Cartesian coordinate for all gravitational fields in nature.
Assuming that $f = 0$, the Riemannian Laplacian operator in two - dimensions(r, θ, t) reduces to Einstein or Minkowski coordinates which is given explicitly by [1- 4]

$$\nabla_R^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2} \tag{6}$$

The time-dependent Schrodinger's equation in two - dimension is given explicitly by [4, 5]

$$i\hbar \frac{\partial}{\partial t}\psi(r,\theta,t) = \left\{ -\frac{\hbar^2}{2m_0}\nabla_E^2 + \frac{1}{2}m_0\omega_0^2 r^2 \right\}\psi(r,\theta,t) \tag{7}$$

where, ∇_E^2 is the Euclidean Laplacian operator which were derived based upon the Euclidean geometry.
Applying Riemannian Laplacian operator in equation (7) becomes

$$i\hbar \frac{\partial}{\partial t}\psi(r,\theta,t) = \left\{ -\frac{\hbar^2}{2m_0}\nabla_R^2 + \frac{1}{2}m_0\omega_0^2 r^2 \right\}\psi(r,\theta,t) \tag{8}$$

where,
∇_R^2 is the Riemannian Laplacian operator which were derived based upon the
Riemannian geometry.
$\psi(r, \theta, t)$ is wave function of the quantum system
i is imaginary unit
$\hbar$ is the reduced Planck constant which is given by $\hbar = \frac{h}{2\pi}$
$\frac{\partial}{\partial t}$ ispartial derivative $w.r.t$ time t
r and θ are the position vector
t is the time
Putting equation (6) into (8) we obtain

$$i\hbar \frac{\partial}{\partial t}\psi(r,\theta,t) = \left\{ -\frac{\hbar^2}{2m_0}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r^2} \right) + \frac{1}{2}m_0\omega_0^2 r^2 \right\}\psi(r,\theta,t) \tag{9}$$

Let us seek the method of separation of variables as;

$$\psi(r,\theta,t) = R(r)\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] \tag{10}$$

where, E is the quantum mechanical energy and $R(r)$ is the quantum mechanical energy wave function.
Differentiating partially $w.r.t\ x, \theta$ and t respectively, we have

$$\frac{\partial \psi}{\partial r} = \frac{dR}{dr}\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] \tag{11}$$

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{d^2R}{dr^2}\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] \tag{12}$$

$$\frac{\partial \psi}{\partial \theta} = iER(r)\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] \tag{13}$$

$$\frac{\partial^2 \psi}{\partial \theta^2} = -E^2R(r)\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] \tag{14}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -R(r)\frac{E^2}{\hbar^2}\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] \tag{15}$$

Putting equations (11) – (15) into equation (9), we obtain

$$i\hbar\left\{ \frac{-iE}{\hbar}\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] R(r) \right\} =$$

$$-\frac{\hbar^2}{2m_0}\left\{ \frac{-R(r)\frac{E^2}{\hbar^2}\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right]}{c^2} - R(r)E^2\exp\left[iE\left(\theta - \frac{t}{\hbar} \right) \right] \right.$$

$$+\frac{d^2R}{dr^2}\exp\left[iE\left(\theta-\frac{t}{\hbar}\right)\right]+\frac{1}{2}m_0\omega_0^{\,2}r^2\Bigg\}R(r)\exp\left[iE\left(\theta-\frac{t}{\hbar}\right)\right] \tag{16}$$

Dividing (11) equation through by $\exp\left[iE\left(\theta-\frac{t}{\hbar}\right)\right]$ and simplifying we obtain

$$-\frac{\hbar^2}{2m_0}R''(r)+\left\{\left(\frac{E^2}{2m_0c^2}+\frac{\hbar^2E^2}{2m_0}-E\right)+\frac{1}{2}m_0\omega_0r^2\right\}R(r)=0 \tag{17}$$

Let ξ be a new independent variable defined by;

$$\xi=\left(\frac{m_0\omega_0}{\hbar}\right)^{\frac{1}{2}}r \tag{18}$$

$$\xi^2=\frac{m_0\omega_0}{\hbar}r^2 \tag{19}$$

$$r^2=\xi^2\frac{\hbar}{m_0\omega_0} \tag{20}$$

$$r=\xi\left(\frac{\hbar}{m_0\omega_0}\right)^{\frac{1}{2}} \tag{21}$$

Using the chain rule

$$\frac{dR}{d\xi}=\frac{dR}{dr}.\frac{dr}{d\xi} \tag{22}$$

but

$$\frac{dx}{d\xi}=\left(\frac{\hbar}{m_0\omega_0}\right)^{\frac{1}{2}} \tag{23}$$

$$\frac{d^2R}{dr^2}=\frac{d^2R}{d\xi^2}\frac{m_0\omega_0}{\hbar} \tag{24}$$

Using equation (24) into equation (17) and simplifying we have

$$R^{''}(\xi)+\left\{\frac{2E}{\hbar m_0}-\frac{E^2}{m_0\omega_0^{\,2}}-\frac{E^2}{m_0\omega_0^{\,2}\hbar c^2}-\xi^2\right\}R(\xi)=0 \tag{25}$$

or

$$R^{''}(\xi)+\left\{\lambda-\xi^2\right\}R(\xi)=0 \tag{26}$$

where,

$$\lambda=\frac{2E}{\hbar m_0}-\frac{E^2}{m_0\omega_0^{\,2}}-\frac{E^2}{2m_0\omega_0^{\,2}\hbar c^2} \tag{27}$$

For large ξ the function $\Phi(\xi)=\exp\left(\frac{-\xi^2}{2}\right)$ satisfy equation (26)

$$\Phi'(\xi)=-\xi\Phi(\xi) \tag{28}$$

$$\Phi^{''}(\xi)=-\Phi(\xi)+\xi^2\Phi(\xi) \tag{29}$$

Substituting $R(\xi)=F(\xi)\Phi(\xi)$ in (14)

$$F^{11}(\xi)-2\xi F^1(\xi)+(\lambda-1)F(\xi)=0 \tag{30}$$

This is the Hermite -Eigen equation with eigenvalue;

$$(\lambda-1)=2n;\;\; n=0,1,2,... \tag{31}$$

or

$$\lambda=2n+1;\;\; n=0,1,2,... \tag{32}$$

It follows from equation (26) and (31) that

$$2E^2-\frac{2E}{k}m_0c^2+\frac{\hbar m_0\omega_0c^2}{k}(2n+1)=0 \tag{33}$$

Solving equation (33) quadratically, we obtain two solutions in which one is positive (mathematically correct and physically realistic) and the other is negative (mathematically correct and but physically not realistic). The mathematically correct and physically realistic solution is given explicitly as

$$E_n=\frac{1}{2}(2n+1)\hbar\omega_0-\frac{1}{8}\frac{(2n+1)^2\hbar^2\omega_0}{m_0c^2}-\frac{1}{8}\frac{(2n+1)^2\hbar^3\omega_0}{m_0}+... \tag{34}$$

where, E_n is the Riemannian Eigen energies of the simple harmonic oscillator.

When, $n = 0, 1, 2, 3, 4, \ldots$

$$E_0 = \frac{1}{2}\hbar\omega_0 - \frac{1}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \frac{1}{8}\frac{\hbar^3\omega_0}{m_0} - \ldots \quad (35)$$

$$E_1 = \frac{3}{2}\hbar\omega_0 - \frac{9}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \frac{9}{8}\frac{\hbar^3\omega_0}{m_0} - \ldots \quad (36)$$

$$E_2 = \frac{5}{2}\hbar\omega_0 - \frac{25}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \frac{25}{8}\frac{\hbar^3\omega_0}{m_0} - \ldots \quad (37)$$

$$E_3 = \frac{7}{2}\hbar\omega_0 - \frac{49}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \frac{49}{8}\frac{\hbar^3\omega_0}{m_0} - \ldots \quad (38)$$

$$E_4 = \frac{9}{2}\hbar\omega_0 - \frac{81}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \frac{81}{8}\frac{\hbar^3\omega_0}{m_0} - \ldots \quad (39)$$

where, E_0, E_1, E_2, E_3 and E_4 are the ground, first, second, third and fourth energy levels of the simple harmonic oscillator. Equation (34) is the generalized Riemannian Eigen energies of simple linear harmonic oscillator. The ground energy level E_0, first energy level E_1, second energy level E_2, third energy level E_3 and fourth energy level E_4 are given by equations (35), (36), (37), (38) and (39) respectively. The leading terms on the right hand side of these equations are the well - known Schrodinger's quantum Eigen energies of linear simple harmonic oscillator while the remaining terms are additional correction terms which are not found in the well - known Schrodinger's quantum Eigen energies and [4]. It must be noted that the third term on the right hand side of these equations is $c^2\hbar$ times the second term.

The consequences of these additional correction terms are that they can be applied in the areas of theoretical and experimental Physics (Solid state Physics, Thermal Physics and Elementary particle Physics) as well as to Schrodinger's mechanical wave equation such as for finite, infinite and rectangular potential well.

References

[1] Howusu, S.X.K. (2010). Complete Dynamic Theory of Physics, Jos University Press Ltd, Pp. 71- 90

[2] Howusu, S.X.K. (2013). Riemannian Revolution in Physics and Mathematics,Jos University Press Ltd. Pp. 50 – 190

[3] Lumbi, W. L., Howusu, S.X.K. and Liman, M.S. (2014). Generalized Dynamical Gravitational Field Equation for Static Homogeneous Spherical Massive Distribution of Mass. *International Journal of Modern Theoretical Physics 3(1): 37- 43*

[4] John, V. N. (1955). Mathematical Foundation of Quantum Mechanics, Princeton University Press. ISBN 0691028931

[5] Omonile, J. F., Abalakav, J. and Howusu, S. X. K. (2015).Contribution of Riemannian Geometry to Quantum Theory of Linear Simple Harmonic Oscillator Problem. *International Journal of Advance Research in Engineering and Management (IJAREM) Vol. 01: 9 – 12.*