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NEW THIRD DERIVATIVE LINEAR MULTISTEP METHODS FOR STIFF IVPs

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Abstract

In this article, a class of stiffly stable third derivative linear multistep methods (TDLMM) is presented and analysed. The newly proposed method is a modification of the third derivative backward differential formula (TDBDF). The TDBDF is inefficient for order $p = 12$. The proposed class of method is stable for order $p \leq 12$. Numerical tests on linear and nonlinear stiff systems of initial value problems show that, the proposed method compares favourably with TDBDF.

Keywords: Stability region, Stiffly stable, Order.

1.0 Introduction

Consider a stiff system given as

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0,\ f : R \times R^N \to R^N, y_0 \in R^N. \qquad (1)$$

Methods for the numerical approximation of the solution of (1) are expected to possess wide regions of absolute stability R_A, which contain the entire or large enough left half of the complex plane [1,2,3]. Of which methods, whose region of absolute stability contains the entire left half of the complex plane are known as A-stable methods [4,5]. However, A-stable methods are implicit and cannot exceed order $p = 2$ [1-5]. This barrier is popularly known as the second Dahlquist order barrier [2]. To overcome this barrier, different research approaches have been devised and utilized, such inclusion of higher derivative term directly into the method, and addition of future points in the method approach [5]. In [6], the relationship between the mth derivative term present in a method and the maximum order an A-stable method can attain is established. This established relationship is known as the Daniel Moore's conjecture and it states thus: the maximum order of an A-stable linear multistep method (LMM) with m derivative is $2m$. The simple interpretation is that, to derive say for example higher derivative methods whose maximum order $p = 6$, then the third derivative of the solution component must be infused directly into the method. The backward differentiation formula (BDF) is generalized in [7]. Of a particular interest in this article is the third derivative BDF. The third derivative BDF is unstable for order $p \geq 12$. This article seeks to improve on the efficiency of the third derivative BDF. The arrangement of this article is as follows: in section 2 is the construction of methods, while the stability analysis of proposed methods is in section 3. Section 4, is on numerical experiments and conclusion is in section 5.

2.0 Construction of Method

The general k-step third derivative linear multistep method (TDLMM) for solving the IVP (1) is of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\sum_{j=0}^{k}\gamma_j f'_{n+j} + h^3\sum_{j=0}^{k}\varphi_j f''_{n+j} \qquad (2)$$

where $\alpha_k = 1$, $f'_{n+j} \equiv \left.\frac{df(x, y(x))}{dx}\right|_{x=x_{n+j}}$, $f''_{n+j} \equiv \left.\frac{df'(x, y(x))}{dx}\right|_{x=x_{n+j}}$, $\alpha_j, \beta_j,\ \gamma_j$ and φ_j, $j = 0, 1, \ldots, k$ are parameters to be

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determined. The TDLMM (2) is explicit if β_k, γ_k and φ_k are zero at the same time, else it is implicit. The TDLMM (2) can be written in polynomial notation as

$$\rho(E)y_n = h\sigma(E)f_n + h^2\pi(E)f_n' + h^3\eta(E)f_n'' \quad (3)$$

where $\rho(E)=\sum_{j=0}^{k}\alpha_j E^j,\ \sigma(E)=\sum_{j=0}^{k}\beta_j E^j,\ \pi(E)=\sum_{j=0}^{k}\gamma_j E^j$ and $\eta(E)=\sum_{j=0}^{k}\varphi_j E^j.$ (4)

The polynomials: $\rho(E); \sigma(E); \pi(E)$ and $\eta(E)$ are called the first, second, third and fourth characteristics polynomials respectively. The linear difference operator associated with the TDLMM (2) is given as

$$L[\mathrm{y(x}_n);\mathrm{h}] = \sum_{j=0}^{k}[\alpha_j y(x_n+jh) - h\beta_j y'(x_n+jh) - h^2\gamma_j y''(x_n+jh) - h^3\varphi_j y'''(x_n+jh)] \quad (5)$$

It is assumed that (5) is differentiable as often as required, Taylor expanding (5) about x_n yields

$$L[\mathrm{y(x}_n);\mathrm{h}] = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \ldots + C_q h^q y^q(x_n) + \ldots \quad (6)$$

so that

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k}\alpha_j \\ C_1 &= \sum_{j=0}^{k}(j\alpha_j - \beta_j) \\ C_2 &= \sum_{j=0}^{k}\left(\frac{j^2}{2!}\alpha_j - j\beta_j - \gamma_j\right) \\ C_3 &= \sum_{j=0}^{k}\left(\frac{j^3}{3!}\alpha_j - \frac{j^2}{2!}\beta_j - j\gamma_j - \varphi_j\right) \\ &\vdots \\ C_q &= \sum_{j=0}^{k}\left(\frac{j^q}{q!}\alpha_j - \frac{j^{q-1}}{(q-1)!}\beta_j - \frac{j^{q-2}}{(q-2)!}\gamma_j - \frac{j^{q-3}}{(q-3)!}\varphi_j\right),\quad \mathrm{q}=3,4,\cdots \end{aligned}\right\} \quad (7)$$

The TDLMM (2) and its associated linear difference operator (5) is of order p if $C_0 = C_1 = \ldots = C_p = 0,$ and $C_{p+1} \neq 0.$ $C_{p+1} \neq 0$ is called the error constant.

Consider the third derivative backward differentiation formula (TDBDF) given as

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\beta_k^{(1)} f_{n+k} + h^2\beta_k^{(2)} f'_{n+k} + h^3\beta_k^{(3)} f''_{n+k}, \quad (8)$$

In polynomial notation, (8) can be written as

$$\rho(E) = h\sigma(E) + h^2\vartheta(E) + h^3\tau(E), \quad (9)$$

where $\rho(E)=\sum_{i=0}^{k}\alpha_i E^i,\ \sigma(E)=\beta_k^{(1)}E^k,\ \vartheta(E)=\beta_k^{(2)}E^k,\ \tau(E)=\beta_k^{(3)}E^k$ are the first, second, third and fourth characteristics polynomials of the third derivative backward differentiation formulae (TDBDF) respectively [1,4]. The TDBDF (8) has been shown to be *A*-stable for order $p \le 6$, $A(\alpha)$ -stable for order $p \le 11$ and unstable for order $p \ge 12,$ [8]. In the spirit of Muka and Obiorah (2016), a non-zero coefficient is introduced into the second characteristics polynomials of (8), thereby resulting into a new class of methods. This is done by introducing a non-zero coefficient $\beta_{k-\mu}^{(1*)}$ of the function f evaluated at the point $x_{n+k-\mu}$. This is a modification of the second characteristics polynomial of the scheme (8) in the following way

$$\sigma(E) = \beta_k^{(1)}E^k + \beta_{k-\mu}^{(1*)}E^{k-\mu}. \quad (10a)$$

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h(\beta_k^{(1)} f_{n+k} + \beta_{k-\mu}^{(1*)} f_{n+k-\mu}) + h^2\beta_k^{(2)} f'_{n+k} + h^3\beta_k^{(3)} f''_{n+k}, \quad (10b)$$

where $\beta_k^{(v)},\ \mathrm{v}=1,2,3,\ \beta_{k-\mu}^{(1*)},\ \mu=1,2,...,k$ are non-zero parameters to be determined. The parameters of methods (10b) are determined using Taylor's series expansion and method of undetermined coefficients. Parameters

α_i, i = 1, 2,..., k, $\beta_k^{(v)}$, v = 1, 2, 3, $\beta_{k-\mu}^{(1*)}$, $\mu = 1, 2, ..., k$ are determined completely by requiring (10b) to be of order $p = k+3$. If $\beta_{k-\mu}^{(1*)} = 0$ in (10a), then (10a) reduces to the TDBDF (8). The linear difference operator $L\ [x_n, y(x_n); h]$ associated with the TDLMM (10b) is given as

$$L\ [x_n, y(x_n); h] = \sum_{i=0}^{k} \alpha_i y(x_{n+i}) - h(\beta_k^{(1)} f(x_{n+k}, y(x_{n+k})) + \beta_{k-\mu}^{(1*)} f(x_{n+k-\mu}, y(x_{n+k-\mu}))) - h^2 \beta_k^{(2)} f'(x_{n+k}, y(x_{n+k})) \quad (11)$$
$$- h^3 \beta_k^{(3)} f''(x_{n+k}, y(x_{n+k})),$$

Expanding (11) about x_n yields

$$L\ [x_n, y(x_n); h] = C_0 y(x_n) + C_1 y'(x_n) + C_2 y''(x_n) + ... + C_q\, y^q(x_n) + \ldots \quad (12)$$

The coefficients $C_o, C_1, \ldots, C_q$ are given as follows

$$\left.\begin{aligned}
C_0 &= \sum_{i=0}^{k} \alpha_i \\
C_1 &= \sum_{i=0}^{k} i\alpha_i - \beta_k^{(1)} - \beta_{k-\mu}^{(1*)} \\
C_2 &= \sum_{i=0}^{k} \frac{i^2}{2!}\alpha_i - k\beta_k^{(1)} - (k-\mu)\beta_{k-\mu}^{(1*)} - \beta_k^{(2)} \\
C_3 &= \sum_{i=0}^{k} \frac{i^3}{3!}\alpha_i - \frac{k^2}{2!}\beta_k^{(1)} - \frac{(k-\mu)^2}{2!}\beta_{k-\mu}^{(1*)} - k\beta_k^{(2)} - \beta_k^{(3)} \\
&\vdots \\
C_q &= \sum_{i=0}^{k} \frac{i^q}{q!}\alpha_i - \frac{k^{(q-1)}}{(q-1)!}\beta_k^{(1)} - \frac{(k-\mu)^{(q-1)}}{(q-1)!}\beta_{k-\mu}^{(1*)} - \frac{k^{(q-2)}}{(q-2)!}\beta_k^{(2)} - \frac{k^{(q-3)}}{(q-3)!}\beta_k^{(3)}, \quad q \geq 4
\end{aligned}\right\} \quad (13)$$

TDLMM (10b) of order $p = k+3$ can be obtained by solving the first $(k+4)$ system of linear equations (13) using the linear solver of MATHEMATICA 10 suite. By solving the first $(k+4)\times(k+4)$ system of linear equations in (13) will yield parameters that are functions of μ. Therefore, for each k-step and varying $\mu = 1, 2, \ldots, k$ will lead to the derivation of k number of methods of k-step TDLMM.

For $k = 1$ and $\mu = 1$, (10b) becomes

$$y_{n+1} + \alpha_0 y_n = h(\beta_1^{(1)} f_{n+1} + \beta_0^{(1*)} f_n) + h^2 \beta_1^{(2)} f'_{n+1} + h^3 \beta_1^{(3)} f''_{n+1} \quad (14)$$

By setting $C_q = 0, \quad q = 0(1)4$ in (13) yields the following order conditions

$$\left.\begin{aligned}
C_0 &= 1 + \alpha_0 = 0 \\
C_1 &= 1 - \beta_1^{(1)} - \beta_0^{(1*)} = 0 \\
C_2 &= \frac{1}{2} - \beta_1^{(1)} - \beta_1^{(2)} = 0 \\
C_3 &= \frac{1}{6} - \frac{\beta_1^{(1)}}{2} - \beta_1^{(2)} - \beta_1^{(3)} = 0 \\
C_4 &= \frac{1}{24} - \frac{\beta_1^{(1)}}{6} - \frac{\beta_1^{(2)}}{2} - \beta_1^{(3)} = 0
\end{aligned}\right\} \quad (15)$$

Solving the system of linear equations (15), yields $\alpha_0 = -1, \beta_0^{(1*)} = \frac{1}{4}, \beta_1^{(1)} = \frac{3}{4}, \beta_1^{(2)} = -\frac{1}{4}, \beta_1^{(3)} = \frac{1}{24}$.

Inserting the values of the parameters as determined from (15), yields

$$y_{n+1} = y_n + \frac{h}{4}(3f_{n+1} + f_n) - \frac{h^2}{4} f'_{n+1} + \frac{h^3}{24} f''_{n+1} \quad (16)$$

The one-step $\mu = 1,$ TDLMM (16) has error constant $C_5 = \frac{-1}{480}$.

For $k=2,$ (10b) becomes

$$y_{n+2}+\alpha_1 y_{n+1}+\alpha_0 y_n = h(\beta_2^{(1)} f_{n+2}+\beta_{2-\mu}^{(1*)} f_{n+2-\mu})+h^2\beta_2^{(2)} f'_{n+2}+h^3\beta_2^{(3)} f''_{n+2} \qquad (17)$$

Setting $C_q=0, \quad q=0(1)5$ in (13) yields

$$\left.\begin{aligned}
&C_0=1+\alpha_1+\alpha_0=0\\
&C_1=2+\alpha_1-\beta_2^{(1)}-\beta_{2-\mu}^{(1*)}=0\\
&C_2=2+\frac{\alpha}{2}-2\beta_2^{(1)}-(2-\mu)\beta_{2-\mu}^{(1*)}-\beta_2^{(2)}=0\\
&C_3=\frac{4}{3}+\frac{\alpha_1}{6}-2\beta_2^{1}-\frac{(2-\mu)^2}{2}\beta_{2-\mu}^{(1*)}-2\beta_2^{(2)}-\beta_2^{(3)}=0\\
&C_4=\frac{2}{3}+\frac{\alpha_1}{24}-\frac{4}{3}\beta_2^{(1)}-\frac{(2-\mu)^3}{6}\beta_{2-\mu}^{(1*)}-2\beta_2^{(2)}-2\beta_2^{(3)}=0\\
&C_5=\frac{4}{15}+\frac{\alpha_1}{120}-\frac{2}{3}\beta_2^{(1)}-\frac{(2-\mu)^4}{24}\beta_{2-\mu}^{(1*)}-\frac{4}{3}\beta_2^{(2)}-2\beta_2^{(3)}=0
\end{aligned}\right\} \qquad (18)$$

For $\mu=1$, solving the system (18), yields

$$\alpha_0=-\frac{1}{49},\ \alpha_1=-\frac{48}{49},\ \beta_2^{(1)}=\frac{34}{49},\ \beta_1^{(1*)}=\frac{16}{49},\ \beta_2^{(2)}=-\frac{10}{49},\ \beta_2^{(3)}=\frac{4}{147};$$

and inserting these values in (17) to obtain

$$y_{n+2}=\frac{48}{49}y_{n+1}+\frac{1}{49}y_n-\frac{h}{49}(34f_{n+2}+16f_{n+1})+\frac{10h^2}{49}f'_{n+2}-\frac{4h^3}{147}f''_{n+2}, \qquad (19)$$

The Two-step $\mu=1,$ TDLMM (19) has error constant $C_6=\frac{-1}{2205}$.

Similarly, for $\mu=2$, solving the system (18), we obtain

$$\alpha_0=\frac{3}{13},\ \alpha_1=-\frac{16}{13},\ \beta_2^{(1)}=\frac{11}{13},\ \beta_0^{(1*)}=-\frac{1}{13},\ \beta_2^{(2)}=-\frac{4}{13},\ \beta_2^{(3)}=\frac{2}{39};$$

Inserting these values in (17) yields

$$y_{n+2}=\frac{16}{13}y_{n+1}-\frac{3}{13}y_n-\frac{h}{13}(11f_{n+2}-f_n)+\frac{4h^2}{13}f'_{n+2}-\frac{2h^3}{39}f''_{n+2}, \qquad (20)$$

Two-step $\mu=2,$ TDLMM (20) has error constant $C_6=\frac{-1}{585}$.

Other members of the class of TDLMM (10b) can be derived in like manner. Coefficients of the TDLMM (10b) are presented for $k=1,2,3,4$ along with their error constant C_{p+1} and order p in Table 1.

Table 1: Coefficients of TDLMM (10b) for each k and μ values.

k	μ	p	$\beta_k^{(3)}$	$\beta_k^{(2)}$	$\beta_k^{(1)}$	$\beta_{k-\mu}^{(1*)}$	α_0	α_1	α_2	α_3	α_4	C_{p+1}
1	1	4	$\frac{1}{24}$	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	-1	1				$-\frac{1}{480}$
2	1	5	$\frac{4}{147}$	$-\frac{10}{49}$	$\frac{34}{49}$	$\frac{16}{49}$	$-\frac{1}{49}$	$-\frac{48}{49}$	1			$-\frac{1}{2205}$
	2	5	$\frac{2}{39}$	$-\frac{4}{13}$	$\frac{11}{13}$	$-\frac{1}{13}$	$\frac{3}{13}$	$-\frac{16}{13}$	1			$-\frac{1}{585}$
3	1	6	$\frac{36}{1697}$	$-\frac{306}{1697}$	$\frac{1122}{1697}$	$\frac{648}{1697}$	$\frac{4}{1697}$	$-\frac{81}{1697}$	$-\frac{1620}{1697}$	1		$-\frac{9}{59395}$
	2	6	$\frac{9}{239}$	$-\frac{63}{239}$	$\frac{381}{478}$	$-\frac{81}{478}$	$\frac{4}{239}$	$\frac{81}{239}$	$-\frac{324}{239}$	1		$-\frac{9}{16730}$
	3	6	$\frac{36}{797}$	$-\frac{234}{797}$	$\frac{666}{797}$	$\frac{24}{797}$	$-\frac{68}{797}$	$\frac{243}{797}$	$-\frac{972}{797}$	1		$-\frac{27}{27895}$
4	1	7	$\frac{288}{16153}$	$-\frac{2664}{16153}$	$\frac{10308}{16153}$	$\frac{6912}{16153}$	$-\frac{9}{16153}$	$\frac{128}{16153}$	$-\frac{1296}{16153}$	$-\frac{14976}{16153}$	1	$-\frac{36}{565355}$
	2	7	$\frac{288}{9451}$	$-\frac{2232}{9451}$	$\frac{7212}{9451}$	$-\frac{2592}{9451}$	$-\frac{27}{9451}$	$\frac{512}{9451}$	$\frac{3888}{9451}$	$-\frac{13824}{9451}$	1	$-\frac{72}{330785}$
	3	7	$\frac{288}{7969}$	$-\frac{2088}{7969}$	$\frac{6372}{7969}$	$\frac{768}{7969}$	$-\frac{81}{7969}$	$-\frac{1408}{7969}$	$\frac{3888}{7969}$	$-\frac{10368}{7969}$	1	$-\frac{108}{278915}$
	4	7	$\frac{144}{3671}$	$-\frac{1008}{3671}$	$\frac{2994}{3671}$	$-\frac{54}{3671}$	$\frac{153}{3671}$	$-\frac{512}{3671}$	$\frac{1296}{3671}$	$-\frac{4608}{3671}$	1	$-\frac{72}{128485}$

3.0 STABILITY ANALYSIS

In this section, the stability properties of the TDLMM (10b) is examined.

Definition 1: The $k-$ *step TDLMM (10b) is zero-stable if the roots* $w_j,\quad j=1,2,\ldots,k$ *of the first characteristics polynomial* $\rho(w)$ *are such that* $|w_j|\le 1,\quad j=1,2,\ldots,k$ *and* $|w_j|=1$ *being simple.*

Applying the TDLMM (10b) to the scalar test equation (21)

$$y'(x)=\lambda y \tag{21}$$

yields the stability polynomial

$$\pi(w,z)=\sum_{i=0}^{k}\alpha_i w^i - z(\beta_k^{(1)}w_k+\beta_{k-\mu}^{(1*)}w^{k-\mu})-z^2\beta_k^{(2)}w^k-z^3\beta_k^{(3)}w^k, \tag{22}$$

equating (22) to zero, yields the characteristics equation

$$\pi(w,z)=\sum_{j=0}^{k}\alpha_j w^j=0 \tag{23}$$

The roots of the first characteristics polynomial of the TDLMM (10b) are verified to be zero stable for all μ, k-step members, except for $k=9,\mu=2,3,4$ which are zero unstable. The boundary locus method is used to obtain the polynomial which describes the stability regions of the TDLMM (10b), by inserting $w=e^{t\theta},\quad t=0,1,2,\ldots k,$ in (22) and equating to zero, yields a polynomial of degree three in z. The three roots of z (a function of θ) describe the stability region of TDLMM (10b). The stability characteristics of the TDLMM (10b) are presented in Table 2. The search of stable TDLMM (10b) is carried out with the aid of MATHEMATICA 10. The entries of Table 2 labelled "-" corresponds to unstable members of the TDLMM (10b). The stability plots of the TDLMM (10b) are presented for k = 1,2,3 and are given in Figures 1 - 6. The α-values of TDLMM (10b) are summarized in Table 2.

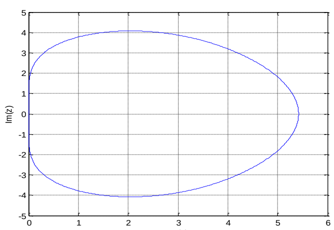

Fig. 1: R_A **of TDLMM (10b) for** $k=1,\mu=1$

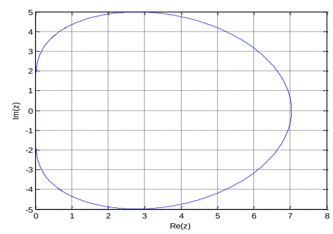

Fig. 2: R_A **of TDLMM (10b) for** $k=2,\mu=1$

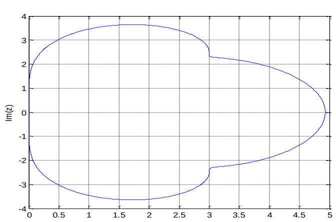

Fig. 3: R_A **of TDLMM (10b) for** $k=2,\mu=2$

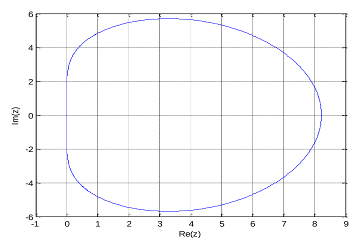

Fig. 4: R_A **of TDLMM (10b) for** $k=3,\mu=1$

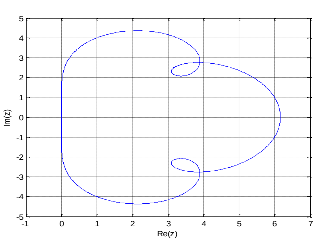

Fig. 5: R_A **of TDLMM (10b) for** $k=3,\mu=2$

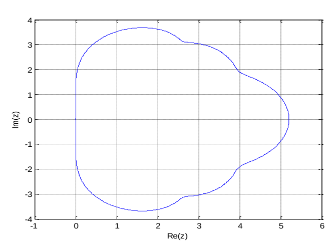

Fig. 6: R_A **of TDLMM (10b) for** $k=3,\mu=3$

Table 2: α-values of TDLMM (10b)

k \ μ	1	2	3	4	5	6	7	8	9
1	90°								
2	90°	90°							
3	89°	89°	89°						
4	88°	89°	90°	89°					
5	88.5°	87°	88°	89°	89°				
6	86°	83°	87°	85°	80°	86°			
7	80°	76°	78°	75°	80°	82°	82°		
8	76°	62°	-	69°	73°	74°	79°	80°	
9	71°	-	-	-	-	64°	69°	67°	68°

4 NUMERICAL EXPERIMENT

Numerical tests are carried out on three standard problems using the TDLMM (20) and the TDBDF (8). The Newton-Raphson iterative scheme is used so as to resolve the implicitness in the methods with a fixed stepsize, h.

Problem 1

The linear system of stiff initial value problem [9]

$$\begin{aligned} y_1' &= -y_1, \\ y_2' &= -10y_2, \\ y_3' &= -100y_3, \\ y_4' &= -1000y_4, \end{aligned} \quad , y(0) = (1,1,1,1)^T, \;\; x \in [0,1]$$

whose exact solution are: $y_1(x) = e^{-x}, y_2(x) = e^{-10x}, y_3(x) = e^{-100x}, y_4(x) = e^{-1000x}$

The Numerical results are presented in Table 3 for the given problem solved with $h = 0.0001$

Table 3: Absolute errors of y_1 and y_2 solution components for problem 1

x	y_i	$\lvert y(x) - y_{TDLMM\,(21)}\rvert$	$\lvert y(x) - y_{TDBDF}\rvert$
0.2	y_1	$5.6424540091026820E-5$	$8.2116040260982180E-5$
	y_2	$9.3793039498996090E-5$	$1.3500694687401670E-4$
0.4	y_1	$4.6444592754935066E-5$	$6.6982840909002300E-5$
	y_2	$1.2691008150999411E-5$	$1.8273697646002673E-5$
0.6	y_1	$3.7623928158980746E-5$	$5.5242600008997830E-5$
	y_2	$1.7140412139999770E-6$	$2.4765753170001570E-6$
0.8	y_1	$3.1065850371048140E-5$	$4.4966832158988000E-5$
	y_2	$2.3124759900001015E-7$	$3.3589058199998090E-7$
1.0	y_1	$2.5905055525965448E-5$	$3.6345239889989944E-5$
	y_2	$3.1368503999999490E-8$	$4.5385289000001610E-8$

From Table 3, it can be observed that the absolute errors from the TDLMM (20) is smaller compared with that of the TDBDF (8).

Problem 2

Consider the nonlinear equation, a special case of the Riccati equation [5] $y' = -\frac{y^3}{2}, \quad y(0) = 1, \quad x \in [1,10]$

The exact solution is $y = \frac{1}{\sqrt{x+1}}$

The numerical results are presented in Table 4 when the problem is solved using different stepsizes of $h = 1, 0.1, 0.01, 0.001$ respectively.

Problem 3

The linear stiff system of equations [5]

$$\begin{aligned} y_1' &= -8y_1 + 7y_2, \quad y_1(0) = 1 \\ y_2' &= 42y_1 - 43y_2, \quad y_2(0) = 8 \end{aligned}, \; x \in [0,1]$$

whose exact solution are: $y_1(x) = 2e^{-x} - e^{-50x}$, $y_1(x) = 2e^{-x} + 6e^{-50x}$. The numerical results are presented in Table 5 with stepsize $h = 0.0001$.

Table 4: Absolute errors for problem 2.

x	h	$\lvert y(x) - y_{TDLMM\,(21)} \rvert$	$\lvert y(x) - y_{TDBDF} \rvert$
5	1	$1.1197152249049990E-2$	$3.3276352762323990E-2$
	0.1	$2.1627696300399735E-3$	$3.4497928421990400E-3$
	0.01	$2.3362480664196328E-4$	$3.4066981905800375E-4$
	0.001	$2.3533782585960505E-5$	$3.4025300106044210E-5$
10	1	*5.5278534429083480E − 3*	*1.4470642548685630E − 2*
	0.1	*8.8613990637437250E − 4*	$1.4009908383366132E-3$
	0.01	*9.4267357315380720E − 5*	$1.3734762859962668E-4$
	0.001	*9.4819958023628640E − 6*	*1.3708021298630690E − 5*

In Table 4, the numerical results show that the error of the TDLMM (20) is smaller when compared to the TDBDF (8). As the step size reduces, there is a positive improvement in the errors of the TDLMM (20).

Table 5: Absolute errors of y_1 and y_2 solution components for problem 3

x	y_i	$\lvert y(x) - y_{TDLMM\,(21)} \rvert$	$\lvert y(x) - y_{TDBDF} \rvert$
0.2	y_1	$1.1318807158500199E-4$	$1.6351085990296000E-4$
	y_2	$1.1427204003000000E-4$	$1.6510249595700000E-4$
0.4	y_1	$9.2797107253073110E-5$	$1.3405774272201576E-4$
	y_2	$9.2797156162172100E-5$	$1.3405781528197380E-4$
0.6	y_1	$7.5975668266226040E-5$	$1.0975738806884294E-4$
	y_2	$7.5975668267114220E-5$	$1.0975738807283975E-4$
0.8	y_1	$6.2203466298260680E-5$	$8.9861898761811610E-5$
	y_2	$6.2203466298260680E-5$	$8.9861898761811610E-5$
1.0	y_1	$5.0927768169284350E-5$	$7.3572822662626440E-5$
	y_2	$5.0927768169284350E-5$	$7.3572822662626440E-5$

The errors of the TDLMM (20) are smaller compared with the TDBDF (8).

5 CONCLUSION

In this article, a new class of third derivative linear multistep method is introduced through the inclusion of a nonzero term in the conventional third derivative backward differentiation formula (TDBDF). The proposed TDLMM (20) is stable for order $\rho \leq 12$, while the TDBDF is stable for order $\rho \leq 11$ and inefficient incase where method of order $\rho = 12$ is required. The numerical experiments show that the new TDLMM (20) gives a better approximation compared to the TDBDF, in solving linear and nonlinear systems of initial value problems in ordinary differential equations.

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