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ORTHOGONAL DECOMPOSITION FOR THE REDUCED ORDER MODEL SYSTEM OF ODE

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Abstract

The paper presents orthogonal decomposition (POD) for reduced order model for system of ordinary differential equation obtained from using four stage –fourth order Runge-Kutta method subject to least squares approach. We give the Wallis factor for these phenomena and showed that the solutions to the slope matrices for the Runge-Kutta method in the subspace integration satisfy the Polarization identity. It is established that the rank deficient matrix arising there from in these slope matrices could be amenable to Tikhonov regularization parameter subject to Givens orthogonal transformation for the singular values decomposition (SVD).The procedures for de-noising solution space in the data have been highlighted. Cholesky Factorization used on the reduced symmetric matrix to tridiagonal matrix by the Givens orthogonal matrix similarity transformation is given. Numerical example is discussed with the described methods with huge success.

Keywords: system of ODE, Runge-Kutta method, slope matrix, polarization identity, Least squares method
AMS Subject Category: 65L10, 65L99, 65 F20, 65G50

1.0 Introduction

The paper considers solving system of ordinary differential equation (ODE_s) using proper orthogonal decomposition on the reduced order model facilitated by the Runge-Kutta method.
The system of ordinary differential equation (ODEs) is given in the form:

$$\frac{dy}{dt} = f(t,y),\ y(t_0) = y_0\ , \qquad (1.1)$$

Where $f : R^n \times R \to R^n$. Such a system of Equation (1.1) has existence and uniqueness theorem which is given below.

Theorem 1.1,[1] . Let f be defined and continuous on the strip $S = \{(t,y) | a \le t \le b, y \in R^n\}$ where a,b finite. Assuming that there is a constant L such that $\| f(t,y_1) - f(t,y_2)\| \le L\|y_1 - y_2\|, \forall t \in [a,b]$ and $y_1, y_2 \in R^n$ (Lipschitz condition). Then, for every $t_0 \in [a,b]$ and $y_0 \in R^n$, there exists exactly one function $y(t)$ such that :

(i) $y(t)$ is continuous and continuously differentiable for $t \in [a,b]$;

(ii) $y'(t) = f(t, y(t))$ for $t \in [a,b]$;

(iii) $y(t_0) = y_0$.

Theorem 1.2, [1]. Let the function $f : S \to R^n$ be continuous on the strip $S = \{(t,y) | a \le t \le b, y \in R^n\}$, which satisfies the Lipschitz condition $\| f(t,y_1) - f(t,y_2)\| \le L\|y_1 - y_2\|$ for all $(t,y) \in S$. The solution $y(t,s)$ of the initial value problem

$y' = f(t,y), y(t,s) = S$, $\|y(t,s_1) - y(t,s_2)\| \le e^{L\|t-t_0\|}\|s_1 - s_2\|$ holds true.

Definition 1.1; A solution which is stable on $[t_0, \infty]$, i.e., stable on $[t_0, t_p]$ for each t_p and with δ

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Independent of t_p is said to be stable in the sense of Lyapunov. If in addition, $\lim_{t\to\infty}\|v(t)-w(t)\|=0$, then the solution $y=v(t)$ is called asymptotically stable where $v(t),w(t)\in S$.

Wherever it is not explicitly stated it is that Meanvalue theorem holds which guarantees the existence of solution as well as continuity of the function f .

A general One- step method could be written in the form:

$y_{k+1}=y_k+h\phi(t_k,y_k,h)$, ($k=0,1,...;$. with y_0 given); (1.2)

Where,

$\phi(t_k,y_k,h)$ is a continuous function of its variables. The global error is denoted $e_k=y(t_k)-y_k$.

Theorem 1.3,[2]. For a given general One-step method

$y_{k+1}=y_k+h\phi(t_k,y_k,h)$, where it is assumed that ϕ is a continuous function of its arguments and in addition ϕ satisfies the Lipschitz condition with respect to its arguments, viz: there exists a positive constant L such that, for each $0\le h\le h_0$ and for the same region R^n as in Picard's theorem,

$\|\phi(t,y,h)-\phi(t,z,h)\|\le L_\phi\|y-z\|, \forall(t,y),(y,z)\in R$ with the fact that $|y_k-y_0|\le y_m$, then we have

$$|e_k|\le e^{L_0(t_k-t_0)}|e_0|+\left(\frac{e^{L_\phi(t_k-t_0)}}{L_\phi}-1\right)T,\ k=0,1,...,n \tag{1.3}$$

Where, $T=\max_{0\le k\le n-1}|T_k|$ for $|y(t_k)-y_k|\to 0, t_k\to t\in[a,b]$.

To implement Equation (1.2), the Runge-Kutta fourth order method [2] may be executed .Starting with Taylor series for two variables, the infinite series is represented in the form

$f(t+h,y+k)=\sum_{i=0}^{\infty}\frac{1}{i!}\left(h\frac{\partial}{\partial t}+k\frac{\partial}{\partial y}\right)^i f(t,y)$, wherefrom the Runge-Kutta methods can be derived.

Taking into consideration that:

$$(h\frac{\partial}{\partial t}+k\frac{\partial}{\partial y})^1 f(t,y)=h\frac{\partial f}{\partial t}+k\frac{\partial f}{\partial y},$$

$$(h\frac{\partial}{\partial t}+k\frac{\partial}{\partial y})^2 f(t,y)=h^2\frac{\partial^2 f}{\partial t^2}+2hk\frac{\partial^2 f}{\partial t\partial y}+k^2\frac{\partial^2 f}{\partial^2 y},$$

then it can be derived that the Fourth order -Stage four Runge-Kutta method is in the form:

$$y(t,h)=y(t)+\frac{1}{6}(F_1+2F_2+2F_3+F_4)$$

$$F_1=hf(t,y), \tag{1.3}$$

$$F_2=hf\left(t+\frac{1}{2}h,y+\frac{1}{2}F_1\right),$$

$$F_3=hf\left(t+\frac{1}{2}h,y+\frac{1}{2}F_2\right),$$

$$F_4=hf(t+h,y+F_3)$$

We expect solution with their norms to satisfy the Polarization identity**:**

$$\begin{aligned}\|s+u\|^2+\|s-u\|^2&=tr\left((s+u)^T(s+u)\right)+tr\left(s^T(s-u)\right)\\&=2tr\left(s^Ts\right)+2tr\left(u^Tu\right)=2\left(\|s\|^2+\|u\|^2\right)\end{aligned} \tag{1.4}$$

s,u are well defined **.**

The asymptotic contraction is discussed for our purpose. Defined that (T,d) is a non-empty, complete metric space and $f:T\to T$ be such that for each $n\ge 1$ there is a constant α_n for which for all $t,y\in T\left(d\left(f^n(t),f^n(y)\right)\le\alpha_n d(t,y)\right)$,where $\sum_{n=1}^{\infty}\alpha_n<\infty$.

By using the Picard iteration sequences it is implicated that $t\in T$ and for $i\in N$ gives that:

$$\lim_{n\to\infty}\sup d\left(f^{n}(t), f^{n+1}(t)\right) = \lim\sup d\left(f^{n+i}(t), f^{n+i+1}(t)\right) \le \lim_{n\to\infty}\varphi_n\left(d\left(f^{i}(t), f^{i+1}(t)\right)\right)$$
$$=\varphi\left(d\left(f^{i}(t), f^{i+1}(t)\right)\right)$$

as $i \to \infty$.Thus there follow the inequalities

$$\lim_{n\to\infty}\sup d\left(f^{n}(t), f^{n+1}(t)\right) \le \varphi\left(\lim_{n\to\infty}\sup d\,(f^{n}(t), f^{n+1}(t))\right).$$

This means that

$$\lim_{n\to\infty} d\left(f^{n}(t), f^{n+1}(t)\right) = \lim_{n\to\infty}\sup d\left(f^{n}(t), f^{n+1}(t)\right) = 0$$

It remains to show that Cauchy-Schwartz-Bunyakovskii inequality holds for the function f. To show this, let $f \in L^{p}$, and $g \in L^{q}$ where $\frac{1}{p}+\frac{1}{q}=1$.Then consider $f^{\frac{p}{q}} \in L^{q}$.

We apply the Binomial theorem expansion on the function f for the Holder inequality in the form:

$$\left\| f^{\frac{p}{q}} + \tau g \right\|_{p}^{q} \le \left\| f^{\frac{p}{q}} \right\|_{q}^{q} + q\tau \left\| f^{\frac{p}{q}} \right\|_{q}^{q-1} \|g\|_{q} + O(\tau^{2}) = \|f\|_{p}^{p} + q\tau\|f\|_{p}\|g\|_{q} + O(\tau^{2}). \quad (1.5)$$

Since $\frac{(q-1)}{q}=\frac{1}{p}$, the convexity of t^{p} implies that

$$\left\| f^{\frac{p}{q}} + \tau g \right\|_{q}^{q} = \int \left| f^{\frac{p}{q}} + \tau g \right|^{q} \ge \int \left(f^{\frac{p}{q}} \right)^{q} + q\tau \left(f^{\frac{p}{q}} \right)^{q-1} g \ge \|f\|_{p}^{p} + q\tau \int fg \quad \text{(since } f, g \ge 0\text{).} \quad (1.6)$$

We now describe the reduced order model for the errors facilitated by proper decomposition (P O D) using closely the ideas due to [3] for system of ODE. The studies of P O D are important and have wide readership from fluid mechanics, identification of coherent structures, control and inverse problems as well as for industrial applications. P O D has been widely applied in the modeling of supersonic Jet, turbine flows, thermal processing of foods and dynamic wind pressures acting on a building as reported in some literatures. The P O D provides best approximating affine subspace to a given set of data.

In the meantime, the remaining sections in the paper are arranged as follows: Section 2 in the paper describes the theoretical backgrounds of study based on existing works from literatures. Section 3 gives the core error problem arising from section 2 which highlights the ingredients of least squares supported by the Givens QR-Cholesky decomposition as well as singular values decomposition (SVD). We employ the techniques of [4,5] supported by Tikhonov regularization method technique for deblurring obscured images and unwanted noise from calibration of data with inherited error can be executed. In section 4, numerical example is demonstrated with described methods with high success.

2.0 Theoretical Backgrounds

As is standard, we follow the way of [3],[6] and the cited references therein. In the given system (1.1) in one dimensional ODE, at the m time points $y(t_1), y(t_2),..., y(t_m)$, which is computed by the standard Runge-Kutta method, or Adams Bashforth and Adams-Mouton methods, we give the values of deviations of mean from the data in the form:

$$y = \left((y(t_1)-\bar{y}), (y(t_2)-\bar{y}), ..., (y(t_m)-\bar{y}) \right), \quad (2.1)$$

Where $\bar{y}$ is the arithmetic mean for computed results $y(t_1), y(t_2),..., y(t_m)$.

Using P O D, we seek a subspace $S \in R^{m}$ and the projection matrix A for which the total distance

$$\phi = \|y - At\|_{2}, \quad (2.2)$$

is at minimum!

We then form a system of linear equations in the form:

$$A^{T}At = A^{T}y \quad (2.3)$$

The matrix $A \in R^{m\times n}, y \in R^m$ with m>n is highly ill-conditioned.

Finding solution to Equation (2.2) leads to inverse problems [7,8,9]. Inverse problems emanate from such diverse scientific and engineering disciplines to include medical imaging, oil and gas exploration, land-mine detection and process control. Mathematical tool box for solving inverse problems are embedded in mathematical analysis to include functional analysis, conformal mappings, spectral theory, theory of partial differential equations, integral equations ,micro-local and global analysis. Thus inverse problem is an interesting topic with wide readerships.

The Tikhonov regularization is the minimization problem

$$\min_t \{\|At-y\|_2^2+\mu^2\|Lt\|_2^2\} \qquad (2.4)$$

The μ in Equation 2.4 is the Tikhonov parameter, L may be taken as an identity matrix.

Equation (2.4) has the solution

$$t_\lambda = \left(A^TA+\mu^2 I\right)^{-1}A^Ty. \qquad (2.5)$$

Introducing filter factor

$f_i = \dfrac{\delta_i^2}{\delta_i^2+\mu^2}$, then we have that the regularized solution to Equation (2.4) is in the form

$$t_{reg} = \sum_{i=1}^{rank(\mu)} f_i \frac{u_i^T y}{\delta_i} v_i \qquad (2.6)$$

Where from, we set that

$$f_i = \begin{cases} 1 & \delta_i \ge \mu_i \\ 0 & \delta_i < \mu_i \end{cases}$$

Besides, we also decompose the matrix $A = U\sum V^T$ where, U and V have orthonormal columns, $\sum = diag(\sigma_1, \sigma_2, ..., \sigma_n)$ and $\sigma_1 \ge \sigma_2 \ge ... \ge \sigma_n$.

Using the generic notation $\|Ut\| = \|t\|, \|V^Tt\| = \|t\|$, we have that

$$\min_t \|\sum t-\alpha\|^2+\mu^2\|t\|^2 , \qquad (2.7)$$

and we defined $\alpha_i = u_i^T y$, $t = V^T t$. Using this technique, we obtain

$$(\sum^T \sum + \mu^2 I)t = \sum^T \alpha \qquad (2.8)$$

The Tikhonov solution will be rewritten as

$$t_{tk} = \sum_{i=1}^{n} \frac{\alpha_i\sigma_i}{\sigma_i^2+\mu^2} v_i. \qquad (2.9)$$

We provide the solution to the discrete case, that is, true noise free by the equation

$$t_{true} = \sum_{i=1}^{n} \frac{\alpha_i-\varepsilon_i}{\sigma_i} v_i \text{ , (for } \varepsilon_i = u_i^T e \text{.)} \qquad (2.10)$$

We now minimize the distance between the Tikhonov regularization method and the true solution in the absence of noise.

$$\min_\mu \|t_{tk}-t_{true}\|^2 = \min_\mu f(\mu) \qquad (2.11)$$

We carry out the transformation of Equation (2.11) using equations (2.9) and (2.10) to obtain

$$f(\mu) = \sum_{i=1}^{n}\left(\frac{\alpha_i\sigma_i}{\sigma_i^2+\mu^2} - \frac{\alpha_i-\varepsilon_i}{\sigma_i}\right)^2 . \qquad (2.12)$$

For stationary values , $\dfrac{df}{d\mu} = 0 = -2\sum_{i=1}^{n}\left(\dfrac{\alpha_i\sigma_i}{\sigma_i^2+\mu^2} - \dfrac{\alpha_i-\varepsilon_i}{\sigma_i}\right)\left(\dfrac{2\alpha_i\sigma_i\mu}{\left(\sigma_i^2+\mu^2\right)^2}\right)$

That is, $\dfrac{1}{4}f^{/}(\mu) = -\sum_{i=1}^{n}\left(\dfrac{\alpha_i\sigma_i}{\sigma_i^2+\mu^2} - \dfrac{\alpha_i-\varepsilon_i}{\sigma_i}\right)\left(\dfrac{\alpha_i\sigma_i\mu}{\left(\sigma_i^2+\mu^2\right)^2}\right) = 0$ (2.13)

From Equation (2.13), it shows that either $\sum_{i=1}^{n}\left(\frac{\alpha_i\sigma_i}{\sigma_i^2+\mu^2}-\frac{\alpha_i-\varepsilon_i}{\sigma_i}\right)=0$ or $\sum\frac{\alpha_i\sigma_i\mu}{(\sigma_i^2+\mu^2)}=0$.

Thus,

$$\sum_{i=1}^{n}\frac{\alpha_i\sigma_i}{\sigma_i^2+\mu^2}=\sum_{i=1}^{n}\frac{\alpha_i-\varepsilon_i}{\sigma_i}. \qquad (2.14)$$

We give now extra statistical observations obtainable from Equation (2.3) and are reported here for discussion. It is supposed that δ^2 denotes the variance of each of the element c. The element of c are supposed to be independently identical distributed (i.i.d) of each other, and δ^2 is computed in the form:

$$\delta^2=\phi(m-n) \ , \qquad (2.15)$$

where, $(m-n)$ denotes the difference between number of observations and number of parameters being estimated and is the degrees of freedom of the parameter estimation problem.

The Covariance matrix $C(y)$ is computed in the form:

$$C(y)=\delta^2\left(A^TA\right)^{-1} \qquad (2.16)$$

We give correlation coefficient matrix from Equation (2.16), defined as

$$\rho_{ij}=\frac{\delta_{ij}}{\sqrt{\delta_{ii}.\delta_{jj}}}, \qquad (2.17)$$

Forming a matrix $G=yy^T$ using standard matrix-matrix multiply, the eigenvalues of the correlation matrix are $\lambda_1\geq\lambda_2\geq\ldots\lambda_m\geq 0$ and are of decreasing order of magnitude. The matrix G is decomposed using the SVD in the form:

$$G=U\sum V^T \ . \qquad (2.18)$$

We obtained the P O D subspace , the projection matrix $H=ss^T\in R^{n\times n}$ where s is the matrix projection onto S, the subspace spanned by the reduced basis obtained from SVD.

Finally in the concluding part here, we bring out all the subspace filtering emanating from the vector $y(t_i)$, $(i=1,2,...,n)$ be the n- dimensional vector of clean speech samples (trajectory) and ε_i, (i=1,2,…,n) the zero mean , additive white noise distortion that is uncorrelated with clean speech (trajectory).

We thus represent noisy speech in the trajectory of solution $y(t)$ given by

$\bar{y}(t)=y(t)+\varepsilon(t)$. By denoting the M_y, N_ε and $O_{\bar{y}}$ as representing $n\times n$ covariance matrices .Then we have

$$O_{\bar{y}}=M_y+N_\varepsilon. \qquad (2.19)$$

Thus, their eigen decompositions are given by

$$O_{\bar{y}}=U(\Sigma+\delta_w^2 I)V^T,$$

$$M_y=U\sum V^T,$$

$$N_\varepsilon=U\left(\delta_w^2 I\right)V^T. \qquad (2.20)$$

The δ_w^2 is the white variance, I is the identity matrix.

The implication of this is that the speech and noise are separated in the sense that the speech is restricted to the $r<n$ dimensional subspace which is the signal subspace, whereas the noise occupies the $n-$ dimensional observation space.

In reality, we write that

$$O_{\bar{y}}=\left(U_rU_{n-p}\right)\begin{pmatrix}\Sigma_r & o\\ o & o\end{pmatrix}+\delta_w^2\begin{pmatrix}I_r & o\\ o & I_{n-r}\end{pmatrix} \qquad (2.21)$$

We now move to the full space arbitrary vector $z = \{z_1, z_2, ..., z_m\}$ with $\hat{y}$ as approximate solution to the ODE

$$\frac{d\hat{y}}{dt} = Af\left(\hat{y}, t\right), \quad \hat{y}(t_0) = Ay_0 \quad . \tag{2.22}$$

Compare to an equivalent P O D reduced model, we then have that:

$$\frac{dz}{dt} = p^T f\left(pz, t\right), z(t_0) = p^T y_0 \tag{2.23}$$

The aim is to compute the statistical condition estimate (SCE) taking vector $s \in R^k$ with q selected randomly and uniformly from the unit sphere S_{k-1} where the expected value $q^T s$ is approximation to the norm $\|y\|$. We then compute that:

$$E\left(\left|q^T y\right|\right) = W_k \|s\| . \tag{2.24}$$

We give the Wallis factor which is defined [10] in the form

$$W_1 = 1, \quad W_k = \begin{cases} \dfrac{1.3....(k-2)}{2.4.6....(k-1)}, k = odd \ \ with \quad k = 3,5,7,... \\ \dfrac{2}{\pi}.\dfrac{2.4.6....(k-2)}{1.3.5....(k-1)}, \ k = even, \ with \quad k = 4,6,8,... \end{cases} \tag{2.25}$$

In the limit, the value for W_k is given by:

$$W_k \cong \left(\frac{2}{\pi(k-0.5)}\right)^{\frac{1}{2}}.$$

It follows that estimate for the norm $\|s\|$ satisfies the equation for the error

$$\xi = \frac{\left|q^T s\right|}{W_k} \tag{2.26}$$

Equation (2.26) is the optimal estimate for the relative error for the norm $\|s\|$ that is inversely proportional to the size of the error with guaranteed probability error bound given that τ >1 in the form:

$$\Pr\left(\frac{\|s\|}{\tau} \le \xi \le \tau\|s\|\right) \ge 1 - \frac{2}{\pi\tau} + O(\tau^{-2}) \tag{2.29}$$

After computing for various estimates $\xi_1, \xi_2, ..., \xi_p$ corresponding to p randomly selected vectors $s_1, s_2, ..., s_p$ from the unit sphere S_{k-1} ,we compute the expected value for the length of error vector in the form

$$E\left(\left|q_1^T s\right|^2 + \left|q_2^T s\right|^2 + ... + \left|q_p^T s\right|^2\right)^{\frac{1}{2}} \cong \frac{W_k}{W_p}\|s\| \tag{2.30}$$

3.0 The Derived Error for Proper Orthogonal Decomposition (P O D) From Reduced Model

The essence is comparing errors incurred from the results computed using proper orthogonal decomposition with those obtained from standard theoretical solution. Firstly, we signify our intention by the following notation.

$e = \hat{y} - y$ as total error and this is split into subspace approximation error:

$$e_\perp = p^T y - y \, , \tag{3.1}$$

$$e_s = \hat{y} - p^T y \, , \tag{3.2}$$

is the error computed by the integration in subspace.

By subtracting Equation (3.1) from (3.2), we have that

$$e = \hat{y} - y = \left(\hat{y} - p^T y\right) - \left(p^T y - y\right) = e_s - e_{\perp} \qquad (3.3)$$

The rate of change of approximation error with respect to t (time) is given by the following equation**:**

$$\frac{de}{dt} = Af(\hat{y},t) - f(y,t) = Af(\hat{y},t) - f(y,t) + f(\hat{y},t) - f(y,t) \qquad (3.4)$$

Simple factorization of Equation (3.4) would yield that :

$$\frac{de}{dt} = (A - I) f\left(\hat{y},t\right) - J\left(\hat{y},t\right)\left(y - \hat{y}\right) + O(\|e\|) \qquad (3.5)$$

By further setting as :

$G = I - A$, then we obtain that**:**

$$\frac{de}{dt} = J\left(\hat{y},t\right) e(t) = Gf\left(\hat{y},t\right) = -Gy_0 \qquad (3.6)$$

We noted that the matrix J appearing in equation (3.5) is the Jacobian matrix $\frac{\partial f}{\partial y}$. The matrix $G(t) \in R^{n \times n}$ has transform

$$\frac{dG}{dt} = J\left(\hat{y},t\right) G, \qquad (3.7)$$

with $G(t_0) = I_n$.

Before proceeding further, we shall adopt the following notation:

$\|f\|_p = \left\{\int_t |f|^p d\mu\right\}^p$, $f \in L^p(\mu)$ if $\|f\|_p < +\infty$, and for $t \in R^n$, $\|t\|_p = \left(\sum_{n=1}^{\infty} |t_n|^p\right)^{\frac{1}{p}}$, $\|f\|_\infty = ess.\sup|f(t)|$, where ess.sup$|f(t)|$ is the infimum of sup of $g(t)$ as g ranges over all functions which are equal to f almost everywhere on t.

The following theorem holds verbatim for adoption.

Theorem 3.1

Let p and q be conjugate exponent with $1 < p < \infty$. Let t be a measure space with positive measure μ. If f and g are measurable functions on t with range in $[0,+\infty]$.Then we have that:

(i) $\int_t fg d\mu \le \left(\int_t f^p d\mu\right)^{\frac{1}{p}} \left(\int_t g^q d\mu\right)^{\frac{1}{q}}$; (ii) $\left(\int (f+g)^p d\mu\right)^{\frac{1}{p}} \le \left(\int_t f d\mu\right)^{\frac{1}{p}} + \left(\int_t g^p d\mu\right)^{\frac{1}{p}}$.

The first inequality in (i) is the Holder's inequality while that in (ii) is the Minkowski's. Thus for $p = 2$ and $q = 2$ we have the Schwartz' inequality.

Firstly we represent $e^A = \lim_{s\to\infty}\left(I + \frac{A}{s}\right)^s$ and for $T_{r,s} = \left(\sum_{i=0}^{r} \frac{1}{i!}\left(\frac{A}{s}\right)^i\right)^s$ we define that

$$\left\|e^A - T_{r,s}\right\| \le \frac{\|A\|^{r+1}}{s^r (r+1)!} e^{\|A\|} ,$$

$$\lim_{r\to\infty} T_{r,s}(A) = \lim_{s\to\infty} T_{r,s}(A) = e^A.$$

The matrix exponential for the identity is discussed below.

$$e^{(A+E)t} = e^{At} + \int_0^t e^{A(t-s)} E e^{(A+E)s} ds = e^{At} + \int_0^t e^{A(t-s)} E e^{As} ds + O\left(\|E\|^2\right).$$

The Jacobian for the matrix exponential is

$$J(A,E) = \int_0^1 e^{A(1-s)} E e^{As} ds.$$

Then vec $(J(A,E)) = \int_0^1 \left(e^{A^T s} \otimes e^{A(1-s)}\right) vec(E) ds = \left(I \otimes e^A\right) \int_0^1 \left(e^{A^T} \otimes e^{-As} ds\right) vec(E)$.

Therefore, for any subordinate matrix norm we have $K_{\exp}(A) \le \frac{e^{\|A\|}\|A\|}{\|e^{A}\|}$.

Thus the fundamental matrix solution for the error $e(t_b)$ at the end point b in the interval [a, b] is given by the equation:

$$e(t_b) = \int_{t_0}^{t_b} G(t_b)G^{-1}(\theta)Gf\left(\hat{y}(\theta),\theta\right)d\theta = G(t_b)Gy_a \qquad (3.8)$$

The main purpose is to synchronize Equation (3.8) with estimated condition norm as derived in section 2

Introduce thus into discussion a randomly selected vector s from unit sphere S_{n-1} **for which holds:**

$$s^T e(t_b) = \int_{t_a}^{t_b} s^T G(t_b)G^{-1}(\theta)Gf(\hat{y}(\theta),\theta)d\theta - s^T G(t_b)Gy_a \qquad (3.9)$$

Now the adjoint equation to the defining equation for the error is

$$\frac{d\lambda}{dt} = -J^T\left(\hat{y},t\right)\lambda,\ \lambda(t_b) = q \qquad (3.10)$$

Where

$$\lambda^T(s) = s^T G(t_b)G^{-1}(s) \qquad (3.11)$$

$$\lambda^T(t_a) = z^T G(t_b) \qquad (3.12)$$

Coupling together equations (3.9) through(3.12) we then write that:

$$s^T e(t_b) = -\int_{t_a}^{t_b} \lambda^T(\theta)Gf(\hat{y}(\theta),\theta)d\theta - \lambda^T(t_a)Gy_a \qquad (3.13)$$

We relate Equation (3.13) with statistical condition estimate for the error norm $e(t_b)$ in the form**:**

$$\|e(t_b)\| = \frac{W_p}{W_n}\left((\sum_{j=1}^{p}\left|\int_{t_a}^{t_b} \lambda^T(\theta)Gf(\hat{y}(\theta),\theta)d\theta + \lambda^T(t_0)Gy_0\right|)^2\right)^{\frac{1}{2}} \qquad (3.14)$$

Equation (3.14) is the POD error for the reduced model for system of ODE given earlier in section 2. Similarly, it can be described that the error for the subspace integration as follows:

By setting $e_s = \hat{y} - p^T y$, we have that

$$\frac{de_s}{dt} = \frac{d\hat{y}}{dt} - A\frac{dy}{dt} = A\left(f(\hat{y},t) - f(y,t)\right) = AJ\left(\hat{y},t\right)e(t) = AJ(\hat{y},t)(e_s + e_\perp), \qquad (3.15)$$

where, $\hat{y}(t_a) = y(t_0)$ **is the projection** $p^T y(t_a)$ **of** $y(t_a)$ **onto** s **which gives initial condition** $e_s(t_0) = 0$**.** The rate of change of step size is given by the equation**:**

$$\frac{dh}{dt} = p^T J(\hat{y},t)ph + p^T J(\hat{y},t)e_\perp;\ h(t_0) = 0, \qquad (3.16)$$

with $p^T P = I_k$**.**

For a matrix $\Psi \in R^{n\times n}$**,** we seek that

$$\frac{d\Psi}{dt} = p^T J(\hat{y},t)p\Psi,\ \Psi(t_a) = I_k \qquad (3.17)$$

$$v^T h(t_b) = \int_{t_a}^{t_b}\left(v^T\Psi(t_b)\Psi^{-1}(\theta)p^T J\left(\hat{y}(t),\theta\right)e_\perp(\theta)d\theta\right) \qquad (3.18)$$

The $v = v_1, v_2, \ldots v_q$ is any vector that is randomly and uniformly selected from the unit sphere S_{k-1}.

As before the solution to the adjoint system defined in the form

$$\frac{d\eta}{dt} = -p^T J^T(\hat{y},t)p\eta,\ \ \eta(t_b) = v \qquad (3.19)$$

has the solution

$$\eta^T(\theta) = v^T \Psi(t_b)\Psi^{-1}(\theta), \ where \ \ \theta \in [t_a, t_b] \qquad (3.20)$$

With

$$v^T h(t_b) = \int_{t_a}^{t_b} \eta^T(\theta), p^T J(\hat{y}(\theta), \theta) e_\perp(\theta) d\theta \qquad (3.21)$$

We have the statistical condition number error estimate is in the form:

$$\|e_s(t_b)\| = \|h(t_b)\| = \frac{W_q}{W_n}\left(\sum_{j=1}^{q}\left|\int_{t_a}^{t_b} \eta^T(\theta) p^T J(\hat{y}(\theta), \theta) e_\perp(\theta) d\theta\right|^2\right)^{\frac{1}{2}} \qquad (3.22)$$

We now give the bounded error for the subspace integration

$$\left|\int_{t_a}^{t_b} \eta(\theta) p^T J(\hat{y}(\theta), \theta) e_\perp(\theta) d\theta\right| \le \int_{t_a}^{t_b} \left|\eta^T(\theta) p^T J(\hat{y}(\theta), \theta) e_\perp(\theta)\right| d\theta \le \|J^T p \eta\|_{L_1} \cdot \|e_\perp\|_{L_\infty} \qquad (3.23)$$

The distribution nature of eigenvalues of the rectangular matrix A is obtained by the Jacobi -Givens type similarity transformation [9,11]which is orthogonal and preserves length a useful tool in the resolution of systemic matrix arrays for antennae beam formations. Thus we have

$$Q(p,q,\theta) = \begin{pmatrix} 1 & & \cdots & & & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1 & & & \\ & & & \cos\theta & & \sin\theta & \cdots & 0 \\ & & & \cdots & & \cdots & \\ & & & & 1 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & \cdot \\ \cdots & & & -\sin\theta & & \cos\theta & \cdots & 0 \\ & & & & & 1 & \\ & & & & & & \cdot \\ & & & & & & & \cdot \\ \cdots & & & & & & & \cdot \\ 0 \cdots & & & & & & & 1 \end{pmatrix}$$

Where,

$\bar{A} = Q(p,q,\theta)A$, $\cos(\theta) = \dfrac{p_i}{\sqrt{p_i^2 + q_i^2}}$; $\sin(\theta) = \dfrac{q_i}{\sqrt{p_i^2 + q_i^2}}$.

Applying the Sylvester Inertia Law on the symmetric matrix shows that $\bar{A} = Q^T AQ$, where the Rayleigh Quotient $R[t] = \dfrac{t^T At}{t^T t}$,

for $t \neq 0$ and the field of values defined as $F(A)$ is the set of possible Rayleigh Quotients:

$F(A) = \left\{ t^H At \,\middle|\, t \in \mathbb{C}^n \ \|t\|_2 = 1 \right\}$ for which holds $\frac{1}{2}\|A\|_2 \le r(A) \le \|A\|_2$, and that $\dfrac{r(A)}{\rho(A)} \le K_2(A)$.

Therefore, the matrix A will be far from normality if $r(A) \ge \rho(A)$.

We now ask an important question that if the QR iteration algorithm is shifted by a factor, what can be said about the eigenvalues of A? It is known that the shifted QR algorithm does not alter the eigenvalues of A. This is established by the similarity transformation of A_k following the orthogonality of Q_k in the sense of [11]. After setting as

$QR = A_k - \mu_k I : Q_k R_k = (A_k - \mu_k I)$, it would hold that $A_{k+1} = R_k Q_k + \mu_k I$ .Hence, $R_k = Q_k^T (A_k - \mu_k I)$.Furthermore, in the same way it follows that

$A_{k+1} = R_k Q_k + \mu_k I = Q_k^T (A_k - \mu_k I) Q_k + \mu_k I = Q_k^T (A_k - \mu_k I) Q_k + \mu_k Q_k^T Q_k = Q_k^T A_k Q_k$.

Besides, for the symmetric case for the matrix A, we can reduce the symmetric matrix A to tridiagonal matrix with Givens Similarity transformation [11].We may apply the Cholesky decomposition on T_k in the form

$$T_k = \begin{pmatrix} \alpha_1 & \beta_2 & & & & \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & . & \beta_n. \\ 0 & & & & \beta_n & \alpha_n \end{pmatrix} = L_k D_k L_k^T$$

$$L_k = \begin{pmatrix} 1 & & & 0 \\ m_2 & 1 & & \\ & . & . & \\ & & . & . \\ & & . & . \\ 0 & & m_k & 1 \end{pmatrix}; \qquad D_k = \begin{pmatrix} d_1 & & & & 0 \\ & d_2 & & & \\ & & . & & \\ & & & . & \\ & & & & . \\ & & & & d_k \end{pmatrix}.$$

We calculated the L_k for T_k in the form of elementary matrices wherefrom,

$d_1 = \alpha_1$

for $k = 2,3,\dots,n$

$m_k = \beta_{k-1} / d_{k-1}$

$d_k = \alpha_k - \beta_{k-1} m_k$

We then solve the system of linear equation the LDL^T -Cholesky based method using suitable MATLAB routines.

$A = U \sum V^T$, where, $\sum$ is a diagonal matrix containing $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$. The matrices U and V are the left and right singular vectors u_i and v_i , i=1,2, …, n.

The filtered solution [8] for the system (2.3) is given in the form:

$$t_A(\phi_A) = \sum_{i=1}^{n} (\phi_A)_i \frac{u_i^T b}{\sigma_i} v_i = V\, diag(U^T b) \Sigma^{-1} \phi_A \qquad (3.4)$$

Where $(\phi_A)_i = \dfrac{\sigma_i^2}{(\sigma_i^2 + \lambda^2)}$ for the regularization parameter λ .

We discuss the contribution by the noise ε as follows:

From the system of linear equation assuming

$$As = \bar{b} \ni \|\Delta b\| = \min! \quad , \bar{b} = b + \varepsilon \qquad (3.5)$$

subject to $As = \bar{b} + \Delta b$, where $A \in R^{m \times n}$,$\bar{b} \in R^m$ see e.g., [12], the least squares solution is

$$s_{Ls} = A^+ \bar{b} = \sum_{i=1}^{p} \frac{u_i^T b}{\sigma_i} v_i + \varepsilon \sum_{i=1}^{p} \frac{u_i^T w}{\sigma_i} v_i \qquad (3.6)$$

In Equation (3.6), it is supposed that p is the numerical rank of A. The point is that $\sum_{i=1}^{p} \frac{u_i^T b}{\sigma_i} v_i \to \hat{s}$; while at the same time , $\varepsilon \sum_{i=1}^{p} \frac{u_i^T w}{\sigma_i} v_i \to 0$ where the Picard condition shows that $|u_i^T b \to 0|$ faster compared to σ_i .In Equation (3.5) the noise ε is calculated by the quantity $\varepsilon = \dfrac{\|\Delta b\|}{\|b\|}$; the condition number $K_2(A) = \dfrac{\sigma_1}{\sigma_2}$.

The relative error for the solution to least squares equation is given by the inequality

$$\frac{\left\| s-\bar{s} \right\|}{\left\| s \right\|} \le \varepsilon\left(\frac{2K_2(A)}{\cos(\theta)} + \tan(\theta)K_2^2(A) \right) + O\left(\varepsilon^2\right) \quad (3.7)$$

The angle θ is between b and its projection onto $R(A)$.As a measure of effectiveness, whenever the system of equations is inconsistent, it will always flag off a warning signal that $r = b - As \neq O$ and besides, $\tan\theta \neq 0$. The fact is that a little perturbation of introduced relative error in the least squares solution is directly proportional to the squares of condition number $K_2(A)$.

What can we deduce from the error in high frequencies, that is, the coefficients $< b, V_n >$ corresponding to singular vectors with large n (and small σ_n)? Really, it is that the errors are amplified much stronger than those for low frequencies (larger σ_n). We should mention here that the way how high frequency errors are amplified depends on the operator matrix A.That is, the decay speed of its singular values. The faster the decay is, the more severe the Picard criterion

$\left\| A^+ b \right\|^2 = \sum \frac{< b, v_n >}{\sigma_n^2} < \infty$, where $\sigma = \sqrt{\lambda}$.

4.0 Numerical Examples

We consider the system of ordinary differential equation as problem 1 taken from [13].

Problem 1:

$$\frac{dy}{dt} = t + y + z, \quad y(0) = 1$$

$$\frac{dz}{dt} = 1 + y + z, \quad z(0) = -1$$

Take $h = 0.1$. We implemented Fourth Stage order four Runge-Kutta method

$$k_1 = hf\left(t_i, y_i, z_i\right),$$

$$k_2 = hf\left(t_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_1 + \frac{l_1}{2} \right),$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_2}{2} \right),$$

$$k_4 = hf\left(t_i + h, y_i + k_3, z_i + l_3\right)$$

$$l_1 = hg\left(t_i, y_i, z_i\right),$$

$$l_2 = hg\left(t_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{l_1}{2} \right),$$

$$l_3 = hg\left(t_i + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_2}{2} \right),$$

$$l_4 = hg\left(t_i + h, y_i + k_3, z_i + l_3\right)$$

Then,

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

$$z_{i+1} = z_i + \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

Numerical results for problem 1 are displayed in Table 1 for the first seven ordinate points.

Table 1 Shows Results computed by Four Order Stage Four Runge-Kutta-Method

Iterations K	Time (t_k)	Results for Runge-Kutta method for Problem 1 W_k
1	0.1	$\begin{pmatrix} 1.008720833 \\ -0.895445833 \end{pmatrix}$
2	0.2	$\begin{pmatrix} 1.0423203758 \\ -0.77684629 \end{pmatrix}$
3	0.3	$\begin{pmatrix} 1.1033033571 \\ -0.640863309 \end{pmatrix}$
4	0.4	$\begin{pmatrix} 1.1961502541 \\ -0.482563825 \end{pmatrix}$
5	0.5	$\begin{pmatrix} 1.3278092863 \\ -0.295904807 \end{pmatrix}$
6	0.6	$\begin{pmatrix} 1.5052411176 \\ -0.073472976 \end{pmatrix}$
7	0.7	$\begin{pmatrix} 1.7374728509 \\ 0.1937587579 \end{pmatrix}$

Next we form the system of linear equation from the nodal points , the slopes of the K_i and the step sizes.

$$A = \begin{pmatrix} 0 & 0.01 & 0.0108 & 0.0108 \\ 0.0213 & 0.0330 & 0.0339 & 0.0466 \\ 0.0465 & 0.0602 & 0.0613 & 0.0763 \\ 0.0762 & 0.0924 & 0.0924 & 0.1114 \\ 0.1114 & 0.1305 & 0.1322 & 0.1533 \\ 0.1532 & 0.1760 & 0.1780 & 0.2330 \\ 0.2032 & 0.2305 & 0.2330 & 0.2633 \end{pmatrix}, \quad c = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \end{pmatrix}, s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$$

We solved the under determined system to $As = c$ with solution

$$s = \begin{pmatrix} -9.7846 \\ 25.2305 \\ -20.4109 \\ 6.2866 \end{pmatrix}.$$

The Singular Value Decomposition (SVD) for the slop matrix K_i is

$$U = \begin{pmatrix} -0.0240 & -0.3692 & -0.9222 & -0.1128 & 0.0030 & -0.0004 & -0.0037 \\ -0.1003 & -0.6261 & 0.2133 & 0.3078 & 0.0575 & 0.3003 & 0.6036 \\ -0.1801 & -0.4796 & 0.1652 & 0.2613 & -0.4964 & -0.4323 & -0.4566 \\ -0.2734 & -0.3083 & 0.2380 & -0.8786 & 0.0282 & -0.4325 & 0.0090 \\ -0.3865 & -0.1202 & 0.0372 & 0.2025 & 0.8188 & -0.2197 & -0.2740 \\ -0.5200 & 0.0972 & -0.0354 & 0.0868 & -0.1761 & 0.7383 & -0.3679 \\ -0.6800 & 0.3504 & -0.1325 & 0.0612 & -0.2191 & -0.3589 & 0.4655 \end{pmatrix}$$

$$D = \begin{pmatrix} 0.6866 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0172 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0050 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0008 & 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} -0.4256 & 0.7913 & 0.4350 & 0.0590 \\ -0.4928 & 0.0283 & -0.4312 & -0.7553 \\ -0.4982 & -0.0021 & -0.5721 & 0.6516 \\ -0.5726 & -0.6108 & 0.5455 & 0.0393 \end{pmatrix}$$

The eigenvalues for the Slope matrix $K = (0.8286,\ 0.1311,\ 0.0707,\ 0.0283)$.

Similarly, for the slope L -matrix from the Runge-Kutta method we form the underdetermined linear system.

$Bu = x(t)$, where $L \in R^{m \times n}$, $m > n$ and $x(t) \in R^m$

$$B = \begin{pmatrix} 0.1 & 0.105 & 0.10575 & 0.1058 \\ 0.1132 & 0.1180 & 0.1188 & 0.1266 \\ 0.1265 & 0.1352 & 0.1363 & 0.1463 \\ 0.1462 & 0.1574 & 0.1587 & 0.17135 \\ 0.17136 & 0.1855 & 0.1872 & 0.2033 \\ 0.2032 & 0.2210 & 0.2230 & 0.2433 \\ 0.2432 & 0.2655 & 0.2680 & 0.2933 \end{pmatrix}, \quad x(t) = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

This gives the solution

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -13.2376 \\ 18.8923 \\ -19.2721 \\ 13.9779 \end{pmatrix}.$$

The SVD for this matrix is

$$U = \begin{pmatrix} -0.2193 & 0.8287 & -0.5146 & 0.0083 & -0.0087 & 0.0091 & -0.0125 \\ -0.2503 & 0.2925 & 0.5748 & 0.0565 & -0.1637 & 0.4095 & 0.5690 \\ -0.2870 & 0.1850 & 0.4321 & 0.0819 & -0.4915 & -0.4943 & -0.4517 \\ -0.3342 & 0.0745 & 0.2361 & -0.5967 & 0.6113 & -0.3098 & 0.0343 \\ -0.3943 & -0.0581 & 0.0839 & 0.7209 & 0.5145 & 0.0793 & -0.2089 \\ -0.4699 & -0.2080 & -0.1154 & -0.3334 & -0.1740 & 0.6187 & -0.4454 \\ -0.5647 & -0.3759 & -0.3769 & 0.0575 & -0.2506 & -0.3206 & 0.4785 \end{pmatrix}$$

$$D = \begin{pmatrix} 0.9495 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0104 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0015 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0001 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.4585 & 0.5850 & 0.6668 & 0.0539 \\ -0.4960 & 0.1662 & -0.4272 & -0.7375 \\ -0.5004 & 0.1442 & -0.5250 & 0.6732 \\ -0.5417 & -0.7802 & 0.3117 & 0.0078 \end{pmatrix}$$

The eigenvalues for the slope matrix L is $\lambda_l = (0.9744,\ 0.1020,\ 0.0354,\ 0.01)$.

We computed and verified the polarization identity for the solutions obtained from the slope matrices
With result given by

$$\|s+u\|^2 + \|s-u\|^2 = 2\left(\|s\|^2 + \|u\|^2\right) = 4574.2337\,.$$

5.0 Conclusion

The paper discussed system of ordinary differential equation using Runge-Kutta fourth order method . We give the Wallis factor for the subspace integration for the POD system and the optimal estimate for the relative error for the norm $\|s\|$ that is inversely proportional to the size of the error with guaranteed probability error bound given that τ >1. We made reference to this approach in [10].

We applied the Givens orthogonal plane rotation matrix on the slope matrices appearing in the solution process in the Runge-Kutta method to obtain the singular values decomposition (SVD) .The resulting over determined system of linear equations from the slope matrices were solved by means of least squares equation using the SVD. Since the resulting linear system is over determined and ill-conditioning may occur, the use of Tikhonov regularization parameter was brought into play in this direction. It was also mentioned on how to de-noise the solution space due to huge condition number appearing in the left hand side of the overdetermined linear system using the earlier method discussed in [8] by introducing filter factor into the calculation . This was discussed in section 2. All numerical calculations well carried by using MATLAB windows 07. It was established that the solution space satisfied the polarization identity.

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