Transactions of the Nigerian Association of Mathematical Physics
Volume 7, **(***March, 2018***), pp113 – 120**

NUMERICAL SOLUTION OF SIXTH AND TWELFTH ORDER BOUNDARY VALUE PROBLEM USING MODIFIED VARIATIONAL ITERATION METHOD.

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Abstract

In this paper, we consider the sixth and twelfth order boundary value problems using a modified variational iterative scheme. The modification involves the construction of Canonical polynomials which are used as trial functions in the approximation of the analytic solution. Numerical examples considered show that the method yields the desired accuracy when compared with the variational decomposition method (VDM), variational iteration decomposition method (VIDM), and homotopy perturbation method (HPM). We analyzed calculations in this work using Maple18 software.

Keywords: Variational iteration method, boundary value problems, Canonical polynomials, approximate solutions.

1.0 Introduction

Consider a general boundary value problem (BVP) of the form

$$a_n \frac{d^n}{dx^n} y + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + a_{n-2} \frac{d^{n-2}}{dx^{n-2}} y \ \dots \ a_1 \frac{d}{dx} y + a_0 y = f(x) \qquad (1)$$

With boundary conditions

$y(a) = A_1 \,, y^{'}(a) = A_2 \ , y^{''}(a) = A_3 \ \dots y^{n}(a) = A_n \,,$

$y(b) = B_1, y^{'}(b) = B_2, y^{''}(b) = B_3 \ \dots y^{n}(b) = B_n$

Where $a_n, a_{n-1,}, a_{n-2} \dots a_1, a_0$ are constants and $f(x)$ is continuous on [a,b]. These types of problems are prevalent in many fields of science and engineering such as viscoelastic flow, heat transfer, mathematical modeling and mathematical physics, amongst others. The difficulty in getting analytic solutions to boundary value problems has prompted many research efforts in the area of numerical solution methods. In this paper, we propose a modified version of the variational iteration method for the solution of boundary value problems. In the proposed method, the correction functional is constructed for the given BVP, and the Lagrange multiplier is computed optimally via the variational theory, Canonical polynomials are then used as trial functions. We review some related literature in the next section.

2.0 LITERATURE REVIEW

Much emphasis has been given to the variational iteration method, its variants and combination of other methods and the variational iteration method in a quest for better solution methods for boundary value problems. A recent endeavour in this direction [1], proposed a method that combined the Adomian decomposition method and the variational iteration methods for solving delay differential equations. Some other fairly recent methods in the literature can be found in [2-5].

Ojobor and Ogeh [6] applied a modified variational iteration method with canonical Polynomials to seek the numerical solution to eight order boundary value problems. The method proposed in [7] equally obtained the numerical solution of fifth order boundary value problem using Mamadu-Njoseh polynomials as trial functions. The power series approximation method in [8] was used to seek solution to a generalized BVP. Also, the method of tau and tau-collocation approximation method was vigorously used in [9] to seek solution to first and second order ordinary differential equations. Noor and Mohyud-Din [10-13] adopted the variational iteration decomposition method and the variational iteration homotopy perturbation method for the numerical solution of higher order boundary value problems.

Mirmoradi et al [14] applied the homotopy perturbation method to seek the numerical solution of twelfth order boundary value problem. In like manner, Mohyud-Din and Yildirim [15] combined the homotopy perturbation method with the standard variational iteration method to seek the numerical solution of ninth and tenth order boundary value problems. Siddiqi

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and Iftikhar [16] used the homotopy perturbation method and the variational iteration method to obtain the numerical solution of seventh-order boundary value problems. Fazal.et.al [17] used the collocation method to obtain the numerical solution of sixth-order boundary value problems. The solution method in [18] seeks the numerical solution of fifth order boundary value problem with sixth degree B-spline. The Adomian decomposition method was used in [19] to solve both linear and nonlinear boundary value problems.
In this work we seek solutions to sixth and twelfth order boundary value problems using a modified variational iteration method. The method is applied to both linear and nonlinear BVPs and the resulting numerical evidence show that it is effective, accurate and reliable as compared with existing methods [11],[16] and[14] as available in literature.

3. THE STANDARD VARIATIONAL ITERATION METHOD

Consider the following general differential equation

$$Lu + Nu - g(x) = 0, \qquad (2)$$

Where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term
Constructing the correction functional by variational iteration method, *Equation (2)* becomes;

$$u_{n+1} = u_n(x) + \int_0^x \lambda(t)\big(Lu_n(t) + N\widetilde{u_n(t)} - g(t)\big)dt \qquad (3)$$

Where $\lambda(t)$ is a Lagrange multiplier, which can be identified optimally via variational theory. The subscripts n denote the nth approximation, $\widetilde{u_n(t)}$is considered as a restricted variation. i.e.$\widetilde{u_n}$= 0. *Equation (3)* is known as a correction functional. The solution of the linear problems can be found in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, we need to determine the langrange multiplier $\lambda(t)$ optimally and hence the successive approximation of the solution u will be readily obtained upon using the langrange multiplier and $u_0(x)$. The solution is given by

$$\lim_{n-\infty} u_n = u$$

The Langrange Multiplier plays a major role in the determination of the solution of the problem. Hence we summarize the formula for the Langrange Multiplier respectively as;

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{n-1}$$

Where *n* is the highest order of the given BVP.

4. CONSTRUCTION OF CANONICAL POLYNOMIALS

Given the general order boundary value problem of the form

$$a_r \frac{d^r}{dx^r} y + a_{r-1} \frac{d^{r-1}}{dx^{r-1}} y + a_{r-2} \frac{d^{r-2}}{dx^{r-2}} y \;\ldots\; a_1 \frac{d}{dx} y + a_0 y = f(x)$$

In operator form, we have

$$L = a_r \frac{d^r}{dx^r} y + a_{r-1} \frac{d^{r-1}}{dx^{r-1}} y + a_{r-2} \frac{d^{r-2}}{dx^{r-2}} y \;\ldots\; a_1 \frac{d}{dx} y + a_0 y$$

Let$LQ_r(x) = x^r$, then

$$Lx^r = \left(a_r \frac{d^r}{dx^r} y + a_{r-1} \frac{d^{r-1}}{dx^{r-1}} y + a_{r-2} \frac{d^{r-2}}{dx^{r-2}} y \;\ldots\; a_1 \frac{d}{dx} y + a_0 y\right) x^r$$

This can also be written as

$$Lx^r = \left(\sum_{i=0}^{n} a_r \frac{d^r}{dx^r} y\right) x^r$$

If$n = 6$, we have

$$Lx^r = \sum_{i=0}^{6} a_r\, x^{r-i} \prod_{j=0}^{i-1} (r-i)$$

Defining $LQ_r(x) = x^r$

$$Lx^r = \sum_{i=0}^{6} a_r\, LQ_{r-i}(x) \prod_{j=0}^{i-1} (r-i)$$

Since L is a linear operator, and assume L^{-1} exist, hence we have

$$x^r = \sum_{i=0}^{6} a_r\, Q_{r-i}(x) \prod_{j=0}^{i-1} (r-i)$$

$$Q_r(x) = \frac{1}{a_0}\left[x^r - \sum_{i=0}^{6} a_r\, Q_{r-i}(x) \prod_{j=0}^{i-1}(r-i)\right]$$

Where $r = 0(1)6$, we have the following polynomials

$$Q_0(x) = \frac{1}{a_0}, Q_1(x) = \frac{x}{a_0}, Q_2(x) = \frac{x^2}{a_0}, Q_3(x) = \frac{x^3}{a_0}$$

$$Q_4(x) = \frac{x^4}{a_0}, Q_5(x) = \frac{x^5}{a_0}, Q_6(x) = \frac{x^6 + 720}{a_0}$$

Hence generalizing the above equation we have

$$Q_r(x) = \frac{1}{a_0}\left[x^r - \sum_{i=0}^{n} a_r\, Q_{r-i}(x) \prod_{j=0}^{i-1}(r-i)\right]$$

5. MODIFIED VARIATIONAL ITERATION METHOD USING CANONICAL POLYNOMIALS (MVIM)

Using *Equations (2) and (3)*, we now assume an approximate solution of the form;

$$u_{n,N}(x) = \sum_{i=0}^{N} a_{i,N} Q_{i,N}(x)$$

Where $Q_{i,N}(x)$ are the constructed Canonical Polynomials, $a_{i,N}$ are constants to be determined, and N the degree of approximant. Hence we obtain the following iterative method

$$u_{n+1,N} = \sum_{i=0}^{N} a_{i,N} Q_{i,N}(x) + \int_0^x \lambda(t)\left(L\sum_{i=0}^{N} a_{i,N} Q_{i,N}(t) + \tilde{N}\sum_{i=0}^{N} a_{i,N} Q_{i,N}(t) - g(t)\right)dt$$

6. NUMERICAL APPLICATIONS

In this section we applied the proposed technique to solve examples of which are linear and non-linear BVPs. Numerical results also show the accuracy of the proposed method.

Example 6.1 [11]

Consider the following sixth order linear boundary value problem

$$u^{(vi)}(x) = u(x) - 6e^x,\ 0 \le x \le 1 \qquad (4)$$

subject to boundary conditions

$$u(0) = 1, u'(0) = 0, u''(0) = -1, u(1) = 0, u'(1) = -e, u''(1) = -2e \qquad (5)$$

The exact solution is

$$u(x) = (1-x)e^x \qquad (6)$$

The correction functional for the boundary value problem is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(\frac{d^6 u_n}{dt^6} - u_n(t) + 6e^t\right)dt$$

Making the correction functional stationary, $\lambda(t) = \frac{(t-x)^5}{5!}$ as the Lagrange multiplier, we have the following

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{(t-x)^5}{5!}\left(\frac{d^6 u_n}{dt^6} - u_n(t) + 6e^t\right)dt$$

Applying the proposed method, we assume an approximate solution of the form

$$u_{n,6}(x) = \sum_{i=0}^{6} a_{i,6} Q_{i,6}(x)$$

$$u_{n+1,N}(x) = \sum_{i=0}^{6} a_{i,6} Q_{i,6}(x) + \int_0^x \frac{(t-x)^5}{5!}\left(\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6} Q_{i,6}(x)\right) - \sum_{i=0}^{6} a_{i,6} Q_{i,6}(x) + 6e^t\right)dt$$

$$u_{n+1,N}(x) = -a_{0,6} - a_{1,6}x - a_{2,6}x^2 - a_{3,6}x^3 - a_{4,6}x^4 - a_{5,6}x^5 - a_{6,6}x^6$$
$$- 720a_{6,6}\int_0^x \frac{(t-x)^5}{5!}\left(\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6} Q_{i,6}(x)\right) - \sum_{i=0}^{6} a_{i,6} Q_{i,6}(x) + 6e^t\right)dt$$

Iterating and applying the boundary condition in *Equation (5)* the values of the unknown constants can be determined as follows;

$a_{0,6} = -12.03701107, \quad a_{1,6} = 0, \quad a_{2,6} = 0.5000000$
$a_{3,6} = 0.333333410, \quad a_{4,6} = 0.12499982, \quad a_{5,6} = 0.033333527$
$a_{6,6} = 0.01532918294$

Consequently, the series solution is given as;

$$u(x) = 1.000000 - 0.5000000x^2 - 0.333333410x^3 - 0.12499982x^4 - 0.033333527x^5 - 0.00694444443x^6 - 0.00119047619x^7 - 0.0001736111111x^8 - 0.00002204585664x^9 - 0.000002480157604x^{10} - 0.0000002505213654x^{11} + O(x)^{12}$$

The corresponding results are shown in *Table 1;*

Table 1: The result of the proposed method compared with Variational decomposition method

X	Exact solution	Approximate Solution	Present Method Error	VDM Error
0.0	1.0000000	1.0000000	0.0000e+00	0.0000e+00
0.1	0.9946538	0.9946538	0.0000e+00	4.0933e-04
0.2	0.9771222	0.9771222	2.0000e-10	7.7820e-04
0.3	0.9449012	0.9449012	1.2000e-09	1.0704e-03
0.4	0.8950948	0.8950948	1.7000e-09	1.2578e-03
0.5	0.8243606	0.8243606	2.0000e-09	1.3223e-03
0.6	0.7288475	0.7288475	1.7000e-09	1.2578e-03
0.7	0.6041258	0.6041258	1.0000e-09	1.0740e-03
0.8	0.4451082	0.4451082	1.0000e-10	7.7820e-04
0.9	0.2459603	0.2459603	8.0000e-10	4.0933e-04
1.0	0.0000000	0.0000000	1.7000e-09	0.0000e+00

Example 6.2 [16]

Consider the following sixth order linear boundary value problem of the form

$$u^{(vi)}(x) = u(x) - 6cosx \quad ,0 \le x \le 1, \qquad (7)$$

subject to boundary conditions

$$u(0) = 0, u'(0) = -1, u''(0) = 2u(1) = 0, u'(1) = \sin(1), u''(1) = 2\cos(1) \quad (8)$$

The exact solution is

$$u(x) = (x-1)\sin(x) \qquad (9)$$

The correction functional for the boundary value problem is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(\frac{d^6u_n}{dt^6} - u_n(t) + 6\cos(t)\right)dt$$

Making the correction functional stationary, $\lambda(t) = \frac{(t-x)^5}{5!}$ as the Lagrange multiplier, we have the following

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{(t-x)^5}{5!}\left(\frac{d^6u_n}{dt^6} - u_n(t) + 6\cos(t)\right)dt$$

Applying the proposed method, we assume an approximate solution of the form

$$u_{n,6}(x) = \sum_{i=0}^{6} a_{i,6}Q_{i,6}(x)$$

$$u_{n+1,N}(x) = \sum_{i=0}^{6} a_{i,6}Q_{i,6}(x) + \int_0^x \frac{(t-x)^5}{5!}\left(\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6}Q_{i,6}(x)\right) - \sum_{i=0}^{6} a_{i,6}Q_{i,6}(x) + 6\cos(t)\right)dt$$

$$u_{n+1,N}(x) = -a_{0,6} - a_{1,6}x - a_{2,6}x^2 - a_{3,6}x^3 - a_{4,6}x^4 - a_{5,6}x^5 - a_{6,6}x^6 - 720a_{6,6} + \int_0^x \frac{(t-x)^5}{5!}\left(\frac{d^6}{dt^6}\left(\sum_{i=0}^{6} a_{i,6}\varphi_{i,6}(x)\right) - \sum_{i=0}^{6} a_{i,6}\varphi_{i,6}(x) + 6\cos(t)\right)dt$$

Iterating and applying the boundary condition in *Equation (8)* the values of the unknown constants can be determined as follows;

$a_{0,6} = -4.507915208, \quad a_{1,6} = 1, \quad a_{2,6} = -1$
$a_{3,6} = -0.1666660509, \quad a_{4,6} = 0.166665282, \quad a_{5,6} = 0.0083341252$
$a_{6,6} = 0.006260993344$

Finally the series solution is

$$u(x) = -x + x^2 + 0.1666660509x^3 - 0.166665282x^4 - 0.0083341252x^5 + 0.008333333336x^6 + 0.0001984127x^7 - 0.0001984127x^8 - 0.000002755721741x^9 - 0.000002755722764x^{10} - 0.00000002505448894x^{11} + O(x)^{12}$$

Table 2: The result of the proposed method compared with Variational Iteration Decomposition method

X	Exact solution	Approximate Solution	Present Method Error	VIDM Error
0.0	0.0000000	0.0000000	0.0000e+00	0.0000e+00
0.1	-0.0898501	-0.0898501	4.9000e-10	4.0933e-06
0.2	-0.1589355	-0.1589355	3.0000e-09	7.7820e-06
0.3	-0.2068641	-0.2068641	7.3000e-09	1.0704e-05
0.4	-0.2336510	-0.2336510	1.2100e-08	1.2578e-05
0.5	-0.2397128	-0.2397128	1.5200e-08	1.3223e-05
0.6	-0.2258570	-0.2258570	1.5000e-08	1.2578e-05
0.7	-0.1932653	-0.1932653	1.1588e-08	1.0740e-05
0.8	-0.1434712	-0.1434712	6.1000e-09	7.7820e-04
0.9	-0.0783327	-0.0783327	1.7400e-09	4.0933e-04
1.0	0.0000000	-0.0000000	0.0000e+00	0.0000e+00

Example 6.3 [14] Consider the following twelfth order linear boundary value problem

$$u^{(xii)}(x) = -xu(x) - e^x\,(120 + 23x + x^3), 0 \le x \le 1 \qquad (10)$$

With boundary conditions

$$u(0) = 0, u'(0) = 1, u''(0) = 0u'''(0) = -3, u^{iv}(0) = -8,$$
$$u^{v}(0) = -15u(1) = 0, u'(1) = -e, u''(1) = -4e, \qquad (11)$$
$$u'''(1) = -9e, u^{iv}(1) = -16, u^{v}(1) = -25e$$

The exact solution is

$$u(x) = x(1-x)e^x \qquad (12)$$

The correct functional for the boundary value problem

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\left(\frac{d^{12}u_n}{dt^{12}} - tu_n(t) + e^t(120 + 23t + t^3)\right)dt$$

Making the correct functional stationary, $\lambda(t) = \frac{(t-x)^{11}}{11!}$ as the Lagrange multiplier, we have the following

$$u_{n+1,N}(x) = u_{n,N}(x) + \int_0^x \frac{(t-x)^{11}}{11!}\left(\frac{d^{12}u_n}{dt^{12}} - tu_{n,N}(t) + e^t(120 + 23t + t^3)\right)dt$$

Applying the proposed method, we assume an approximate solution of the form

$$u_{n,12}(x) = \sum_{i=0}^{12} a_{i,12}Q_{i,12}(x)$$

$$u_{n+1,N}(x) = \sum_{i=0}^{12} a_{i,12}Q_{i,12}(x) + \int_0^x \frac{(t-x)^{11}}{11!}\left(\frac{d^{12}u_n}{dt^{12}} - t\sum_{i=0}^{12} a_{i,12}Q_{i,12}(t) + e^t(120 + 23t + t^3)\right)dt$$

$$u_{n+1,N}(x) = -a_{0,12} - a_{1,12}x - a_{2,12}x^2 - a_{3,12}x^3 - a_{4,12}x^4 - a_{5,12}x^5 - a_{6,12}x^6 - a_{7,12}x^7 - a_{8,12}x^8 - a_{9,12}x^9 - a_{10,12}x^{10} - a_{11,12}x^{11} - a_{12,12}x^{12} - 479001600a_{12,12} + \int_0^x \frac{(t-x)^{11}}{11!}\left(\frac{d^{12}u_n}{dt^{12}} - t\sum_{i=0}^{12} a_{i,12}Q_{i,12}(t) + e^t(120 + 23t + t^3)\right)dt$$

Iterating and applying the boundary condition *Equation (11)* the values of the unknown constants can be determined as follows

$a_{0,12} = -263.8636390,\quad a_{1,12} = -1,\quad a_{2,12} = 0, a_{3,12} = 0.5000000$
$a_{4,12} = 0.33333333,\quad a_{5,12} = 0.12500000,\quad a_{6,12} = 0.0333333$
$a_{7,12} = 0.0069442,\quad a_{8,12} = 0.001190,\quad a_{9,12} = 0.000179, a_{10,12} = 0.0000222$
$a_{11,12} = 0.0000025,\quad a_{12,12} = 0.00000055086$

Finally the series solution is

$$u(x) = x - 0.5000000x^3 - 0.33333333x^4 - 0.12500000x^5 - 0.0333333x^6 - 0.0069442x^7 - 0.001190x^8 - 0.000174x^9 - 0.0000222x^{10} - 0.0000025x^{11} + O(x)^{12}$$

Table 3: The result of the proposed method compared with Homotopy Perturbation Method

x	Exact solution	Approximate Solution	Present Method Error	HPM Error
0.0	0.0000000	0.0000000	0.0000e+00	0.0000e+00
0.1	0.0994654	0.0994654	2.0000e-11	3.0000e-11
0.2	0.1954244	0.1954244	0.0000e+00	0.0000e+00
0.3	0.2834708	0.2834708	0.0000e+00	1.0000e-10
0.4	0.3580379	0.3580379	1.0000e-10	2.0000e-10
0.5	0.4121803	0.4121803	1.0000e-09	1.1000e-09
0.6	0.4373085	0.4373085	1.1700e-09	4.4000e-09
0.7	0.4228881	0.4228881	1.1300e-08	1.3500e-08
0.8	0.3560865	0.3560866	2.9500e-08	3.6800e-08
0.9	0.2213643	0.2213644	1.2900e-08	9.0100e-08
1.0	0.0000000	0.0000000	1.9120e-07	2.0270e-07

7.0 DISCUSSION OF RESULTS

The numerical solutions of the proposed solution technique in this work are compared to those obtained for the same problems and the results are presented in *Tables 1, 2, and 3.* As can be seen from *Table 1* the proposed method generated the same values as the exact solution for the test problem with smaller error terms (maximum error of order 10^{-10}) when compared with the error terms (maximum error of order 10^{-4}) obtained using the variational decomposition method. *Table 2* shows the results of the proposed method when compared to the test problem in [16], where the variational iteration decomposition method is used. Again, we see that the new method proposed in this work is consistent with the exact solution and produces very small deviations in the region of 10^{-9}, with the implication of faster convergence when compared with the variational iteration decomposition method. The comparison between the solutions obtained for a test problem using the homotopy perturbation method and the proposed method is presented in *Table 3*. As we find in *Table 3*, the homotopy perturbation method has higher error terms for all values for the test problem. The value for the approximate solution from the proposed method is consistent with the exact solution and converges faster when compared with the homotopy perturbation method.

8.0 CONCLUSION

In this paper, a new modification of the variational iteration method has been used to obtain the numerical solution of boundary value problems. The modification involves the construction of Canonical polynomials coupled with the standard variational iterative scheme. The polynomials are used as trial functions in the approximation method employed. The results show that the proposed method converges faster to the exact solution than VDM, VIDM and HPM.

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