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EXTENDED GENERALIZED ADAMS METHODS FOR STIFF SYSTEMS.

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Abstract

In this article, we propose a class of generalized Adams methods with future points for stiff ordinary differential equations (ODEs). For reference purposes these methods will be referred to as extended generalized Adams methods (EGAMs). The EGAMs are boundary value methods (BVMs) of order $k+2$ which are $0_{v,\,k+1-v}-$stable and $A_{v,\,k+1-v}-$stable with $(v, k+1-v)$-boundary conditions for $k \geq 1$.

Keywords: Adams methods, **S**tiff ODEs, $A_{v,\,k+1-v}-$stable $0_{v,\,k+1-v}-$stable.

AMS subject classification: 65L04, 65L05

1. Introduction

Boundary value methods are linear multistep methods used with a fixed number of initial and final conditions. These methods generate stable discrete boundary value schemes for the solution of initial and boundary value ordinary differential equations of the form

$$y'(x) = f(x, y(x)),\ x \in [a,b], \tag{1}$$

that could be subject to either initial $(y(a) = y_0)$ or boundary $(g(y(a), y(b)) = 0)$

Conditions; $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$ is sufficiently smooth function. Examples of boundary value methods (BVMs) can be found in [1-13].

In [2], Brugnano and Trigiante examined the family of boundary value methods (BVMs) based on the particular class of k-step linear multistep formulae (LMF) of the form

$$y_{n+j} - y_{n+j-1} = h\sum_{i=0}^{k} \beta_i f_{n+i} \qquad j = 1,\dots,k \tag{2}$$

For $j = k$, one obtains the Adams-Moulton formulae which are all 0-stable and are used as initial value methods (IVMS). These methods have been intensively used mainly for the approximation of the solution of non-stiff ODEs. The trapezoidal rule which has order 2 is the only Adams-Moulton method appropriate for solving stiff problems because it has an unbounded Absolute stability region. This corresponds to the case k= 1 :

$$y_{n+1} - y_n = \tfrac{h}{2}(f_{n+1} + f_n) \tag{3}$$

But for $k \geq 2$the Absolute stability regions of these methods are all bounded and become smaller and smaller as $k$increases. In an attempt to obtain methods with better stability regions, Brugnano and Trigiante, in [2], derived for j=1the Reverse Adams Methods; a family of $0_{1,\,k-1}-$stable methods which must be used as BVMs with $(1, k-1)$ -boundary condition and given by

$$y_n - y_{n+1} = -h\sum_{i=0}^{k} \beta_{i,1} f_{n+i} \tag{4}$$

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The boundaryloci of the Reverse Adams methods are the reflection about the imaginary axis of those of the corresponding Adams-Moulton methods. The Reverse Adams methods can also be used to approximate the solution of stiff problems since for $k \leq 8$the$(1, k-1)$-Absolute stabilityregions are unbounded. But even though they have better stability properties than the Adams-Moulton methods they do not provide very high order methods suitable for stiff problems. By choosing in (2) $j = v$, Brugnano and Trigiante derived for $k \geq 1$ a family of methods with thebest stability properties called generalized Adams methods (GAMs) of order$k+1$ which are $0_{v,\, k-v}-$stable and $A_{v,\, k-v}-$stable with$(v, k-v)$-boundary conditions. These methods can be written as

$$y_{n+v} - y_{n+v-1} = h\sum_{i=0}^{k} \beta_i f_{n+i} \tag{5}$$

$$v = \begin{cases} \dfrac{k+1}{2} & for\ odd\ k \\ \dfrac{k}{2} & for\ even\ k \end{cases} \tag{6}$$

They are conveniently used with the following set of additional initial methods,

$$y_j - y_{j-1} = h\sum_{i=0}^{k} \beta_i^{(j)} f_i, \quad j = 1, \ldots, v-1, \tag{7}$$

and final ones,

$$y_j - y_{j-1} = h\sum_{i=0}^{k} \beta_{k-i}^{(j)} f_{N-i}, \quad j = N-k+v+1, \ldots, N. \tag{8}$$

For odd values of$k$, the methods are regarded as generalization of the basic trapezoidal rule and referred to as the extended trapezoidal rule (ETRs) because they share the same stability properties with the trapezoidal rule. That is their boundary loci coincide with the imaginary axis which makes them perfectly $A_{v,\, v-k}-$stable.

A potentially good numerical method for the solution of stiff systems of ODEs must have good accuracy and some reasonably wide region of absolute stability [14]. A-stability requirement puts a severe limitation on the choice of suitable methods for stiff problems. The search for higher order A-stable multistep methods is carried out in the two main directions:

-use higher derivative of the solution;

-throw in additional stages, off-step points, super-future points and the like. This leads into the large field of general linear methods [15].

Here we introduce a new classof extended generalized Adams methods (EGAMs)for stiff ordinary differential equations (ODEs). The methods have good stability properties and great advantage in accuracy.

The article is organized as follows. In the next section the new class of EGAMs is introduced. In the third section the stability behavior of our method is analyzed. The implementation procedure is given in the forth section. Numerical results are shown in the final section.

2. The new class of EGAMs

The new class of EGAMs for the solution of stiff initial value problems (IVPs) in (1) takes the following general form

$$y_{n+v} - y_{n+v-1} = h\sum_{j=0}^{k+1} \beta_j f_{n+j} \tag{9}$$

In the method (9)

$$v = \begin{cases} \dfrac{k+1}{2} & for\ odd\ k \\ \dfrac{k}{2} & for\ even\ k \end{cases} \tag{10}$$

with $(v, k+1-v)$-boundary conditions for $k \geq 1$. The numerical solution y_n is an approximation to $y(x_n)$. $y'(x) = f(x, y)$ isthe first derivative function while y_{n+k} is the output solution at the point x_{n+k} of the method (9). $h = x_{n+1} - x_n$ is the mesh size and$k$ is the step number. The β_j are the coefficients of the method.

In the spirit of [10-13] the coefficients, error constant and the order of the EGAMs for $k = 1(2)9$ are given in Table 1.

3. Stability analysis

The method (9) can be written compactly as 6

$$\rho(E)y_n = h\sigma(E)f_n \qquad (11)$$

Where$\rho(\xi) = \xi^v - \xi^{v-1}$, $\sigma(\xi) = \sum_{j=0}^{k+1} \beta_j \xi^j$ are the first and second characteristic polynomials respectively, $\xi \in C$ and $E^j y_n = y_{n+j}$ is the shift operator.

The stability analysis is done through linearization in the spirit of Hairer and Wanner [15] where we consider the usual test equations

$$y' = \lambda y, \quad y'' = \lambda^2 y \qquad (12)$$

which is applied to the form (11) to yield the characteristic equation

$$\xi^v - \xi^{v-1} - \sum_{j=0}^{k+1}(z\beta_j)\xi^j, \; z = \lambda h, \; z \in C \qquad (13)$$

Inserting $\xi = e^{i\theta}$ inequation (13)we obtain the stability regions of the class of methods (9). The boundary loci of these methods are shown in Figures 1 and 2.

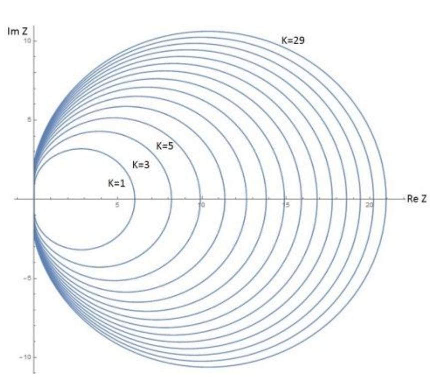

Figure1: Stability region (exterior of closed curves) of (9), k=1(2)29

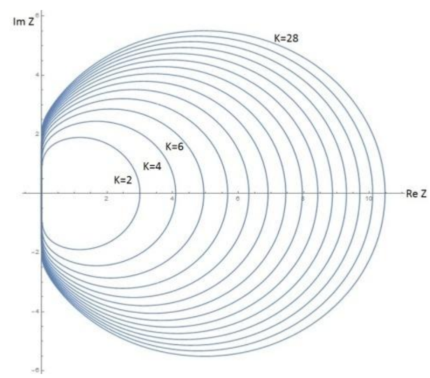

Figure2: Stability region (exterior of closed curves) of (9), k=2(2)28

Table 1: The Coefficients, Error Constant (EC) and Order p of the EGAMs (9) for $k = 1(1)10$

k	v	p	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7
1	1	3	$\frac{5}{12}$	$\frac{2}{3}$	$-\frac{1}{12}$					
2	1	4	$\frac{3}{8}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$				
3	2	5	$-\frac{19}{720}$	$\frac{173}{360}$	$\frac{19}{30}$	$-\frac{37}{360}$	$\frac{11}{720}$			
4	2	6	$-\frac{3}{160}$	$\frac{637}{1440}$	$\frac{511}{720}$	$-\frac{43}{240}$	$\frac{77}{1440}$	$-\frac{11}{1440}$		
5	3	7	$\frac{271}{60480}$	$-\frac{23}{504}$	$\frac{10273}{20160}$	$\frac{586}{945}$	$-\frac{2257}{20160}$	$\frac{67}{2520}$	$-\frac{191}{60480}$	
6	3	8	$\frac{13}{4480}$	$-\frac{4183}{120960}$	$\frac{6403}{13440}$	$\frac{9077}{13440}$	$-\frac{20227}{120960}$	$\frac{803}{13440}$	$-\frac{191}{13440}$	$\frac{191}{120960}$
7	4	9	$-\frac{3233}{3628800}$	$\frac{18197}{1814400}$	$-\frac{108007}{1814400}$	$\frac{954929}{1814400}$	$\frac{13903}{22680}$	$-\frac{212881}{1814400}$	$\frac{63143}{1814400}$	$-\frac{12853}{1814400}$
8	4	10	$-\frac{7}{12800}$	$\frac{10063}{1451520}$	$-\frac{42767}{907200}$	$\frac{225623}{453600}$	$\frac{2381791}{3628800}$	$-\frac{583073}{3628800}$	$\frac{5779}{90720}$	$-\frac{17663}{907200}$
9	5	11	$\frac{90817}{479001600}$	$-\frac{292531}{119750400}$	$\frac{493837}{31933440}$	$-\frac{1394959}{19958400}$	$\frac{14296081}{26611200}$	$\frac{379571}{623700}$	$-\frac{3216337}{26611200}$	$\frac{163459}{3991680}$
10	5	12	$\frac{443}{3942400}$	$-\frac{218059}{136857600}$	$\frac{2149187}{191600640}$	$-\frac{2607167}{45619200}$	$\frac{11672473}{22809600}$	$\frac{11429669}{17740800}$	$-\frac{24994867}{159667200}$	$\frac{235723}{3548160}$

Table 2: Table 1 continued

K	V	p	β_8	β_9	β_{10}	β_{11}	C_{p+1}
1	1	3					$\frac{1}{24}$
2	1	4					$-\frac{19}{720}$
3	2	5					$-\frac{11}{1440}$
4	2	6					$\frac{271}{60480}$
5	3	7					$\frac{191}{120960}$
6	3	8					$-\frac{3233}{3628800}$
7	4	9	$\frac{2497}{3628800}$				$-\frac{2497}{7257600}$
8	4	10	$\frac{27467}{7257600}$	$-\frac{2497}{7257600}$			$\frac{90817}{479001600}$
9	5	11	$-\frac{1746433}{159667200}$	$\frac{32309}{17107200}$	$-\frac{14797}{95800320}$		$\frac{14797}{191600640}$
10	5	12	$-\frac{7562041}{319334400}$	$\frac{1959493}{319334400}$	$-\frac{192361}{191600640}$	$\frac{14797}{191600640}$	$-\frac{9959263}{237758976000}$

4. Implementation procedure of the EGAMs

The EGAMs are conveniently used with the following set of additional Initial methods

$y_r - y_{r-1} = h\sum_{j=0}^{k+1}\beta_j\, f_j \qquad r = 1, \dots, v-1$ (14)

and final methods

$y_{N+r} - y_{N+r-1} = h\sum_{j=-v}^{(k+1)-v}\beta_{j+v}\, f_{N+j-1} \qquad r = 0 \dots, K-v$ (15)

The class of methods (9) requires $v-1$ initial and $k+1-v$ final additional methods for its implementation since y_0 is already provided by the problem to be solved.

The fourth order EGAMs (9)

$y_n - y_{n-1} = h(\frac{3f_{n-1}}{8} + \frac{19f_n}{24} - \frac{5f_{n+1}}{24} + \frac{f_{n+2}}{24})$

$n = 1, \dots, N-1$

can be used with the following two final additional methods,

$y_N - y_{N-1} = h(-\frac{f_{N-2}}{24} + \frac{13f_{N-1}}{24} + \frac{13f_N}{24} - \frac{f_{N+1}}{24})$

and

$y_{N+1} - y_N = h(\frac{f_{N-2}}{24} - \frac{5f_{N-1}}{24} + \frac{19f_N}{24} + \frac{3f_{N+1}}{8})$

The fifth order EGAMs (9)

$y_n - y_{n-1} = h(-\frac{19f_{n-2}}{720} + \frac{173f_{n-1}}{360} + \frac{19f_n}{30} - \frac{37f_{n+1}}{360} + \frac{11f_{n+2}}{720})$

$n = 2, \dots, N-1$

can be used with the following initial method,

$y_1 - y_0 = h\left(\frac{251}{720}f_0 + \frac{323}{360}f_1 - \frac{11}{30}f_2 + \frac{53}{360}f_3 - \frac{19}{720}f_4\right)$

and the two final additional methods,

$y_N - y_{N-1} = h\left(\frac{11}{720}f_{N-3} - \frac{37}{360}f_{N-2} + \frac{19}{30}f_{N-1} + \frac{173}{360}f_N - \frac{19}{720}f_{N+1}\right)$

and

$y_{N+1} - y_N = h(-\frac{19}{720}f_{N-3} + \frac{53}{360}f_{N-2} - \frac{11}{30}f_{N-1} + \frac{323}{360}f_N + \frac{251}{720}f_{N+1})$

5. Numerical results

In this section, all numerical computations were carried out using MATLAB.

Problem 1: Linear system solved by Brugnano and Trigiante [2]

$y_1' = -21y_1 + 19y_2 - 20y_3$, $y_1(0) = 1$, $\quad y_2' = 19y_1 - 21y_2 + 20y_3$, $\quad y_2(0) = 0$,

$y_3' = 40y_1 - 40y_2 + 40y_3$, $\quad y_3(0) = -1$.

The analytical solution of the system is given by

$y_1(x) = \frac{1}{2}(e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x)))$

$y_2(x) = \frac{1}{2}(e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x)))$

$y_3(x) = -e^{-40x}(\cos(40x) - \sin(40x))$.

The numerical results for problem 1 are presented in Table 3. It is seen from Table 3 that the implementation using EGAMs (9) is better than the GAMs of Brugnano and Trigiante [2]. In all cases, the rate of convergence of our method is consistent with the order of the method.

Table 3: Maximum Errors for Example 1

H	EGAMs error K = 2 ,p=4	Rate	EGAMs error K = 3, p=5	Rate	GAMs[2] error K = 4, p=5	rate	GAMs [2] error K = 6, p=7	Rate
1.e-1	8.453e-5	-------	1.644e-2	-------	5.283e-1	---------	3.500e-1	---------
5.e-2	2.307e-7	8.52	2.1049e-6	12.93	2.249e-1	1.75	1.266e-1	2.47
2.5e-2	1.496e-8	3.95	8.381e-11	14.62	4.413e-2	3.31	1.449e-2	2.40
1.25e-2	9.443e-10	3.99	2.315e-12	5.18	6.490e-3	3.21	1.508e-3	5.72
6.25e-3	5.933e-11	3.99	6.775e-14	5.10	8.859e-4	5.05	1.114e-4	7.27
3.125e-3	3.718e-12	4.00	2.0817e-15	5.02	9.881e-5	5.59	4.877e-6	7.46

Problem 2: Nonlinear stiff system proposed by Kaps[16]

$y_1' = -1002y_1 + 1000y_2^2$, $\qquad y_1(0) = 1, y_2' = y_1 - y_2(1 + y_2)$, $\quad y_2(0) = 1$,

$0 \le t \le T$ the smaller is, the more serious the stiffness of the system.

The exact solution is

$y_1(t) = y_2^2(t)$, $y_2(t) = e^{-t}$

The results of Akinfenwa et al [17] and Akinfenwa and Jator [18] are reproduced in Table 4 and compared with that obtained using the EGAMs. It can be seen in Table 4 that the results obtained for EGAMs for $k = 3$ is superior to those of Akinfenwa et al [17] for $k = 4$ and 5 and that of Akinfenwa and Jator [18] for $k = 4$. While for $k = 3$ our method is highly competitive with that of Akinfenwa and Jator [18] for $k = 5$.

Table 4: Comparison of methods at $T = 10$for Problem 2

h	$CBBDF_4$[17] $k = 4, p = 4$	$CBBDF_5$[17] $k = 5, p = 5$	$ECBBDF_4$[18] $k = 4, p = 5$	$ECBBDF_5$[18] $k = 5, p = 6$	$EGAMs_3$ $k = 3, p = 5$
0.02	4.88×10^{-16} 5.39×10^{-12}	8.37×10^{-18} 9.16×10^{-14}	2.48×10^{-19} 3.75×10^{-16}	1.33×10^{-20} 1.35×10^{-16}	2.47e-019 3.71e-016
0.01	3.13×10^{-17} 3.45×10^{-13}	3.39×10^{-21} 1.23×10^{-17}	2.68×10^{-19} 2.93×10^{-15}	2.87×10^{-22} 2.93×10^{-19}	2.43e-021 3.80e-018
0.002	5.14×10^{-20} 5.67×10^{-14}	4.64×10^{-21} 5.16×10^{-17}	1.11×10^{-21} 1.09×10^{-17}	2.32×10^{-21} 5.55×10^{-17}	8.87e-022 9.84e-018

6. Conclusion

This article is concerned with the solution of systems of stiff (IVPs) in (ODEs). This has been achieved by the construction and implementation of a class of EGAMs. It has also been shown that these methods are highly competitive with existing methods cited in the literature.

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