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IMPLICIT-EXPLICIT SECOND DERIVATIVE LINEAR MULTISTEP METHODS FOR ADDITIVELY SEPARABLE STIFF ODEs

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Abstract

The implicit-explicit (IMEX) linear multistep methods are now emerging technique for the numerical solutions of ordinary differential equations (ODEs) which arises from the discretization of partial differential equations (PDE) by method of lines (MOL), and chemical reaction models amongst other sources in which the resultant stiff ODEs admits an additively separable structure. In fact, the Prothero-Robinson stiff ODE is a typical example. In this paper, the purpose is to extend the implicit-explicit linear multistep methods to implicit-explicitsecond derivative linear multistep methods (IMEX SDLMM) for the numerical solution of additively separable stiff ODEs. The new IMEX SDLMMare based on the second derivative backward differentiation formulas (SDBDF). The IMEX methods developed herein are constructed by combining an extrapolated explicit method with its implicit method. The IMEX methods are shown to be stable on the conventional Dahlquist test problem. Numerical results are presented on the notable Prothero-Robinson stiff problem.

Mathematics subject classification: 65L05, 65L06.
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1.0 Introduction

Large systems of ordinary differential equations (ODEs) arise amongst other sources from chemical reaction models and discretization in space of partial differential equations (PDEs) by method of lines (MOL).These differential equations (DEs) are sometimes models from real life applications. For such systems, there are often natural splitting of the right hand side of the differential system into two or more parts; such system can be written in the general form;

$$y^{/}(t)=\sum_{j=1}^{s}F_j(t,y(t)),\ y(t_0)=y_0,\ t\in[t_0,T] \qquad (1.1)$$

where each of the $F_j(t,y(t));\quad j=1(1)s$ may represent a process in the model. However, the interest will be in when $s=2$. In particular, consider the two term additive splitting,

$$y^{/}(t)=f(t,y(t))+g(t,y(t)),\ y(t_0)=y_0,\ t\in[t_0,T] \qquad (1.2)$$

where $f(t,y(t))$ represents the non-stiff process and suitable for explicit time integration, for example advection and $g(t,y(t))$ represents the stiff process and suitable for implicit time integration for example diffusion or chemical reaction models [1]. The implicit-explicit (IMEX) integration approach discretizes the non-stiff part $f(t,y(t))$ with an explicit method, and the stiff part $g(t,y(t))$ with an implicit stable method. This strategy seeks to ensure the numerical stability of the solution of (1.2) while reducing the implicitness and therefore the overall computational effort in solving the ODE (1.2). This is the computational advantage of IMEX LMM. Hence in solving ODE (1.1), numerical schemes which integrate the $g(t,y(t))$ term implicitly and $f(t,y(t))$ term explicitly are highly desired, such implicit-explicit methods are referred to as IMEX schemes [2]. One of the simplest examples is the IMEX Euler method

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$$y_{n+1} = y_n + hf(t_n, y(t_n)) + hg(t_{n+1}, y(t_{n+1}))$$

which is obtained by applying the implicit Euler formula to the g term and the explicit Euler formula to the f term. Here h is the step size, $t_n = t_0 + nh,$ and y_n denotes an approximate value of the theoretical solution of $y(t_n)$ of (1.2). The IMEX linear multistep method was introduced in [3] and [4] and further analysis of its stability is done in [5].

In this paper, the purpose is to extend the implicit-explicit linear multistep methods to implicit-explicitsecond derivative linear multistep methods (IMEX SDLMM) based on the second derivative backward differentiation formulas (SDBDF). To achieve this aim, the following objectives are outlined. The first of these is to develop IMEX SDLMM based on the second derivative backward differentiation formula (SDBDF) up to the ninth order. The second objective is to analyze the basic properties of the methods in terms of its order, zero stability and region of absolute stability. The last objective is to numerically validate the proposed IMEX second derivative schemes.

This paper is arranged as follows. In section 2, a variety of *k*-step IMEX SDLMM are derived. The stability of the IMEX SDLMM are analyzed and discussed in section 3. Section 4 presents the numerical experiments on the Prothero-Robinson problem amongst others.

2.0 Derivation of IMEX SDLMM

Consider the additive splitting of the ODEs (1.1) into two parts in (1.2). The general SDLMM is,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \lambda_j f'_{n+j} \qquad (2.1)$$

we shall derive an IMEX SDLMM based on the SDBDF

$$\frac{h^2}{2} f'_{n+1} = \left(\sum_{i=1}^{k} \frac{1}{i}\right) hf_{n+1} - \sum_{i=1}^{k}\left(\sum_{j=1}^{k}\frac{1}{j}\right)\frac{\nabla^j y_{n+1}}{j}; \quad k \le 9, \qquad p = k+1 \qquad (2.2)$$

where

$$\nabla^j y_{n+1} = \sum_{r=1}^{k} \binom{j}{r} (-1)^r y_{n+1-r}, \quad k = 1(1)9$$

which are A-stable/A(α)-stable for $k \le 9$. The IMEX SDLMM to be derived are such that the explicit parts are obtained by extrapolation of the implicit terms f_{n+k}, f'_{n+k} of the SDLMM (2.2).

2.1 The Design of IMEX SDLMM

The fully implicit second derivative linear *k*-step method is

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j \left(f(t_{n+j}, y_{n+j}) + g(t_{n+j}, y_{n+j})\right) +$$
$$h^2 \sum_{j=0}^{k} \lambda_j \left(f'(t_{n+j}, y_{n+j}) + g'(t_{n+j}, y_{n+j})\right); \qquad \alpha_k = 1 \qquad (2.3)$$

with respect to the additive splitting in (1.2). An IMEX SDLMM can be derived by reducing $f(t_{n+k}, y_{n+k})$, $f'(t_{n+k}, y_{n+k})$ through extrapolation as follows,

$$\Phi(t_{n+k}) = \sum_{j=0}^{k-1} \gamma_j \Phi(t_{n+j}) + O(h^q); \qquad \Phi(t) = f(t, y(t))$$
$$\Phi'(t_{n+k}) = \sum_{j=0}^{k-1} \gamma_j \Phi'(t_{n+j}) + O(h^q); \qquad \Phi'(t) = f'(t, y(t)) \qquad (2.4)$$

This leads to the *k*-step IMEX SDLMM

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k-1} \beta^*_j f(t_{n+j}, y_{n+j}) + h^2 \sum_{j=0}^{k-1} \lambda^*_j f'(t_{n+j}, y_{n+j})$$
$$+ h\sum_{j=0}^{k} \beta_j g(t_{n+j}, y_{n+j}) + h^2 \sum_{j=0}^{k} \lambda_j g'(t_{n+j}, y_{n+j}) \qquad (2.5)$$

with

$$\beta^*_j = \beta_j + \beta_k \gamma_j; \qquad \lambda^*_j = \lambda_j + \lambda_k \gamma_j; \qquad \beta_k, \lambda_k \ne 0$$

The order of Prothero-Robinson convergence of the IMEX SDLMM to be derived will be stated and this is captured in what follows.

2.2 Prothero-Robinson Convergence of the IMEX SDLMM (2.5)

Let the extrapolated IMEX SDLMM schemes (2.5) be of order *p* when applied to the stiff system of differential equations (1.2). Following [6], consider the Prothero-Robinson convergence of the scheme in (2.5) on the Prothero-Robinson test problem [6].

$$\begin{cases} y^{/}(t) = \lambda(y(t) - q(t)) + q^{/}(t), \quad t \geq 0, \\ y(0) = q(t), \end{cases} \tag{2.2.1}$$

where $\lambda \in \not\subset$ has a large and negative real part and $q(t)$ is a slowly varying function. The solution to (2.2.1) is $y(t) = q(t)$. The IMEX scheme (2.5) is said to be Prothero-Robinson (PR) convergent if the application of (2.5) to the equation (2.2.1) leads to the solution $y(t_{n+k})$ whose global error satisfies

$$\left\| \sum_{j=0}^{k} \alpha_j y(t_{n+j}) - \left(h\sum_{j=0}^{k-1} \beta^{*}{}_j f(t_{n+j}, y(t_{n+j})) + h^2 \sum_{j=0}^{k-1} \lambda^{*}{}_j f^{/}(t_{n+j}, y(t_{n+j})) \right. \right.$$

$$\left. \left. + h\sum_{j=0}^{k} \beta_j g(t_{n+j}, y(t_{n+j})) + h^2 \sum_{j=0}^{k} \lambda_j g^{/}(t_{n+j}, y(t_{n+j})) \right) \right\| = O\left(h^{p+1}\right), \quad p = \min\left(r, q\right) \tag{2.2.2}$$

where r is the order of underlying implicit second derivative method in (2.2), q is the order of resultant explicit extrapolation (2.4) and $f = q^{/}(t)$, $g = \lambda(y(t) - q(t))$ or $f = -\lambda q(t) + q^{/}(t)$, $g = \lambda y(t)$ the non-stiff and stiff parts respectively. The methods in (2.5) are PR-convergent and the order of its Prothero-Robinson convergence is captured in the following theorem

Theorem

Assume the implicit SDLMM (2.3) has order r and the extrapolation procedure (2.4) has order q. Then the IMEX SDLMM (2.5) has order p=min(r,q) as $h \to 0$, $\lambda h \to -\infty$, *and* $\lambda h \in R_{AS}$. *Here* R_{AS} *is the region of absolute stability of the SDLMM (2.3)*

Proof. Let $\Phi(t) = f(t, y(t))$ and $\Phi^{/}(t) = f^{/}(t, y(t))$, the local truncation error can be written as

$$\frac{1}{h^2} \sum_{j=0}^{k} (\alpha_j y(t_{n+j}) - h\beta_j y^{/}(t_{n+j}) - h^2 \lambda_j y^{//}(t_{n+j})) + \frac{1}{h} \left(\beta_k \Phi(t_{n+k}) - \sum_{j=0}^{k-1} \gamma_j \Phi(t_{n+j}) \right) +$$

$$\lambda_k \Phi^{/}(t_{n+k}) - \sum_{j=0}^{k-1} \gamma_j \Phi^{/}(t_{n+j})$$

$$= C_1 h^r y^{(r)}(t_n) + O(h^{r+1}) + (\beta_k + \lambda_k) C_2 h^q \Phi^{(q)}(t_n) + O(h^{q+1}) \tag{2.2.3}$$

with constants C_1, C_2 determined by the coefficients of the SDLMM (2.5) and extrapolation procedure (2.4). The order *p* follows therefrom (2.2.3)

2.3 Derivation of the IMEX SDLMM (2.5) based on the SDBDF (2.2)

As an application of this theorem, the derivation of the IMEX SDLMM based on the SDBDF (2.2) for the case of *k*=1,2 appears trivial, but consider for example when *k*=3, the SDBDF for *k*=3 is

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = \frac{66}{85} h g_{n+3} - \frac{18}{85} h^2 g^{/}{}_{n+3}, \quad r = 4 \tag{2.6}$$

According to (2.3), we have

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = \frac{66}{85} h(f_{n+3} + g_{n+3}) - \frac{18}{85} h^2 (f^{/}{}_{n+3} + g^{/}{}_{n+3})$$

and the explicit term for $f_{n+3}, f^{/}{}_{n+3}$ are obtained from the extrapolation procedure (2.4). The extrapolated explicit method of (2.6) becomes

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = \frac{198}{85} hf_{n+2} - \frac{198}{85} hf_{n+1} + \frac{66}{85} hf_n - \frac{54}{85} h^2 f^{/}{}_{n+2} + \frac{54}{85} h^2 f^{/}{}_{n+1} - \frac{18}{85} h^2 f^{/}{}_{n},$$

$q = 3$

The IMEX method from SDBDF (2.2) for *k*=3 according to (2.5) becomes

$$y_{n+8} - \frac{5644800}{3144919} y_{n+7} + \frac{4939200}{3144919} y_{n+6} - \frac{4390400}{3144919} y_{n+5} + \frac{3087000}{3144919} y_{n+4} - \frac{1580544}{3144919} y_{n+3} + \frac{548800}{3144919} y_{n+2}$$

$$- \frac{115200}{3144919} y_{n+1} + \frac{11025}{3144919} y_n = h\left(\frac{15341760}{3144919} f_{n+7} - \frac{53696160}{3144919} f_{n+6} + \frac{107392320}{3144919} f_{n+5} - \frac{134240400}{3144919} f_{n+4} \right.$$

$$\left. + \frac{107392320}{3144919} f_{n+3} - \frac{53696160}{3144919} f_{n+2} + \frac{15341760}{3144919} f_{n+1} - \frac{1917720}{3144919} f_n + \frac{1917720}{3144919} g_{n+8} \right) + h^2 \left(-\frac{2822400}{3144919} f^{/}{}_{n+7} \right.$$

$$+ \frac{9878400}{3144919} f^{/}{}_{n+6} - \frac{19756800}{3144919} f^{/}{}_{n+5} + \frac{24696000}{3144919} f^{/}{}_{n+4} - \frac{19756800}{3144919} f^{/}{}_{n+3} + \frac{9878400}{3144919} f^{/}{}_{n+2} - \frac{2822400}{3144919} f^{/}{}_{n+1}$$

$$\left. + \frac{352800}{3144919} f^{/}{}_{n} - \frac{352800}{3144919} g^{/}{}_{n+8} \right) \tag{2.14}$$

$$y_{n+3}-\frac{108}{85}y_{n+2}+\frac{27}{85}y_{n+1}-\frac{4}{85}y_n=\frac{198}{85}hf_{n+2}-\frac{198}{85}hf_{n+1}+\frac{66}{85}hf_n-\frac{54}{85}h^2f'_{n+2}+$$
$$\frac{54}{85}h^2f'_{n+1}-\frac{18}{85}h^2f'_n+\frac{66}{85}hg_{n+3}-\frac{18}{85}h^2g'_{n+3}, \qquad p=3$$

The IMEX SDBDF for k=1(1)9based on the SDBDF (2.2) are obtained using (2.3) and following the extrapolation process in (2.4). The resultant IMEX SDBDF methods are now listed below.They will be referred to as IMEX SDBDFk subsequently. The k indicates the step number of the SDBDF in (2.2) from which it was obtained.

IMEX SDBDF1, *p*=1

$$y_{n+1}-y_n=h\left(f_n+g_{n+1}\right)+h^2\left(-\frac{1}{2}f'_n-\frac{1}{2}g'_{n+1}\right) \qquad (2.7)$$

IMEX SDBDF2,*p*=2

$$y_{n+2}-\frac{8}{7}y_{n+1}+\frac{1}{7}y_n=h\left(\frac{12}{7}f_{n+1}-\frac{6}{7}f_n+\frac{6}{7}g_{n+2}\right)+h^2\left(-\frac{4}{7}f'_{n+1}+\frac{2}{7}f'_n-\frac{2}{7}g'_{n+2}\right) \qquad (2.8)$$

IMEX SDBDF3, *p*=3

$$y_{n+3}-\frac{108}{85}y_{n+2}+\frac{27}{85}y_{n+1}-\frac{4}{85}y_n=h\left(\frac{198}{85}f_{n+2}-\frac{198}{85}f_{n+1}+\frac{66}{85}f_n+\frac{66}{85}g_{n+3}\right)+h^2\left(-\frac{54}{85}f'_{n+2}+\frac{54}{85}f'_{n+1}\right.$$
$$\left.-\frac{18}{85}f'_n-\frac{18}{85}g'_{n+3}\right) \qquad (2.9)$$

IMEX SDBDF4,*p*=4

$$y_{n+4}-\frac{576}{415}y_{n+3}+\frac{216}{415}y_{n+2}-\frac{64}{415}y_{n+1}+\frac{9}{415}y_n=h\left(\frac{240}{83}f_{n+3}-\frac{360}{83}f_{n+2}+\frac{240}{83}f_{n+1}-\frac{60}{83}f_n\right. \qquad (2.10)$$
$$\left.+\frac{60}{83}g_{n+4}\right)+h^2\left(-\frac{288}{415}f'_{n+3}+\frac{432}{415}f'_{n+2}-\frac{288}{415}f'_{n+1}+\frac{72}{415}f'_n-\frac{72}{415}g'_{n+4}\right)$$

IMEX SDBDF5, *p*=5

$$y_{n+5}-\frac{18000}{12019}y_{n+4}+\frac{9000}{12019}y_{n+3}-\frac{4000}{12019}y_{n+2}+\frac{1125}{12019}y_{n+1}-\frac{144}{12019}y_n=h\left(\frac{41100}{12019}f_{n+4}\right.$$
$$-\frac{82200}{12019}f_{n+3}+\frac{82200}{12019}f_{n+2}-\frac{41100}{12019}f_{n+1}+\frac{8220}{12019}f_n+\left.\frac{8220}{12019}g_{n+5}\right)+h^2\left(-\frac{9000}{12019}f'_{n+4}\right.$$
$$\left.+\frac{18000}{12019}f'_{n+3}-\frac{18000}{12019}f'_{n+2}+\frac{9000}{12019}f'_{n+1}-\frac{1800}{12019}f'_n-\frac{1800}{12019}g'_{n+5}\right) \qquad (2.11)$$

IMEX SDBDF6,*p*=6

$$y_{n+6}-\frac{21600}{13489}y_{n+5}+\frac{13500}{13489}y_{n+4}-\frac{8000}{13489}y_{n+3}+\frac{3375}{13489}y_{n+2}-\frac{864}{13489}y_{n+1}+\frac{100}{13489}y_n=h\left(\frac{7560}{1927}f_{n+5}\right.$$
$$-\frac{18900}{1927}f_{n+4}+\frac{25200}{1927}f_{n+3}-\frac{18900}{1927}f_{n+2}+\frac{7560}{1927}f_{n+1}-\frac{1260}{1927}f_n+\left.\frac{1260}{1927}g_{n+6}\right)+h^2\left(-\frac{10800}{13489}f'_{n+5}+\right.$$
$$\left.\frac{27000}{13489}f'_{n+4}-\frac{36000}{13489}f'_{n+3}+\frac{27000}{13489}f'_{n+2}-\frac{10800}{13489}f'_{n+1}+\frac{1800}{13489}f'_n-\frac{1800}{13489}g'_{n+6}\right) \qquad (2.12)$$

IMEX SDBDF7,*p*=7

$$y_{n+7}-\frac{1234800}{726301}y_{n+6}+\frac{926100}{726301}y_{n+5}-\frac{686000}{726301}y_{n+4}+\frac{385875}{726301}y_{n+3}-\frac{148176}{726301}y_{n+2}+\frac{34300}{726301}y_{n+1}$$
$$-\frac{3600}{726301}y_n=h\left(\frac{3201660}{726301}f_{n+6}-\frac{9604980}{726301}f_{n+5}+\frac{16008300}{726301}f_{n+4}-\frac{16008300}{726301}f_{n+3}+\frac{9604980}{726301}f_{n+2}\right.$$
$$-\frac{3201660}{726301}f_{n+1}+\frac{457380}{726301}f_n+\left.\frac{457380}{726301}g_{n+7}\right)+h^2\left(-\frac{617400}{726301}f'_{n+6}+\frac{1852200}{726301}f'_{n+5}-\frac{3087000}{726301}f'_{n+4}\right.$$
$$\left.+\frac{3087000}{726301}f'_{n+3}-\frac{1852200}{726301}f'_{n+2}+\frac{617400}{726301}f'_{n+1}-\frac{88200}{726301}f'_n-\frac{88200}{726301}g'_{n+7}\right) \qquad (2.13)$$

IMEX SDBDF8,*p*=8

$$y_{n+8}-\frac{5644800}{3144919}y_{n+7}+\frac{4939200}{3144919}y_{n+6}-\frac{4390400}{3144919}y_{n+5}+\frac{3087000}{3144919}y_{n+4}-\frac{1580544}{3144919}y_{n+3}+\frac{548800}{3144919}y_{n+2}$$
$$-\frac{115200}{3144919}y_{n+1}+\frac{11025}{3144919}y_{n}=h\left(\frac{15341760}{3144919}f_{n+7}-\frac{53696160}{3144919}f_{n+6}+\frac{107392320}{3144919}f_{n+5}-\frac{134240400}{3144919}f_{n+4}\right.$$
$$\left.+\frac{107392320}{3144919}f_{n+3}-\frac{53696160}{3144919}f_{n+2}+\frac{15341760}{3144919}f_{n+1}-\frac{1917720}{3144919}f_{n}+\frac{1917720}{3144919}g_{n+8}\right)+h^{2}\left(-\frac{2822400}{3144919}f'_{n+7}\right.$$
$$+\frac{9878400}{3144919}f'_{n+6}-\frac{19756800}{3144919}f'_{n+5}+\frac{24696000}{3144919}f'_{n+4}-\frac{19756800}{3144919}f'_{n+3}+\frac{9878400}{3144919}f'_{n+2}-\frac{2822400}{3144919}f'_{n+1}$$
$$\left.+\frac{352800}{3144919}f'_{n}-\frac{352800}{3144919}g'_{n+8}\right) \qquad (2.14)$$

IMEX SDBDF9,*p*=9

$$y_{n+9}-\frac{57153600}{30300391}y_{n+8}+\frac{57153600}{30300391}y_{n+7}-\frac{59270400}{30300391}y_{n+6}+\frac{50009400}{30300391}y_{n+5}-\frac{32006016}{30300391}y_{n+4}+\frac{14817600}{30300391}y_{n+3}$$
$$-\frac{4665125}{30300391}y_{n+2}+\frac{893025}{30300391}y_{n+1}-\frac{78400}{30300391}y_{n}=h\left(\frac{161685720}{30300391}f_{n+8}-\frac{646742880}{30300391}f_{n+7}+\frac{1509066720}{30300391}f_{n+6}\right.$$
$$-\frac{2263600080}{30300391}f_{n+5}+\frac{2263600080}{30300391}f_{n+4}-\frac{1509066720}{30300391}f_{n+3}+\frac{646742880}{30300391}f_{n+2}-\frac{161685720}{30300391}f_{n+1}$$
$$\left.+\frac{17965080}{30300391}f_{n}+\frac{17965080}{30300391}g_{n+9}\right)+h^{2}\left(-\frac{28576800}{30300391}f'_{n+8}+\frac{114307200}{30300391}f'_{n+7}-\frac{266716800}{30300391}f'_{n+6}\right.$$
$$+\frac{400075200}{30300391}f'_{n+5}-\frac{400075200}{30300391}f'_{n+4}+\frac{266716800}{30300391}f'_{n+3}-\frac{114307200}{30300391}f'_{n+2}+\frac{28576800}{30300391}f'_{n+1}$$
$$\left.-\frac{3175200}{30300391}f'_{n}-\frac{3175200}{30300391}g'_{n+9}\right) \qquad (2.15)$$

3.0 The stability of the IMEX SDLMM (2.5)

Consider applying the IMEX SDLMM (2.5) to the ODE (1.2) where $f(t,y(t))$ is the non-stiff part and $g(t,y(t))$ is the stiff part of the system. Considering the scalar test problem $y'(t)=\lambda y(t)+\mu y(t),$ it was determined in [5] the conditions under which λh and μh lying in the region of stability of their respective methods are sufficient condition for the IMEX method to be asymptotically stable. But that the independent stability of the explicit and implicit methods does not imply the stability of the IMEX scheme [5]. However the context of that consideration in [5] is for LMM without derivative of the ODE (1.2). The application of this approach of stability analysis is difficult and complicated for multiple part splitting and for a LMM with step number $k\geq 2$. However the approach in [3] is simpler and have been adopted in this work, but with extension to the more generalized Cauchy test problem,

$$y'(t)=\left(\sum_{j=1}^{s}e_j\right)\lambda y(t)-\lambda\sum_{j=1}^{s-1}\left(e_j y(t)\right) \qquad (3.1)$$

as suggested by the additive splitting (1.1). In the consideration herein $e_s=v$ will represent the stiff part of the ODE (1.1), while the rest $\{e_j\}_{j=1(1)s-1}$ are the non-stiff parts. We shall use this to study the stability of the IMEX methods derived in (2.7)-(2.15), where in (3.1)

$$\begin{cases}\mathrm{Re}(\lambda)<0, & \lambda\in\mathbb{C}\\ v=e_s>0,\quad 0<e_j<v, & e_j\in R;\quad j=1(1)s\end{cases}$$

In particular when s=2 as in (1.2),the Cauchy test problem for this is

$$\begin{cases}y'(t)=[g]+[f]=[(e+v)\lambda y(t)]+[-e\lambda y(t)]\\ y''(t)=[g]+[f]=[((e+v)^2-2e(e+v))\lambda^2 y(t)]+[e^2\lambda^2 y(t)]\\ y(0)=1,\quad \mathrm{Re}(\lambda)<0,\quad \lambda\in\mathbb{C};\quad e,v>0\end{cases} \qquad (3.2)$$

The square brackets indicates the nature of the splitting. Notice that this reduces to the Dahlquist test problem $y'(t)=v\lambda y(t)$, with exact solution $y(t)=e^{v\lambda t}$. The stability polynomial $\prod(r,z;e,v)$ is obtained by applying the IMEX method (2.5) on the generalized Dahlquist test problem (3.1), thus

$$\prod(r,z;e,v)= \tag{3.3}$$
$$\sum_{j=0}^{k}\alpha_j r^j+\left(\frac{ez}{v}\right)\sum_{j=0}^{k-1}\beta_j^* r^j-\left(\frac{e^2z^2}{v^2}\right)\sum_{j=0}^{k-1}\lambda_j^* r^j-(e+v)\frac{z}{v}\sum_{j=0}^{k}\beta_j r^j-[(e+v)^2-2e(e+v)]\frac{z^2}{v^2}\sum_{j=0}^{k}\lambda_j r^j$$

where $z=h\lambda v$. From this polynomial $\prod(r,z;e,v)$, the stability of the IMEX SDLMM (2.5) is therefore parameterized by the stability variables *e* and *v*. In fact, these variables suggest the separable additive splitting in (1.2) into a stiff part $g(t,y(t))$ by *v* and non-stiff part $f(t,y(t))$ by *e*. If $e\to 0$ and $v\to 1$ then the stability of the IMEX-SDLMM (2.5) approaches the stability of the underlying implicit second derivative method (2.2). It is indeed exactly so when *e*=0 and *v*=1. The A-stability region of a method (2.5) can be illustrated by plotting its boundary locus curve of the stability polynomial $\prod(r,z;e,v)$, that is the values of $z=h\lambda v$ corresponding to the boundary locus defined by

$$|\prod(r,z;e,v)|\le 1;\ r=e^{i\theta},\ 0<\theta\le 2\pi,\ z=h\lambda v,\ i=\sqrt{-1} \tag{3.4}$$

of its stability polynomial $\prod(r,z;e,v)$ with roots with respect to zlying within or on the unit circle. The collection of *z* for which (3.4) holds defines the absolute stability region of an IMEX method in (2.5). Following Jorgenson [7], the essence here is to explore IMEX methods which, when applied to the Cauchy test problem (3.2), display stable behavior. We shall be considering IMEX methods (2.5) which apply the implicit scheme to the first part of (3.2) and its explicit scheme to the second part. A priori it is not obvious which mixed method will exhibit stable behavior, and if so, whether the stability properties of the implicit or explicit part will dominate in the IMEX method. Consider the IMEX SDBDF (2.7); to see the stability plot of this IMEX SDLMM in terms of step size and roots of its stability polynomial, take $y_{n+i}=r^{n+i},\ z=h\lambda v, z^2=h^2\lambda^2v^2$ and applying to the Cauchy test problem (3.2), we have

$$y_{n+1}-y_n=-e\lambda hy_n-\frac{1}{2}e^2\lambda^2h^2y_n+(e+v)\lambda hy_{n+1}-\frac{1}{2}\left[(e+v)^2-2e(e+v)\right]\lambda^2h^2y_{n+1} \tag{3.5}$$

This implies,

$$\prod(r,z;e,v)=r-1-\left(z\left(-\frac{e}{v}\right)-z^2\frac{1}{2}\left(\frac{e^2}{v^2}\right)+z\left(\frac{e+v}{v}\right)r-\frac{z^2}{2}\left(\frac{(e+v)^2}{v^2}-\frac{2e(e+v)}{v^2}\right)r\right)$$

as in (3.3). So that,

$$r=\frac{1+z\left(\frac{-e}{v}\right)-\frac{z^2}{2}\left(\frac{e^2}{v^2}\right)}{1-z\left(\frac{e+v}{v}\right)+\frac{z^2}{2}\left(\frac{(e+v)^2}{v^2}-\frac{2e(e+v)}{v^2}\right)}$$

This $\prod(r,z;e,v)$ is the stability polynomial in *r* plotted for the parameter *z* according to (3.4). The stability plot for the method (3.5) for *v*=0.1 with *e* allowed to vary is shown in Figs (1.0b,c), the stability plots of the independent extrapolated explicit and its implicit method of the SDBDF is also shown in Fig (1.0a). The implicit method, IM-SDBDF1 is

$$y_{n+1}-y_n=hg_{n+1}-\frac{h^2}{2}g'_{n+1};\qquad r=2$$

from which its extrapolated explicit method, EX-SDBDF1 is

$$y_{n+1}-y_n=hf_n-\frac{1}{2}h^2f'_n;\qquad q=1$$

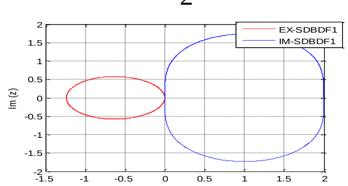

Fig1.0a: Boundary locus of EX-SDBDF1 and IM-SDBDF1

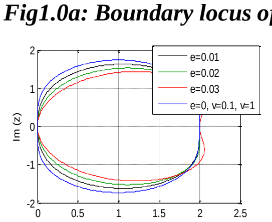

Fig 1.0b

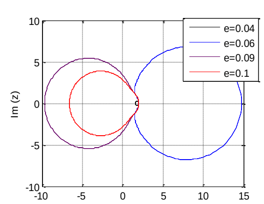

Fig 1.0c

Fig(1.0b,c): Boundary Locus of IMEX SDBDF1 with various e, v=0.1

The plot in the Fig (1.0b) above shows that the stability region of IMEX SDBDF1 is growing with *e*; that is, the region of absolute stability grows as the scaling of the explicit part of the method approaches that of the implicit. Clearly we see that the method is A-stable from *e*=0.01 to *e*=0.03 in Fig (1.0b), but the method becomes unstable from *e*=0.04 to *e*=0.1 in Fig (1.0c). But notice that if *e*=0 for *v*=1, we recover the stability plot of the implicit SDBDF1. The stability plots for the independent implicit and extrapolated explicit SDBDF and the IMEX SDBDF from *k*=2(1)9 are shown in Figs (1.1)-(1.8) with *v*=0.1 and for various values of *e*, the first graphs(a) will be showing the independent explicit and implicit SDBDF, the second graphs(b) will be showing the stability plots for which the IMEX SDBDF method is stable and the third graphs(c) will be showing the stability plots for which the IMEX SDBDF method is exhibiting instability. It is to be noted that for a stable explicit method the interior of the curve is the stability region and for the implicit method that is A/A(α)-stable the exterior of the closed boundary curve is the region of stability. In fact, for *k*=2, the IM-SDBDF2 is,

$$y_{n+2}-\frac{8}{7}y_{n+1}+\frac{1}{7}y_n=\frac{6}{7}hg_{n+2}-\frac{2}{7}h^2g'_{n+2}; \qquad r=3$$

and the EX-SDBDF is,

$$y_{n+2}-\frac{8}{7}y_{n+1}+\frac{1}{7}y_n=h\left(\frac{12}{7}f_{n+1}-\frac{6}{7}f_n\right)+h^2\left(-\frac{4}{7}f'_{n+1}+\frac{2}{7}f'_n\right); \qquad q=2$$

and similarly for the other IM-SDBDF*k*and EX-SDBDF*k*methodsfor*k*=3(1)9 which the boundary loci have been plotted in Figs (1.1)-(1.8).

IMEX SDBDF2 (2.8):

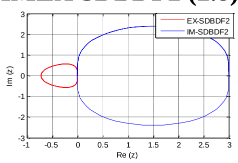

Fig1.1a: Boundary locus of EX-SDBDF2 and IM-SDBDF2

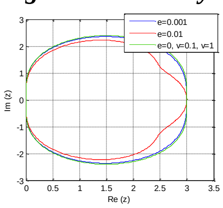

Fig 1.1b

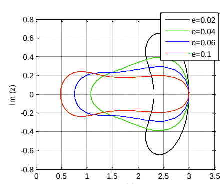

Fig 1.1c

Fig(1.1b,c): Boundary Locus of IMEX SDBDF2 with various e, v=0.1

IMEX SDBDF3 (2.9):

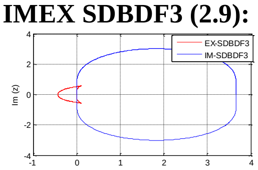

Fig1.2a: Boundary locus of EX-SDBDF3 and IM-SDBDF3

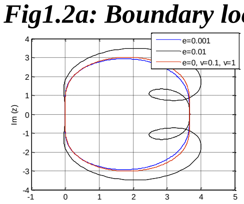

Fig 1.2b

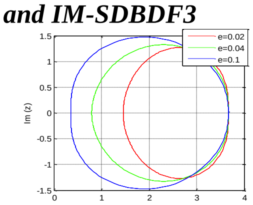

Fig 1.2c

Fig(1.2b,c): Boundary Locus of IMEX SDBDF3 with various e, v=0.1

IMEX SDBDF4 (2.10):

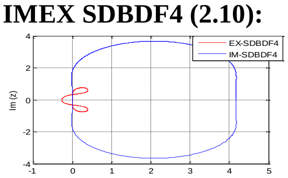

Fig1.3a: Boundary locus of EX and IM SDBDF4

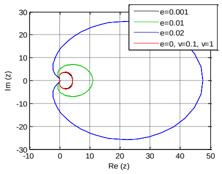

Fig 1.3b

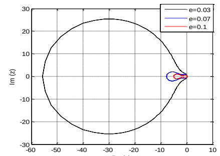

Fig 1.3c

Fig(1.3b,c): Boundary Locus of IMEX SDBDF4 with various e, v=0.1

IMEX SDBDF5 (2.11):

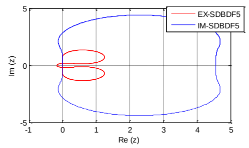

Fig1.4a: Boundary locus of EX-SDBDF5 and IM-SDBDF5

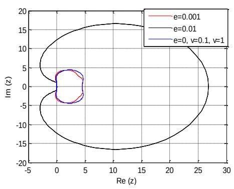

Fig 1.4b

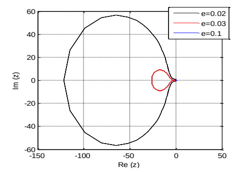

Fig 1.4c

Fig(1.4b,c): Boundary Locus of IMEX SDBDF5 with various e, v=0.1

IMEX SDBDF6 (2.12):

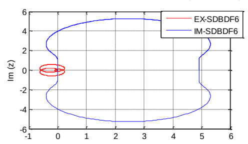

Fig1.5a: Boundary locus of EX-SDBDF6 and IM-SDBDF6

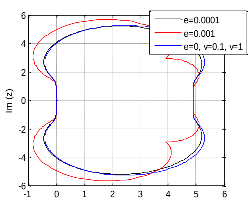

Fig 1.5b

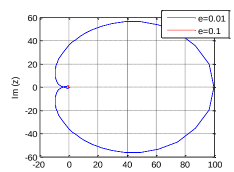

Fig 1.5c

Fig(1.5b,c): Boundary Locus of IMEX SDBDF6 with various e, v=0.1

IMEX SDBDF7 (2.13):

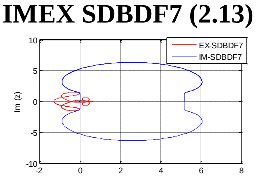

Fig1.6a: Boundary locus of EX-SDBDF7 and IM-SDBDF7

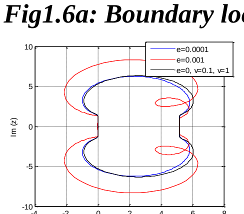

Fig 1.6b

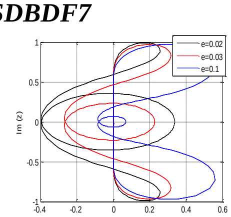

Fig 1.6c

Fig(1.6b,c): Boundary Locus of IMEX SDBDF7 with various e, v=0.1

IMEX SDBDF8 (2.14):

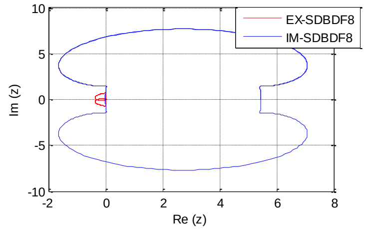

Fig1.7a: Boundary locus of EX-SDBDF8 and IM-SDBDF8

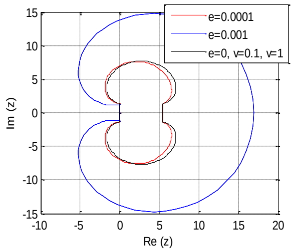

Fig 1.7b

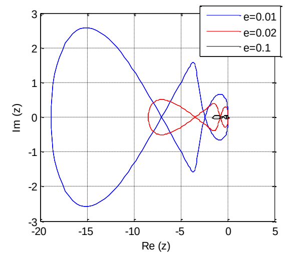

Fig 1.7c

Fig(1.7b,c): Boundary Locus of IMEX SDBDF8 with various e, v=0.1

IMEX SDBDF9 (2.15):

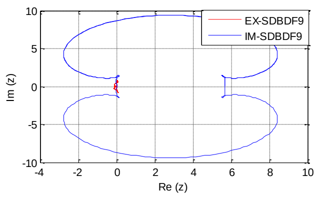

Fig1.8a: Boundary locus of EX-SDBDF9 and IM-SDBDF9

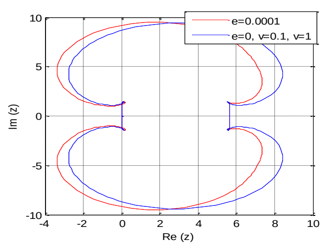

Fig 1.8b

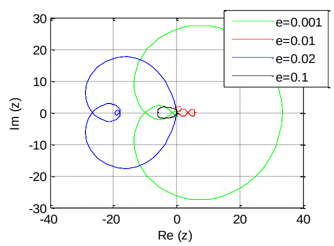

Fig 1.8c

Fig(1.8b,c): Boundary Locus of IMEX SDBDF9 with various e, v=0.1

Observe that for k=5(1)9, the extrapolated explicit methods from the SDBDF (2.2) shows instability, but yet the arising IMEX SDBDF, *k*=1(1)9 shows stability for some values of *e*. This is in line with the fact that the stability of independent explicit and implicit method does not imply the stability of the IMEX method as remarked in [5]. For further insight see [8-23].

4. Numerical experiments and applications of the IMEX SDLMM.

Consider some numerical experiments on the following initial value problems with different additive splitting.

Problem (1): Cauchy test problem [7]

(a) $$y'(t)=[g]+[f]=[(e+v)\lambda y(t)]+[-e\lambda y(t)] \qquad (4.1)$$
$$y(0)=1,\ \lambda=-10,\ e=0.03,\ v=0.1$$

has the exact solution $y(t)=e^{v\lambda t}$

(b) $$y'(t)=-10\frac{1}{2}y(t)=[g]+[f]=[-10y(t)]+\left[-\frac{1}{2}y(t)\right] \qquad (4.2)$$
$$y(0)=1$$

with its exact solution as $y(t)=e^{-10\frac{1}{2}t}$

Problem (2): Prothero-Robinson test problem, see [24] and [25]with the additive splitting (1.2) as

$$y^{/}(t)=[g]+[f]=[\lambda(y(t)-q(t))]+[q^{/}(t)] \qquad (4.3)$$
$$t\geq 0,\ y(0)=q(0),\ \lambda<0$$

The exact solution is given by $y(t)=q(t)$, here choose that

$$q(t)=\sin\left(\frac{\pi}{4}+t\right) \qquad and \quad \lambda=-10^{2}$$

The application of the method (2.5) to problems 1 and 2respectively leads to solving an implicit equation for the solution component y_{n+k} which is resolved by applying the Newton-Raphson scheme on (2.5) to get,

$$y_{n+k}^{[s+1]}=y_{n+k}^{[s]}-\left(J\left(y_{n+k}^{[s]}\right)\right)^{-1}F\left(y_{n+k}^{[s]}\right),\quad s=0,1,2,...,w \qquad (4.4)$$

where $J\left(y_{n+k}^{[s]}\right)$ is the Jacobian matrix from

$$F\left(y_{n+k}^{[s]}\right)=y_{n+k}^{[s]}-\left[\sum_{j=0}^{k-1}\alpha_j y_{n+j}+h\sum_{j=0}^{k-1}\beta^{*}{}_{j}f(t_{n+j},y_{n+j})+h^{2}\sum_{j=0}^{k-1}\lambda^{*}{}_{j}f^{/}(t_{n+j},y_{n+j})\right.$$
$$\left.+h\sum_{j=0}^{k-1}\beta_j g(t_{n+j},y_{n+j})+h^{2}\sum_{j=0}^{k-1}\lambda_j g^{/}(t_{n+j},y_{n+j})\right]+h\beta_k g(t_{n+k},y_{n+k}^{[s]})+h^{2}\lambda_k g^{/}(t_{n+k},y_{n+k}^{[s]}), \qquad s=0,1,2,...,w$$

The solution y_{n+k} is thus given by

$$y_{n+k}=-\sum_{j=0}^{k-1}\alpha_j y_{n+j}+h\sum_{j=0}^{k-1}\beta^{*}{}_{j}f(t_{n+j},y_{n+j})+h^{2}\sum_{j=0}^{k-1}\lambda^{*}{}_{j}f^{/}(t_{n+j},y_{n+j})+h\sum_{j=0}^{k-1}\beta_j g(t_{n+j},y_{n+j})$$
$$+h^{2}\sum_{j=0}^{k-1}\lambda_j g^{/}(t_{n+j},y_{n+j})]+h\beta_k g(t_{n+k},y_{n+k}^{[w]})+h^{2}\lambda_k g^{/}(t_{n+k},y_{n+k}^{[w]})$$

from(2.5). The starting values for (4.4) is from the explicit SDBDF

$$y_{n+1}^{[0]}=y_n+hf_n+\frac{h^{2}}{2}f^{/}{}_{n} \qquad p=2,\quad C_3=\frac{1}{6}$$

Problem 1 and 2 will be solved with the IMEX method (2.7-2.10) in the interval [0,1] with step size $h=0.001$ and $h=0.0125$ respectively. Note that the stiff term is $g=(e+v)\lambda y(t)$ and the non-stiff term is $f=-e\lambda y(t)$ and $g=-10y(t),\quad f=-\frac{1}{2}y(t)$ for (4.1) and (4.2) respectively. While for (4.3), $g=\lambda(y(t)-q(t)),\ f=q^{/}(t).$ The numerical solutions $\bar{y}(1)$ of the IMEX methods (2.7-2.10) for each problem and its corresponding absolute error $|\bar{y}(1)-y(1)|$ with output at *t*=1 will be shown in the tables 1,2,3 and the graphs of the solutions from the IMEX methods (2.7)-(2.10) for each problem at each point of $t\in[0,1]$ will be shown as graphs in figs (4.1-4.3).

Table1. Problem (1a) (4.1), $y(1)=4.53999297624848e-005;\qquad h=0.001;\quad \lambda=-100$

Methods*k*	$\lvert\bar{y}(1)-y(1)\rvert=error$	
	IMEX SDBDF*k*	SDBDF*k*
1	9.13061971219859*e*-007	4.44293392277166*e*-007
2	8.91508411362085*e*-007	8.91497007016978*e*-007
3	1.33435517756372*e*-006	1.33435529383837*e*-006
4	1.77280735728846*e*-006	2.51081575422581*e*-006

Table2. Problem (1b) (4.2), $y(1)=2.75364493497472\text{e-}005;\qquad h=0.001$

Methods*k*	$\lvert\bar{y}(1)-y(1)\rvert=error$	
	IMEX SDBDF*k*	SDBDF*k*
1	4.24607454292506*e*-007	2.82398260454951*e*-007
2	5.66984608429898*e*-007	5.66985883467895*e*-007
3	8.48678136848717*e*-007	8.48678123208292*e*-007
4	1.12742828404115*e*-006	1.61893514155383*e*-006

Table3. Problem(2) (4.3), $y(1) = 0.977061263899476; \quad h = 0.0125; \quad \lambda = -100$

Methods k	$\lvert \bar{y}(1) - y(1) \rvert = error$	
	IMEX SDBDFk	SDBDFk
1	1.19757561528899e-003	1.19757561528933e-003
2	3.20900378985900e-003	3.53066817859904e-003
3	3.61490600078684e-003	3.90243650617472e-003
4	3.43457374914502e-003	3.66932598718350e-003

The IMEX SDBDF$k$methods in section (2.3) resolves the implicitness in the numerical solution of (1.1) in a more cost effective way when compared with the SDBDFk (2.2). And yet on the problems solved, the IMEX SDBDF$k$and SDBDFk(2.2)gives the same numerical order of accuracy as the graphs of the numerical and exact solutions in Figs (4.1)-(4.3) and Tables 1,2,3 will show.

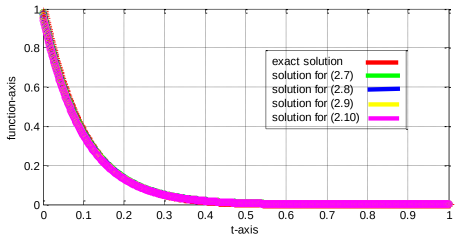

Fig 4.1: Exact and numerical solution of problem4.1 with IMEX methods(2.7)-(2.10)

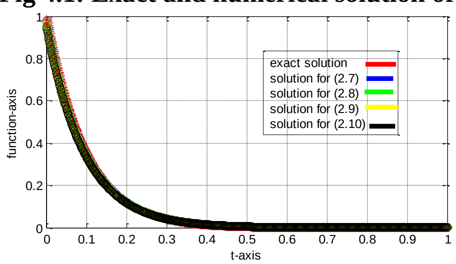

Fig 4.2: Exact and numerical solution of problem4.2 with IMEX methods(2.7)-(2.10)

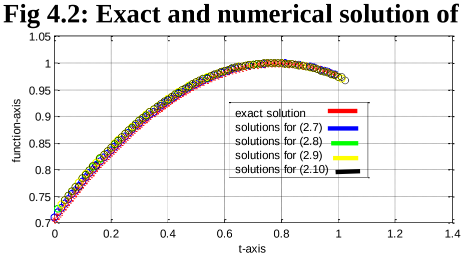

Fig 4.3: Exact and numerical solution of problem4.3 with IMEX methods(2.7)-(2.10)

Conclusively a family of variable order IMEX SDLMM (2.7)-(2.15) for the direct solution of IVPs in ODEs is considered for additively separable ODEs (1.1). The methods are based on the SDBDF (2.2). The boundary loci in Figs(1.0) – Fig (1.8) respectively shows that the proposed schemes from (2.5) based on the SDBDF (2.2) for step length k=1(1)9 for some values of e are stable on the test problem (3.1). Furthermore, the numerical results in Tables 1,2,3 respectively shows that the IMEX SDBDFk algorithm in section (2.3) compares favorably with the exact solutions of each problem. IMEX methods which are based on SDLMM of Enright (1982) [26] can also be considered.

Acknowledgment
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