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FOUR-STAGE RUNGE-KUTTA METHOD IN THE SOLUTION OF INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this work, we have considered the numerical treatment of a highly stable four-stage Runge-kutta method for initial value problems (IVPs) in ordinary differential equations (ODEs). For this purpose, we have engaged an explosive study of the family of Runge-kutta methods up to the four – stage. The four-stage Runge-kutta method was derived and experimented on both linear and nonlinear IVPs for reliability through a manual and computer aided procedure called Maple 18 software to solve such IVPs. The resulting numerical evidences were presented graphically and in tables and were compared with the three stage-Runge-Kutta method for convergence.

Keywords: Differential Equation, Runge-Kutta Method, IVP

1. Introduction

When real-life situation is analyzed and modeled, it usually results to differential equation. Basically, differential equations can be implemented to model real-life questions, and graphs and computer calculations provides the real-life answers. However, in most real-life situations, the differential equation that models the problem is often complicated to solve. Thus, most of these problems are solved or analyzed along certain conditions known as "initial value conditions" which the stated problem must satisfy. Due to the complexity associated in seeking the solution of differential equations, such solutions can be obtained in two approaches [1]. There are few known analytics methods for seeking the solution of initial value problems (IVPs) such as the d-expansion method, the perturbation method, the Lyapunov parameter method [2], Thus, iterative techniques have been developed and implemented by various researchers over the years to effectively solve these problems. Basically, iterative methods for initial value problems are categorized into two groups. The first are the single step methods (Euler method, improved Euler method, Heun method, the Runge-Kutta methods, etc), and the second are the multi-step methods (Adams-Bashforth methods, Adams-Moulton methods, Milne's method, etc). However, the quest of generalizing the Euler's method, by allowing evaluations (number of evaluation) of the derivatives at a step is attributed to Runge [3]. Further works were made in [4-7]. Also, special iterative methods for computing second order ordinary differential equation were poised [8-10]. The Adams-Moulton method (also called the Adam-Bashforth-Moulton Corrector method when used with Adams-Bashforth method as a predicator-correction pair) is a multistep method derived from the fundamental calculus, [11-13]. The object of Euler's method is to seek computations (approximations) to the well-posed initial value problem [1]. In this study, we consider the four-stage Runge-Kutta method in the solution of initial value problems in ordinary differential equations. For the purpose of recalling, we will confine ourselves to the procedure adopted by [14].

2. RUNGE-KUTTA METHOD

The general stages in the Runge-Kutta methods can be tagged N-stage Runge-Kutta family and is defined as

$u_{i+1} = u_i + \emptyset x_i, u_i; h,$ (1)

where

$\emptyset x, u; h = \sum_{i=0}^{N} w_i\, k_i,$

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$k_1 = hgx_i, u_i,$
$k_i = hgx_i + hc_i, u_i + h\sum_{t=i}^{i-1} b_{it}k_t,\ \ i = 2, \cdots, N,\ \ c_i = \sum_{i=1}^{i-1} b_{it},,\ \ i = 2, \cdots, N.$
We now treat the different stages in the Runge-Kutta methods up to the third stage as follows:

2.1 One-stage Runge-Kutta method

Let $N = 1$ in (1) with$w_1 = 1$, we obtain the one-stage Runge-Kutta method (which is simply the Euler's sinlge step explicit method) as

$$u_{i+1} = u_i + ghx_i, u_i \tag{2}$$

2.2 Two-stage Runge-Kutta method

Let $N = 2$ in (1), then the two-stage Runge-Kutta method is expressed as

$$u_{i+1} = u_i + w_1k_1 + w_2k_2, \tag{3}$$

where

$$k_1 = hgx_i, u_i, \tag{4}$$
$$k_2 = hgx_i + hc_2, u_i + b_{21}k_1, \tag{5}$$
$$u_i = ux_i, \quad x_i = x_0 + ih,$$

where $w_1, w_2, c_2$and b_{21} are unknown constant parameters to be determined. Now expanding ux_{i+1} in a Taylor series through the terms of $o(h^2)$, we get

$$u_{i+1} \approx ux_{i+1} = ux_i + h = ux_i + hu'^{x_i} + \frac{h^2}{2}u''x_i \tag{6}$$

Recall that we have defined higher order derivatives as

$$u' = gx, u, u'' = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{du}{dx} = g_x + gg_u, u''' = \frac{\partial}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial}{\partial u}\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{du}{dx}\frac{du}{dx}$$
$$\Rightarrow u''' = g_{xx} + 2gg_{xu} + g^2g_{uu} + g_u(g_x + gg_u)\text{ , etc,}$$

where g and all partial derivatives are evaluated at (x_i, u_i) and $u_i = ux_i$.
Thus, equation (6) can be expressed as

$$u_{i+1} = ux_i + hg + \frac{h^2}{2}g\frac{\partial g}{\partial x} + \frac{h^2}{2}g\frac{\partial g}{\partial u} \tag{7}$$

Also, expanding equation (7) in a Taylor series through term of $o(h^2)$, we get

$$k_2 = hg + c_2h^2\frac{\partial g}{\partial x} + hb_{21}k_1\frac{\partial g}{\partial u} + (h^2b_{21}k_1c_2)\frac{\partial^2 g}{\partial x\partial u} \tag{8}$$

Now, substituting the expressions for k_1 and k_2 into equation (3.3), we have

$$u_{i+1} = u_i + w_1hg + w_2\, hg + c_2h^2\frac{\partial g}{\partial x} + hb_{21}k_1\frac{\partial g}{\partial u} + (h^2b_{21}k_1c_2)\frac{\partial^2 g}{\partial x\partial u} \tag{9}$$

Comparing the coefficients of h and h^2 with equation (7), we obtain

$$w_1 + w_2 = 1, \qquad w_2c_2 = \frac{1}{2},\ \ w_2b_{21} = \frac{1}{2}.$$

Solving the above equations, we obtain
$b_{21} = c_2$, $w_2 = \frac{1}{2c_2}$ and $w_1 = (1 - \frac{1}{2c_2})$.
Thus, the equations (3-5) can be rewritten as

$$u_{i+1} = u_i + 1 - \frac{1}{2c_2}k_1 + \frac{1}{2c_2}k_2, \tag{10}$$

where

$$k_1 = hgx_i, u_i, \tag{11}$$
$$k_2 = hgx_i + hc_2, u_i + b_{21}k_1, \tag{12}$$
$$u_i = ux_i, \quad x_i = x_0 + ih.$$

Here, c_2 is arbitrary and lies in the interval (0,1). Thus, we have an infinite family of these methods.
Now, if $c_2 = 1$, we obtain $b_{21} = 1$, $w_2 = \frac{1}{2}$ and $w_1 = \frac{1}{2}$. Hence, we get the method

$$u_{i+1} = u_i + \frac{1}{2}k_1 + k_2, \tag{13}$$
$$k_1 = hgx_i, u_i,$$
$$k_2 = hgx_i + h, u_i + k_1,$$
$$u_i = ux_i, \quad x_i = x_0 + ih,$$

which is the Heun's method (or Euler - Cauchy method).
If we choose $c_2 = \frac{1}{2}$, then $w_2 = 1, w_1 = 0$ and $b_{21} = \frac{1}{2}$. Thus, we get the method

$$u_{i+1} = u_i + k_2, \tag{14}$$

$k_1 = hgx_i, u_i,$
$k_2 = hg\, x_i + \frac{h}{2}, u_i + \frac{1}{2}k_1,$
$u_i = ux_i, \quad x_i = x_0 + ih\,,$
which is the modified Euler's method.

Local truncation error of the two-stage Runge-Kutta method
Subtracting (9) from (6), we obtain the truncation error (T.E) as follow:

$$T.E = ux_{i+1} - u_{i+1} = h^3 \frac{1}{6} - \frac{c_2}{4}[gg_{xx} + 2gg_{xu} + g^2 g_{uu}] + \frac{1}{6} g_u [g_x + gg_u]_{x_i} \quad (15)$$

Hence, the method is of order two. That is, it is a second order method. Now, observe that if $c_2 = \frac{2}{3}$ in (15), the first term vanishes. Thus, the standard two-stage Runge-Kutta method is given as

$$u_{i+1} = u_i + \frac{1}{4}k_1 + 3k_2, \quad (16)$$

$k_1 = hgx_i, u_i,$
$k_2 = hg\, x_i + \frac{2}{3}h, u_i + \frac{2}{3}k_1,$
$u_i = ux_i, \quad x_i = x_0 + ih\,,$

2.3 Three-stage Runge-Kutta method
Let $N = 3$ in (1), then the three-stage Runge-Kutta method is expressed as

$$u_{i+1} = u_i + w_1k_1 + w_2k_2 + w_3k_3, \quad (17)$$

where

$$k_1 = hgx_i, u_i, \quad (18)$$
$$k_2 = hgx_i + hc_1, u_i + b_{21}k_1, \quad (19)$$
$$k_3 = hgx_i + hc_2, u_i + b_{31}k_1 + b_{32}k_2, \quad (20)$$

$u_i = ux_i, \quad x_i = x_0 + ih\,,$
where $w_i, k_i,$ for $1 \le i \le 3, c_j$ for $1 \le i \le 2$ and b_{21}, b_{31}andb_{32} are unknown constant parameters to be determined.
Now expanding ux_{i+1} in a Taylor series through the terms of $o(h^3)$, we get

$$u_{i+1} \approx ux_{i+1} = ux_i + h = ux_i + hu'^{x_i} + \frac{h^2}{2}u''^{x_i} + \frac{h^3}{6}u'''x_i \quad (21)$$

Recall that we have defined higher order derivatives as

$$u' = gx, u, u'' = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{du}{dx} = g_x + gg_u, u''' = \frac{\partial}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial}{\partial u}\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{du}{dx}\frac{du}{dx}$$

$\Rightarrow u''' = g_{xx} + 2gg_{xu} + g^2 g_{uu} + g_u(g_x + gg_u)$, etc,
where g and all partial derivatives are evaluated at (x_i, u_i) and $u_i = ux_i$.
Thus, equation (21) can be expressed as

$$u_{i+1} = ux_i + hg + \frac{h^2}{2}gg_x + \frac{h^2}{2}gg_u + \frac{h^3}{6}g_{xx} + 2gg_{xu} + g^2g_{uu} + g_u(g_x + gg_u) \quad (22)$$

Also, expanding equation (19) and (20) in a Taylor series through term of $o(h^3)$ as follows:

$$k_2 = hg + c_1h^2\frac{\partial g}{\partial x} + hb_{21}k_1\frac{\partial g}{\partial u} + h^2 b_{21}k_1c_2\frac{\partial^2 g}{\partial x \partial u} + \frac{h^3c_1^2}{2}\frac{\partial^2 g}{\partial x^2} + \frac{hb_{21}^2k_1^2}{2}\frac{\partial^2 g}{\partial u^2}, \quad (23)$$

$$k_3 = hg + c_1h^2\frac{\partial g}{\partial x} + h(b_{31}k_1 + b_{32}k_2)\frac{\partial g}{\partial u} + h^2c_2(b_{31}k_1 + b_{32}k_2)\frac{\partial^2 g}{\partial x \partial u} + \frac{h^3c_2^2}{2}\frac{\partial^2 g}{\partial x^2} + \frac{h(b_{31}k_1 + b_{32}k_2)^2}{2}\frac{\partial^2 g}{\partial u^2} \quad (24)$$

Now, substituting the expressions for k_1, k_2 and k_3 into equation (16), we have

$u_{i+1} = u_i + w_1hg + w_2\, hg + c_1h^2\frac{\partial g}{\partial x} + hb_{21}k_1\frac{\partial g}{\partial u} + h^2b_{21}k_1c_2\frac{\partial^2 g}{\partial x\partial u} + \frac{h^3c_1^2}{2}\frac{\partial^2 g}{\partial x^2} + \frac{hb_{21}^2k_1^2}{2}\frac{\partial^2 g}{\partial u^2} + w_3\, hg + c_1h^2\frac{\partial g}{\partial x} +$
h(b31k1+b32k2)∂g∂u+h2c2(b31k1+b32k2)∂2g∂x∂u+h3c222∂2g∂x2+ h(b31k1+b32k2)22∂2g∂u2,
(25)

Comparing the coefficients of h, h^2 and h^3 with equation (22), we obtain

$$b_{21}w_2 + b_{31} + b_{32}{}^2w_3 = \frac{1}{2}$$

$$b_{21}^2w_2 + b_{31} + b_{32}{}^2w_3 = \frac{1}{3}$$

$c_1 w_2 + c_2 w_3 = \frac{1}{2}$
$c_1^2 w_2 + c_2^2 w_3 = \frac{1}{3}$
$w_1 + w_2 + w_3 = 1$
$c_2 b_{32} w_3 = \frac{1}{6}$.
Now, using
$b_{21} w_2 + b_{31} + b_{32}{}^2 w_3 = \frac{1}{2}$
$b_{21}^2 w_2 + b_{31} + b_{32}{}^2 w_3 = \frac{1}{3}$
$c_1 w_2 + c_2 w_3 = \frac{1}{2}$
$c_1^2 w_2 + c_2^2 w_3 = \frac{1}{3}$
one can show that
$b_{21} = c_1$
$b_{31} + b_{32} = c_2$
$c_1 w_2 + c_2 w_3 = \frac{1}{2}$
$c_1^2 w_2 + c_2^2 w_3 = \frac{1}{3}$
$w_1 + w_2 + w_3 = 1$
$c_2 b_{32} w_3 = \frac{1}{6}$
It is obvious that there are 6 equations with 8 unknowns. Hence, we keep the two unknown arbitrary. Now if we let $c_1 = c_2$, we get
$c_1 = c_2 = \frac{2}{3}, b_{31} = b_{32} = \frac{2}{3}, b_{31} = 0, w_1 = \frac{1}{4}, w_2 = w_3 = \frac{3}{8}$.
Thus, three stage Runge-Kutta method is given as
$u_{i+1} = u_i + \frac{1}{8}(2k_1 + 3k_2 + 3k_3,$ (26)
where
$k_1 = hgx_i, u_i,$
$k_2 = hg\, x_i + \frac{2}{3}h, u_i + \frac{2}{3}k_1,$
$k_3 = hg\, x_i + \frac{2}{3}h, u_i + \frac{2}{3}k_2,$
$u_i = ux_i, \quad x_i = x_0 + ih$.

2.4 Four-stage Runge-Kutta method

Let $N = 4$ in (1), then the form-stage Runge-Kutta method is expressed as
$u_{i+1} = u_i + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4,$ (27)
where
$k_1 = hgx_i, u_i,$ (28)
$k_2 = hgx_i + hc_1, u_i + b_{21} k_1,$ (29)
$k_3 = hgx_i + hc_2, u_i + b_{31} k_1 + b_{32} k_2,$ (30)
$k_4 = hgx_i + hc_3, u_i + b_{41} k_1 + b_{42} k_2 + b_{43} k_3,$ (31)
$u_i = ux_i, \quad x_i = x_0 + ih$,
where $w_i, k_i,$ for $1 \le i \le 4, c_j$ for $1 \le i \le 3$ and $b_{21}, b_{31}, b_{32}, b_{41}, b_{42}$ and b_{43} are unknown constant parameters to be determined. Now expanding ux_{i+1} in a Taylor series (as Taylor's theorem) through the terms of $o(h^4)$, we get
$u_{i+1} \approx ux_{i+1} = ux_i + h = ux_i + hu'^{x_i} + \frac{h^2}{2}u''^{x_i} + \frac{h^3}{6}u'''^{x_i} + \frac{h^4}{24}u^{iv}x_i$ (32)
Using the higher order partial derivatives on (32), we get
$u_{i+1} = ux_i + hg + \frac{h^2}{2}gg_x + \frac{h^2}{2}gg_u + \frac{h^3}{6}g_{xx} + 2gg_{xu} + g^2 g_{uu} + g_u(g_x + gg_u) + \frac{h^4}{24}g_{xxx} + 3g_x g_{xu} + 3gg_{xx}g_u +$
3g2gxguu+3ggxguu+gugxx+5ggugxu+gxgu2+ggugu2+4g2gugxx+g3guuu (33)

where g and all partial derivatives are evaluated at (x_i, u_i) and $u_i = ux_i$. Again, we have by Taylor series for multi-variable function $k_2 = hg + c_1h^2\frac{\partial g}{\partial x} + hb_{21}k_1\frac{\partial g}{\partial u} + h^2b_{21}k_1c_2\frac{\partial^2 g}{\partial x\partial u} + \frac{h^3c_1^2}{2}\frac{\partial^2 g}{\partial x^2} + \frac{hb_{21}^2k_1^2}{2}\frac{\partial^2 g}{\partial u^2} + h^2b_{21}^2c_1k_1^2\frac{\partial^3 g}{\partial x\partial u^2} + h^3b_{21}k_1c_1^2\frac{\partial^3 g}{\partial x^2\partial u} + \frac{h^4c_1^3}{6}\frac{\partial^3 g}{\partial x^3} + \frac{hk_1^3b_{21}^3}{6}\frac{\partial^3 g}{\partial u^3}$, (34)

$$k_3 = hg + c_1h^2\frac{\partial g}{\partial x} + h(b_{31}k_1 + b_{32}k_2)\frac{\partial g}{\partial u} + h^2c_2(b_{31}k_1 + b_{32}k_2)\frac{\partial^2 g}{\partial x\partial u} + \frac{h^3c_2^2}{2}\frac{\partial^2 g}{\partial x^2} + \frac{h(b_{31}k_1+b_{32}k_2)^2}{2}\frac{\partial^2 g}{\partial u^2} + \frac{h^2c_2(b_{31}k_1+b_{32}k_2)^2}{2}\frac{\partial^3 g}{\partial x\partial u^2} + \frac{h(b_{31}k_1+b_{32}k_2)^3}{6}\frac{\partial^3 g}{\partial u^3} + \frac{h^3c_2^2(b_{31}k_1+b_{32}k_2)}{6}\frac{\partial^3 g}{\partial x^2\partial u} + \frac{h^4c_2^3}{6}\frac{\partial^3 g}{\partial x^3}, \quad (35)$$

and

$$k_4 = hg + c_3h^2\frac{\partial g}{\partial x} + h(b_{41}k_1 + b_{42}k_2 + b_{43}k_3)\frac{\partial g}{\partial u} + h^2c_3(b_{41}k_1 + b_{42}k_2 + b_{43}k_3)\frac{\partial^2 g}{\partial x\partial u} + \frac{h^3c_3^2}{2}\frac{\partial^2 g}{\partial x^2} + \frac{h^4c_3^2}{6}\frac{\partial^3 g}{\partial x^3} + h\frac{(b_{41}k_1+b_{42}k_2+b_{43}k_3)^2}{2}\frac{\partial^2 g}{\partial u^2} + \frac{h^2c_3(b_{41}k_1+b_{42}k_2+b_{43}k_3)}{2}\frac{\partial^2 g}{\partial x\partial u^2} + \frac{h^3c_3^2(b_{41}k_1+b_{42}k_2+b_{43}k_3)}{2}\frac{\partial^2 g}{\partial x^2\partial u} + h\frac{(b_{41}k_1+b_{42}k_2+b_{43}k_3)^2}{62x\partial u}\frac{\partial^3 g}{\partial u^3} \quad (36)$$

Now, substituting the expressions for k_1, k_2, $k_3$andk_4 into equation (27), we have

$u_{i+1} = u_i + w_1hg + w_2\, hg + c_1h^2\frac{\partial g}{\partial x} + hb_{21}k_1\frac{\partial g}{\partial u} + h^2b_{21}k_1c_2\frac{\partial^2 g}{\partial x\partial u} + \frac{h^3c_1^2}{2}\frac{\partial^2 g}{\partial x^2} + \frac{hb_{21}^2k_1^2}{2}\frac{\partial^2 g}{\partial u^2} + h^2b_{21}^2c_1k_1^2\frac{\partial^3 g}{\partial x\partial u^2} +$
h3b21k1c12∂3g∂x2∂u+h4c136∂3g∂x3+hk13b2136∂3g∂u3+w3hg+c1h2∂g∂x+h(b31k1+b32k2)∂g∂u+h2c2(b31k1+b32k2)∂2g∂x∂u+h3c222∂2g∂x2+
h(b31k1+b32k2)22∂2g∂u2+h2c2(b31k1+b32k2)22∂3g∂x∂u2+h(b31k1+b32k2)36∂3g∂u3+h3c22(b31k1+b32k2)6∂3g∂x2∂u+h4c236∂3g∂x3+w4hg+c3h2∂g∂x+h(b41k1+b42k2+b43k3)∂g∂u+h2c3(b41k1+b42k2+b43k3)∂2g∂x∂u+h3c322∂2g∂x2+h4c326∂3g∂x3+h(b41k1+b42k2+b43k3)22∂2g∂u2+
h2c3(b41k1+b42k2+b43k3)2∂2g∂x∂u2+h3c32(b41k1+b42k2+b43k3)2∂2g∂x2∂u+
h(b41k1+b42k2+b43k3)262x∂u∂3g∂u3, (37)

Comparing the coefficients of h, h^2, h^3 and h^4 with equation (33), we obtain

$$b_{21}w_2 + b_{31} + b_{32}w_3 + w_4(b_{41} + b_{42} + b_{43}) = \frac{1}{2}$$
$$b_{21}c_1w_2 + b_{31} + b_{32}c_2w_3 + b_{41} + b_{42} + b_{43}c_3w_4 = \frac{1}{3}$$
$$b_{21}c_1^2w_2 + b_{31} + b_{32}c_2^2w_3 + b_{41} + b_{42} + b_{43}c_3^2w_4 = \frac{1}{3}$$
$$c_1w_2 + c_2w_3 + c_3w_4 = \frac{1}{2}$$
$$c_1^2w_2 + c_2^2w_3 + c_3^2w_4 = \frac{1}{3}$$
$$c_1^3w_2 + c_2^3w_3 + c_3^3w_4 = \frac{1}{4}$$
$$w_1 + w_2 + w_3 + w_4 = 1$$
$$c_1b_{32}w_3 + w_4(c_1b_{42} + c_2b_{43}) = \frac{1}{6}$$
$$c_1^2b_{32}w_3 + w_4(c_1^2b_{42} + c_1^2b_{43}) = \frac{1}{12}$$
$$c_1c_2b_{32}w_3 + w_4(c_1b_{42} + c_2b_{43})c_3 = \frac{1}{8}$$
$$w_4c_1b_{32}b_{43} = \frac{1}{24}$$

Again, using the equations

$$b_{21}w_2 + b_{31} + b_{32}w_3 + w_4(b_{41} + b_{42} + b_{43}) = \frac{1}{2}$$

$b_{21}c_1w_2 + b_{31} + b_{32}c_2w_3 + b_{41} + b_{42} + b_{43}c_3w_4 = \frac{1}{3}$

$b_{21}c_1^2w_2 + b_{31} + b_{32}c_2^2w_3 + b_{41} + b_{42} + b_{43}c_3^2w_4 = \frac{1}{3}$

$c_1w_2 + c_2w_3 + c_3w_4 = \frac{1}{2}$

$c_1^2w_2 + c_2^2w_3 + c_3^2w_4 = \frac{1}{3}$

$c_1^3w_2 + c_2^3w_3 + c_3^3w_4 = \frac{1}{4}$,

One can show that

$c_1 = b_{21}$

$c_2 = b_{31} + b_{32}$

$c_3 = b_{41} + b_{42} + b_{43}$.

Hence the set of equations that one has to solve in order to determine the set of all unknowns is given by

$$\begin{cases} c_1 = b_{21} \\ c_2 = b_{31} + b_{32} \\ c_3 = b_{41} + b_{42} + b_{43} \\ c_1w_2 + c_2w_3 + c_3w_4 = \frac{1}{2} \\ c_1^2w_2 + c_2^2w_3 + c_3^2w_4 = \frac{1}{3} \\ c_1^3w_2 + c_2^3w_3 + c_3^3w_4 = \frac{1}{4} \\ w_1 + w_2 + w_3 + w_4 = 1 \\ c_1b_{32}w_3 + w_4c_1b_{42} + c_2b_{43} = \frac{1}{6} \\ c_1^2b_{32}w_3 + w_4c_1^2b_{42} + c_1^2b_{43} = \frac{1}{12} \\ c_1c_2b_{32}w_3 + w_4c_1b_{42} + c_2b_{43}c_3 = \frac{1}{8} \\ w_4c_1b_{32}b_{43} = \frac{1}{24} \end{cases} \quad (38)$$

There exist 11 equations with 13 unknowns. Thus, there exist two arbitrary parameters. Since the terms up to $o(h^4)$ are compared, a simple solution is given by

$c_1 = c_2 = \frac{1}{2}, \quad c_3 = b_{43} = 1, \quad w_2 = w_3 = \frac{1}{3}, w_1 = w_4 = \frac{1}{6}, \quad b_{21} = b_{32} = \frac{1}{2},\ b_{31} = b_{41} = b_{42} = 0.$

Therefore, the four-stage Runge-kutta method is given as

$u_{i+1} = u_i + \frac{1}{6}k_1 + 2k_2 + 2k_3 + k_4,$

where

$k_1 = hgx_i, u_i,$ $\quad k_2 = hg\, x_i + \frac{1}{2}h, u_i + \frac{1}{2}k_1,$

$k_3 = hg\, x_i + \frac{1}{2}h, u_i + \frac{1}{2}k_2,$ $\quad k_4 = hgx_i + h, u_i + k_3,$

$u_i = ux_i, \quad x_i = x_0 + ih\,,\ i = 0,1,2,3,\cdots.$

3 Numerical Examples

In this section, we solve some selected IVPs to illustrate both the three-stage and four-stage Runge-kutta methods.

Example 1

Consider the initial value problem $u' = -2xu^2$ with $u0 = 1$ and $h = 0.2$, in the interval (0,1), using the four-stage Runge-Kutta method.

The Exact solution is $ux = \frac{1}{1+x^2}$.

Clearly, $gx, u = -2xu^2, x_0 = 0,\ u_0 = 1.$

Table 1: Showing the comparison of results between the exact solution, three-stage and four stage Runge-kutta methods for Example 1

i	x_i	Exact solution	Three-stage Runge-Kutta method	Four-stage Runge-Kutta method	Error byThree-stage Runge-Kutta method	Error by Four-stage Runge-Kutta method
0	0	1.0000	1.0000	1.0000	0.000E+00	0.000E+00
1	0.2	0.9615	0.9614	0.9615	1.000E-04	3.275E-05
2	0.4	0.8621	0.8619	0.8621	2.000E-04	4.758E-05
3	0.6	0.7353	0.7352	0.7353	1.000E-04	2.166E-05
4	0.8	0.6098	0.6096	0.6098	2.000E-04	4.817E-05
5	1.0	0.500	0.4997	0.5000	3.000E-04	7.203E-06

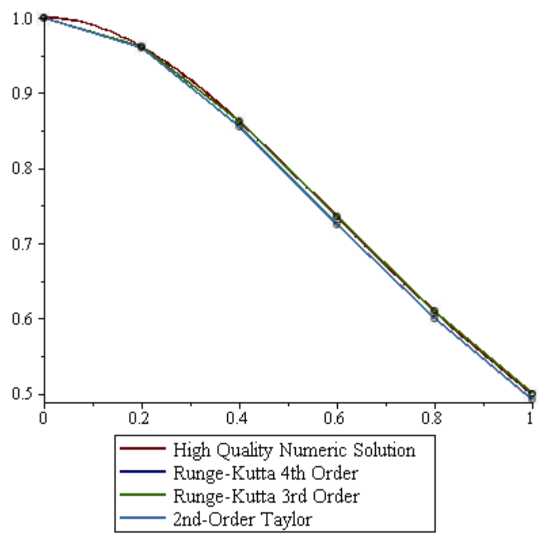

Figure 1: Graphical simulation for Example 1

This procedure operates using floating-point numerics; that is, inputs are first evaluated to floating-point numbers before computations proceed, and numbers appearing in the output will be in floating-point format.

Example 2

Solve the initial value problem $u' = -2x,\ u0 = 1,\ h = 0.2$, on the interval (0,1).

The exact solution is $ux = 1 - x^2$.

Applying the three-stage and four-stage Runge-Kutta methods on the above problem, the results are shown in the table below.

Table 2: Showing the comparison of results between the exact solution, three-stage and four stage Runge-kutta methods for Example 2

i	x_i	Exact solution	Three-stage Runge-Kutta method	Four-stage Runge-Kutta method	Error by Three-stage Runge-Kutta method	Error by Four-stage Runge-Kutta method
0	0	1	1	1	0.0000E+00	0.0000E+00
1	0.2	0.96	0.96	0.96	0.0000E+00	0.0000E+00
2	0.4	0.84	0.84	0.84	0.0000E+00	0.0000E+00
3	0.6	0.64	0.64	0.64	0.0000E+00	0.0000E+00
4	0.8	0.36	0.36	0.36	0.0000E+00	0.0000E+00
5	1.0	0.00	0.00	0.00	0.0000E+00	0.0000E+00

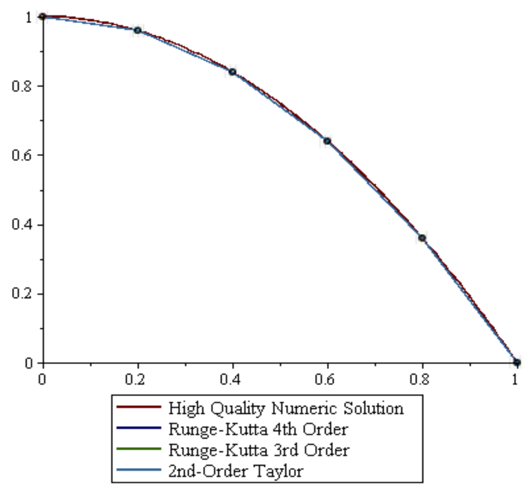

Figure 2: Graphical simulation for Example 2

Example 3

Consider the initial value problem $u' = 1 - \cos(x)$ with $u0 = 1$ and $h = 0.2$, in the interval (0,1), using the four-stage Runge-Kutta method.

The Exact solution is $ux = -\sin x + x + 1$.

Clearly, $gx, u = 1 - \cos(x), x_0 = 0,\ u_0 = 1$.

Table 3: Showing the comparison of results between the exact solution, three-stage and four stage Runge-kutta methods for Example 3

i	x_i	Exact solution	Three-stage Runge-Kutta method	Four-stage Runge-Kutta method	Error by Three-stage Runge-Kutta method	Error by Four-stage Runge-Kutta method
0	0	1	1	1	0.0000E+00	0.000E+00
1	0.2	1.001330669	1.001331359	1.00133055870156	6.9000E-07	1.105E-07
2	0.4	1.010581658	1.010584488	1.01058144109000	2.8300E-06	2.166E-07
3	0.6	1.035357527	1.035363863	1.03535721254087	6.3360E-06	3.141E-07
4	0.8	1.082643909	1.082654977	1.08264351009437	1.1068E-05	3.990E-07
5	1.0	1.158529015	1.158545852	1.15852854715111	1.6837E-05	4.680E-07

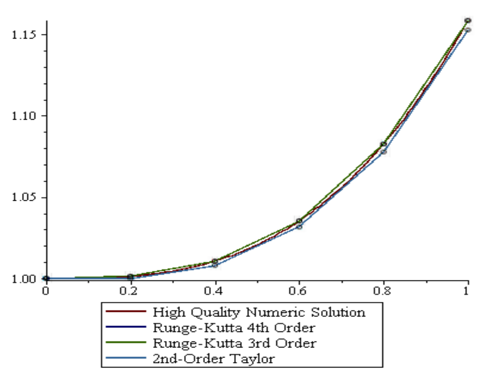

Figure 3: Graphical simulation for Example 3

4. Discussion of Results

The results obtained show that the four-stage Runge-kutta method is a more an excellent solver of initial value problems in ordinary differential equations, and is highly stable. Results obtained as presented graphically and in tables show that approximate solution with four-stage Runge-kutta method at some grid- points are very close to the exact solution as i tends to infinity.

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