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OUT - OF - PLANE EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL COPENHAGEN RESTRICTED THREE-BODY PROBLEM

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Abstract

This paper studies the motion of an infinitesimal body near the out of plane equilibrium points in the photogravitational restricted three-body problem in the case of two equally heavy primary bodies (Copenhagen Problem). These equilibria are determined numerically based on the three-dimensional dynamic equations. The influence of the radiation factors on the positions of these equilibria as well as the allowed regions of motion as determined by the zero velocity curves is studied in a parametric way. It is observed that their positions as well as the topology of the zero velocity curves are affected by the parameters. Finally, the stability of these points is studied and it is found that there is a limited value of the radiation factors for which the equilibrium points are stable. This model has many applications, especially in the dynamics behavior of extremely small objects such as dust grains and interplanetary drifters. It also has interesting applications for artificial satellites, future space colonization or even vehicles and spacecraft parking.

Keywords: Circular restricted three-body problem; Photogravitational Copenhagen problem; Out-of-plane equilibria; Zero velocity curves; Stability.

1.0 Introduction

The restricted three-body problem (R3BP) in which two massive bodies (primaries) revolve around their common center of mass in circular orbits and a third body of infinitesimal mass moves in the gravitational field, is a classical problem; which has attracted the attentions of mathematicians and astronomers for centuries because of its application in dynamics of the solar and stellar systems, lunar theory and artificial satellites. A special version of the classical restricted three-body problem is the so-called 'Copenhagen problem' where the two primary bodies, which rotate with constant angular velocity around their common center of gravity, have equal masses. The third body, which is usually refereed as test particle, moves under the resultant Newtonian gravitational field of the primaries. This problem was initially studied by Elis Strömgren and his colleagues at the Copenhagen Observatory [1]. Due to its simplicity and cosmological relevance the gravitational Copenhagen problem has been studied by some authors ([1-8]). During the last century, several modifications of this classical problem have been introduced in order to make it more relevant and applicable to certain systems of Dynamical Astronomy and hence one cannot survey all the most important works here. One of such modifications of the classical problem is the photogravitational restricted three-body problem (PR3BP) in which the repulsive force of the radiation is also considered in the potential function and it was introduced for studying the specific three-body problem of Sun, planet, and a dust particle [9]. The importance of the radiation influence on celestial bodies has been recognized by many scientists, especially in connection with the formation of concentrations of interplanetary and interstellar dust or grains in planetary and binary star systems, as well as the perturbations on artificial satellites [10, 11]. The majority of these systems consist of a "sun" and a planet or of two planets of nearly equal masses that rotate about their center of mass. Therefore, the Copenhagen case of the R3BP comes again on stage and the scientific interest it arouses is renewed ([12-18]. Their interest was mainly focused on the study of periodic and asymptotic orbits. Unlike the plane problem the three-dimensional R3BP has not been extensively studied. One of the most important steps in the study of a dynamical problem is the determination of the equilibrium states of the system. It is well known that in the classical R3BP five equilibrium points exist [1]. In dynamical astronomy and celestial mechanics these equilibrium points of a system play a role

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of great importance since at these locations the test particle is able to maintain its relative position, with respect to the primary bodies. Also, they are very important especially for plotting the trajectories of spacecrafts. On this basis, knowing the equilibrium points of a system, give us very important information regarding the most intrinsic properties of the dynamical system. The plane perpendicular to the plane of motion of the primaries has been shown to result in out-of-plane equilibrium points (OEPs) when the photogravitational effects of one or both primary bodies are taken into consideration. The existence of the out of the orbital plane equilibrium points, in the PR3BP, was first pointed out by [19] in the cases of Sun-Planet-particle and Galaxy Kernel-Sun-particle. He found two equilibrium points on the (x , z) plane in symmetrical positions with respect to the (x , y) plane. OEPs in the photogravitational circular restricted problems have been studied in many earlier papers [20 – 27]. The existence of these points is of particular astronomical interest in connection with planetary system formation, satellite motion, etc.

In the present study our aim is to numerically investigate how the radiation factors influence the positions of the OEPs as well as the regions allowed to motion of the infinitesimal body in the Copenhagen problem [21]. Seen as whole the problem displays a number of interesting features that are not apparent in previous studies.

The paper is organized in six sections. Section 2 provides the equations of motion for the dynamic model-problem. Section 3 locates the positions of the out of plane equilibrium points. Specifically, it discusses the influence of the system parameters (radiation factors q_1 and q_2 of the primaries) on the equilibrium points in a parametric way. Section 4 is devoted to the surfaces and curves of zero velocity. The regions of allowed motion as determined by the zero velocity curves as well as the positions of out of plane points are given. Section 5 established the linearized stability of these equilibria; while Section 6 discusses the obtained results and conclusion of the paper.

2 Equations of Motion

Consider the motion of an infinitesimal mass m_3 (e.g. interplanetary dust grain) that is influenced by the gravitational and radiation pressure forces of two illuminating primary bodies of mass m_1 and m_2 and radiation pressure factors q_1 and q_2. The two heaviest bodies (the primaries) revolve under their mutual gravitational attraction around their center of mass in circular orbits. The units of measure of mass, length and time are taken so that the sum of the masses and the distance between the primaries is unity, and, also, the Gaussian constant of gravitation G is 1. A rotating rectangular coordinate system whose origin is the center of mass of the primaries and whose Ox-axis contains the primaries is used. The angular velocity of the system is also unity [1]. Then, the equations of motion of the infinitesimal mass in the three-dimensional photogravitational Copenhagen restricted three-body in this coordinate system can be described following [5] and [21] in the dimensionless variables as,

$$\ddot{x}-2\dot{y}=\frac{\partial\Omega}{\partial x}=x-\frac{q_1(1-\mu)(x+\mu)}{r_1^3}-\frac{q_2\mu(x+\mu-1)}{r_2^3},$$
$$\ddot{y}+2\dot{x}=\frac{\partial\Omega}{\partial y}=y-\frac{q_1(1-\mu)y}{r_1^3}-\frac{q_2\mu y}{r_2^3}, \qquad (1)$$
$$\ddot{z}=\frac{\partial\Omega}{\partial z}=-\frac{q_1(1-\mu)z}{r_1^3}-\frac{q_2\mu z}{r_2^3}.$$

where dots denote time derivatives and Ω the potential function in synodic coordinates is given as

$$\Omega=\frac{1}{2}(x^2+y^2)+\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}, \qquad (2)$$

$$r_1^2=(x+\mu)^2+y^2+z^2, \quad r_2^2=(x+\mu-1)^2+y^2+z^2, \qquad (3)$$

where r_1 and r_2 are the distances of the third body from the primaries; μ is the mass-ratio of the smaller primary to the total mass of the primaries and $1-\mu=\mu=0.5$ (the larger primary is located at the position ($-\mu$,0,0) and the second primary at the ($1-\mu$,0,0) correspondingly) and the unit of distance is the distance between the primaries. The radiation pressure parameters of the primaries q_1, q_2 according to Radzievskii theory is expressed by means of the relations $q_i=1-b_i, (i=1,2)$ where b_1 and b_2 are the ratios of the radiation force F_r to the gravitational force F_g which results from the gravitation due to the two primary bodies m_1 and m_2, respectively. It is interesting to note that for $q_1=q_2=1$, we obtain classical circular restricted three-body problem. It is clear that: If q_i=1, radiation pressure has no effect. If $0<q_i<1$, gravitational force exceeds radiation. .If q_i=0, radiation force balances the gravitational one. If $q_i<0$, then radiation pressure overrides the gravitational attraction.

The energy (Jacobi) integral of this problem, is given by the expression

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=2\Omega-C, \tag{4}$$

where C is the Jacobian constant, while $\dot{x}$, $\dot{y}$ and $\dot{z}$ are the velocities.

3 Locations of out-of-plane equilibrium points

The equilibrium points out of the plane Oxy can be found by setting all velocity and acceleration terms to zero and solving the right sides of system (1). Obviously, the second equation of the system (1) is satisfied for $y=0$ and we solve the remaining two equations for $y=0$ and $z\neq 0$,

$$x_0-\frac{q_1(1-\mu)(x_0+\mu)}{r_{10}^3}-\frac{q_2\mu(x_0+\mu-1)}{r_{20}^3}=0, \tag{5}$$

$$\left[\frac{q_1(1-\mu)}{r_{10}^3}+\frac{q_2\mu}{r_{20}^3}\right]z_0=0. \tag{6}$$

where the two radii can be deduced to

$$r_{10}^2=(x_0+\mu)^2+z_0^2, \qquad r_{20}^2=(x_0+\mu-1)^2+z_0^2$$

and the subscript '0' is used to denote the equilibrium values.

From equation (6) we have that

$$\frac{r_{20}}{r_{10}}=\left[\left(\frac{-q_2}{q_1}\right)\frac{\mu}{1-\mu}\right]^{\frac{1}{3}}\equiv k \tag{7}$$

where k is a constant.

From Eq. 7 it can be seen that, for the existence of any real solution, one of the following conditions is necessary to hold:

$$q_1q_2<0 \quad \text{or} \quad q_1=q_2=0 \tag{8}$$

The second condition means that the gravitational attractions balance the corresponding radiation pressure forces. This case will not be considered here. The first condition means that the radiation pressure force of just one of the primaries exceeds its gravitational attraction and implies that Eq. 7 can be satisfied if and only if q_1 and q_2 have different signs, which is not true for the classical case. In this case it is evident from the form of (5) and (6) that solutions, if they exist, occur in pairs, corresponding to $\pm z$, as is necessary from the symmetry of the system. The locations of out-of-plane equilibria are hard to be obtained with analytical expressions. Therefore, approximated solutions (numerical methods) were given in previous publications. Here, only the numerical results are presented to determine the coordinates of these equilibrium points.

We found two points to be located in the (x , z) plane (the equilibria above and below the radiating primary m_1 are denoted by L_1^z(positive $z-$coordinate) and L_2^z(the other one) correspondingly) in symmetrical positions with respect to the (x , y) plane. They are solved by using the software package *Mathematica*. The existence and locations of these equilibria depend on the factors q_1 and q_2 for fixed $\mu=0.5.$ For brevity we shall restrict the analysis of this model problem to the case where $q_1<0$ and $q_2>0.$ Because of the symmetry of the problem, solutions for negative q_2 and positive q_1 can be obtained by the interchange of $\mu\leftrightarrow 1-\mu.$

To what follows we investigate the effects of radiation factors on the positions of the equilibrium points. The admissible regions of the radiation factors q_1 and q_2 lie in the intervals $q_1\in[-1, 0)$ and $q_2\in$ [1, 0) [21]. After working under the assumption that the radiation factors vary in the above given intervals, we have found for various cases, all the equilibrium positions corresponding to each combination of these values. The results obtained are shown numerically and graphically in Tables 1—4 and Figures 1—4, respectively as q_2 varies, for fixed values of q_1. In all the cases, these points exist at various of q_1 and for several values of q_2 in the given intervals and cease to exist for a value outside of the range. Tables 1, 2, 3 and 4, represent the cases when we have set $q_1=-0.1, -0.02, -0.015,$ and -0.002 for varying radiation factor q_2 of the primary body m_2 respectively. As can be seen, as the radiation parameters q_1 and q_2 increase from -1 to zero and from 1 to zero correspondingly, the positions of the out of plane equilibrium points move onto the $Ox-$ axis. In Figures 1, 2, 3 and 4 we show the effects of the radiation parameters on the positions of the two symmetrical, with respect to (x, y) plane, out of plane equilibrium points L_1^Z and L_2^Z, where $q_1=-0.1$, q_2 in the interval [1, 0.11], $q_1=-0.02$, q_2 in the interval [1, 0.03], $q_1=-0.015$, q_2 in the interval [1, 0.12], and $q_1=-0.002$, q_2 in the interval [1, 0.46] correspondingly.

Table 1. Coordinates of the out of plane equilibrium points as function of $q_2 \in$ [1, 0.11] for $q_1 = -0.1$ and $\mu = 0.5$

q_2	x	$\pm z$
1.0	-0.432260	0.482507
0.9	-0.409432	0.487794
0.8	-0.384807	0.493222
0.7	-0.357957	0.499034
0.6	-0.328265	0.505684
0.5	-0.294778	0.514105
0.4	-0.255888	0.526439
0.3	-0.208470	0.548668
0.2	-0.144654	0.605187
0.11	-0.038826	0.985390

Table 2. Coordinates of the out of plane equilibrium points as function of $q_2 \in$ [1, 0.33] for $q_1 = -0.02$ and $\mu = 0.5$

q_2	x	$\pm z$
1.0	-0.477558	0.274714
0.9	-0.456258	0.276413
0.8	-0.433428	0.276789
0.7	-0.408735	0.275585
0.6	-0.381715	0.272427
0.5	-0.351679	0.266762
0.4	-0.317531	0.257743
0.3	-0.277340	0.244058
0.2	-0.227068	0.224091
0.1	-0.154849	0.204535
0.03	-0.051822	0.364787

Table 3. Coordinates of the out of plane equilibrium points as function of $q_2 \in$ [1, 0.12] for $q_1 = -0.015$ and $\mu = 0.5$

q_2	x	$\pm z$
1.0	-0.481525	0.249052
0.9	-0.460347	0.250388
0.8	-0.437657	0.250164
0.7	-0.415678	0.248358
0.6	-0.386315	0.243537
0.5	-0.356534	0.235841
0.4	-0.322724	0.223629
0.3	-0.283016	0.204460
0.2	-0.233536	0.173297
0.12	-0.180169	0.133493

Table 4. Coordinates of the out of plane equilibrium points as function of $q_2 \in$ [1, 0.46] for $q_1 = -0.002$ and $\mu = 0.5$

q_2	$\lambda_{1,2}$	$\lambda_{3,4}$
1.0	-0.495223	0.126306
0.9	-0.474451	0.125643
0.8	-0.452232	0.121207
0.7	-0.428261	0.111598
0.6	-0.402115	0.093670
0.5	-0.373171	0.056632
0.46	-0.360610	0.017567

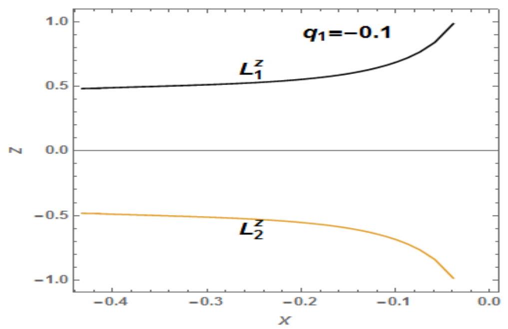

Fig. 1 Position of L_1^z and L_2^z in the ($x-z$) plane as a function of q_2 in the interval [1, 0.11] (with an arbitrary small step), for $q_1=-0.1,$ and $\mu=0.5$ corresponding to Table 1.

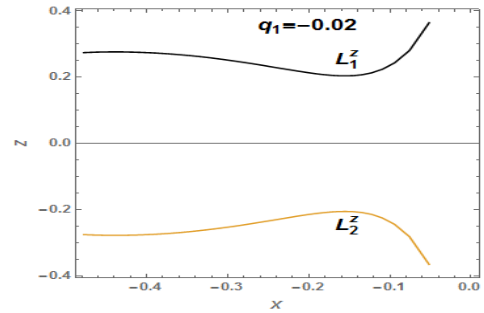

Fig. 2 Position of L_1^z and L_2^z in the ($x-z$) plane as a function of q_2 in the interval [1, 0.03] (with an arbitrary small step), for $q_1=-0.02,$ and $\mu=0.5$ corresponding to Table 2.

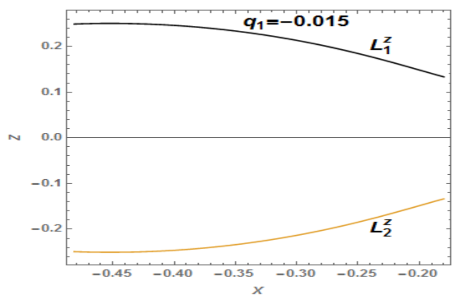

Fig. 3 Position of L_1^z and L_2^z in the ($x-z$) plane as a function of q_2 in the interval [1, 0.12] (with an arbitrary small step), for $q_1=-0.015,$ and $\mu=0.5$ corresponding to Table 3.

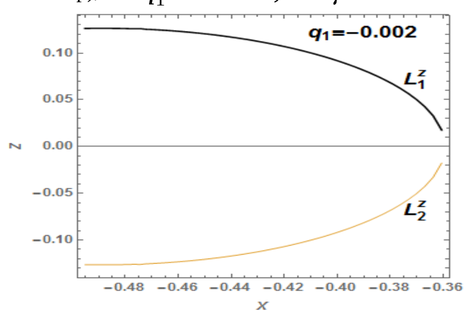

Fig. 4 Position of L_1^z and L_2^z in the ($x-z$) plane as a function of q_2 in the interval [1, 0.46] (with an arbitrary small step), for $q_1=-0.002,$ and $\mu=0.5$ corresponding to Table 4.

From all the above figures it is obvious that as the radiation parameters increase the equilibria decrease to make the positions of out of plane equilibrium points tend to the $Ox-$ axis.
Taking into account all the numerical outcomes given in Tables 1, 2, and 3 as well as in Figures 1, 2, and 3 it is obvious that radiation parameters of the primaries effects the positions of the out of plane equilibrium points significantly.

4 Zero - velocity curves in the (x, z) plane

The usefulness of the Jacobi constant integral in clarifying certain general properties of the relative motion of a small body by the construction and investigation of zero velocity curves in every problem of celestial dynamics was pointed out by many investigators in the past. In this section, we present the contours of the surface (4) on the (x, z) plane, for zero velocity, which provide the zero velocity curves. In Figure 5 the zero velocity curves for $\mu=0.5, q_2=0.9$ when radiation factor q_1 varies (i.e for $q_1=-0.1, q_1=-0.02$ and $q_1=-0.002$) are illustrated. It can be seen that the zero velocity curves between the out of plane equilibrium points form regions not allowed to possible motion which shrink to the bigger primary body m_1 as the radiation factor q_1 increases. In Figure 6 we present the zero velocity curves for $\mu=0.5, q_1=-0.002$ when the values of radiation factor q_2 varies ($q_2=1, q_2=0.55$ and $q_2=0.5$). It is very clear that the locations are affected similarly as depicted in Figure 5. In particular, the zero velocity curves up to the out of plane equilibria go approaching the bigger primary body m_1. Similar phenomenon we observe in the photogravitational restricted four-body problem where for certain negative values of the radiation factor, the symmetrical equilibrium points tend to the dominant primary body m_1 [28].

From the results in Figures 5 and 6 we conclude that radiation pressure has significant effects on the topological structure of the regions allowed to motion of the third infinitesimal particle as determined by the zero velocity surface and the corresponding equipotential curves.

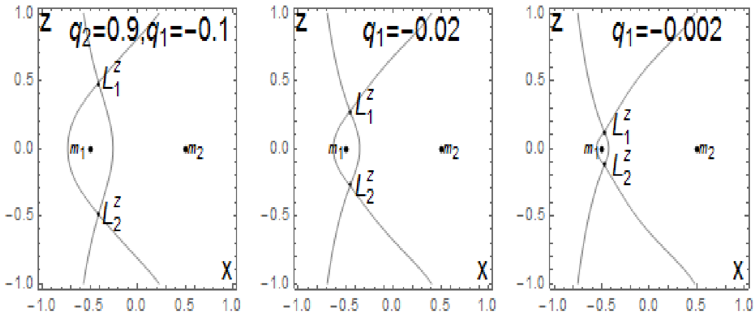

Fig.5. Zero velocity curves in the (x, z) plane and locations of the out of plane equilibria for $q_1 = -0.1$, $q_1 = -0.02$ and $q_1 = -0.002$ correspondingly. The locations of the primary bodies are presented too. Note: The value of $\mu = 0.5$, and $q_2 = 0.9$ are fixed for all cases

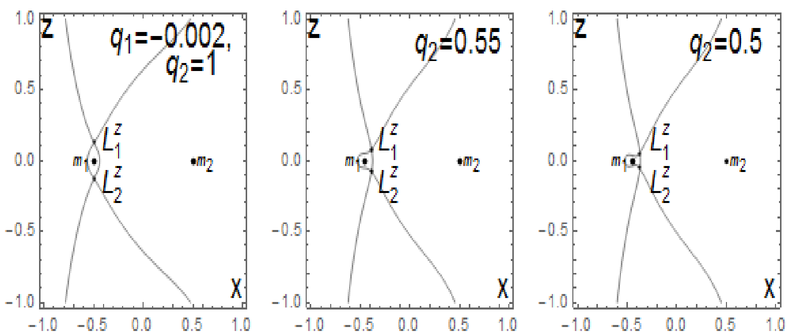

Fig.6. Zero velocity curves in the (x, z) plane and locations of the out of plane equilibria for $q_2 = 1$, $q_2 = 0.55$ and $q_2 = 0.5$ correspondingly. The locations of the primary bodies are presented too. Note: The value of $\mu = 0.5$, and $q_1 = -0.002$ are fixed for all cases

5 Linear stability of the equilibrium points

To determine the linear stability of the out of plane equilibrium points $L_{1,2}^{Z}$ we transfer the origin to ($x_0, 0, z_0$) and linearize the equations of motion, obtaining:

$$\begin{aligned}\ddot{\xi} - 2\dot{\eta} &= \Omega^0_{xx}\xi + \Omega^0_{xy}\eta + \Omega^0_{xz}\zeta \\ \ddot{\eta} + 2\dot{\xi} &= \Omega^0_{yx}\xi + \Omega^0_{yy}\eta + \Omega^0_{yz}\zeta \\ \ddot{\zeta} &= \Omega^0_{zx}\xi + \Omega^0_{zy}\eta + \Omega^0_{zz}\zeta\end{aligned} \tag{9}$$

where the superscript 'o' indicates that the partial derivatives are to be evaluated at the equilibrium point.

Explicitly, the partial derivatives of system (9) are

$$\Omega^0_{xy} = \Omega^0_{yx} = \Omega^0_{yz} = \Omega^0_{zy} = 0, \tag{10}$$

$$\Omega^0_{xx} = 1 - \frac{(1-\mu)q_1}{r_{10}^3} - \frac{\mu q_2}{r_{20}^3} + \frac{3(1-\mu)(x_0+\mu)^2 q_1}{r_{10}^5} + \frac{3\mu(x_0+\mu-1)^2 q_2}{r_{20}^5} \tag{11}$$

$$\Omega^0_{zx} = \Omega^0_{xz} = 3z_0\left(\frac{(1-\mu)(x_0+\mu)q_1}{r_{10}^5} + \frac{\mu(x+\mu-1)q_2}{r_{20}^5}\right) \tag{12}$$

$$\Omega^0_{yy} = 1 - \frac{(1-\mu)q_1}{r_{10}^3} - \frac{\mu q_2}{r_{20}^3} \tag{13}$$

$$\Omega^0_{zz} = -\left[\frac{(1-\mu)q_1}{r_{10}^3} + \frac{\mu q_2}{r_{20}^3} - \frac{3(1-\mu)q_1 z_0^2}{r_{10}^5} - \frac{3\mu q_2 z_0^2}{r_{20}^5}\right] \tag{14}$$

With

$r_{10}^2 = (x_0 + \mu)^2 + z_0^2, \quad r_{20}^2 = (x_0 + \mu - 1)^2 + z_0^2$

The characteristic equation corresponding to system (9) is given by

$$\lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \qquad (15)$$

where

$$a = 4 - \Omega_{xx}^0 - \Omega_{yy}^0 - \Omega_{zz}^0,$$
$$b = \Omega_{xx}^0\Omega_{yy}^0 + \Omega_{yy}^0\Omega_{zz}^0 + \Omega_{zz}^0\Omega_{xx}^0 - 4\Omega_{zz}^0 - (\Omega_{xz}^0)^2, \qquad (16)$$
$$c = (\Omega_{xz}^0)^2\Omega_{yy}^0 - \Omega_{xx}^0\Omega_{yy}^0\Omega_{zz}^0.$$

which is a polynomial of sixth degree in λ.

The eigenvalues of the characteristic equation (15) determine the stability or instability of the respective equilibrium points. An equilibrium point is stable only when all roots of the characteristic equation for λ are pure imaginary. Otherwise, the equilibrium point is unstable.

As a particular example we compute the characteristic roots $\lambda_i, i = 1,2,...,6$ which are shown in Tables 5, 6, 7 and 8 for $\mu = 0.5$ and for a wide range of the radiation factors q_1 and q_2. Here the letters U and S stand for unstable and stable, respectively.

Table 5. Stability table for the equilibrium points as a function of q_2 for $\mu = 0.5, q_1 = -0.1$

q_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$	Stability
1.0	$-0.7200969 \pm 0.8879650i$	$\pm 1.2083523i$	$0.7200969 \pm 0.8879650i$	U
0.5	$-0.5678338 \pm 0.7926044i$	$\pm 1.1783153i$	$0.5678338 \pm 0.7926044i$	U
0.11	$\pm 1.0344721i$	$\pm 0.9564760i$	$\pm 0.1225610i$	S

Table 6. Stability table for the equilibrium points as a function of q_2 for $\mu = 0.5, q_1 = -0.02$

q_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$	Stability
1.0	$-0.7849378 \pm 0.9339548i$	$\pm 1.2197178i$	$0.7849378 \pm 0.9339548i$	U
0.5	$-0.6441616 \pm 0.8380485i$	$\pm 1.1938332i$	$0.6441616 \pm 0.8380485i$	U
0.03	$\pm 1.0515113i$	$\pm 0.9249563i$	$\pm 0.1969260i$	S

Table 7. Stability table for the equilibrium points as a function of q_2 for $\mu = 0.5, q_1 = -0.015$

q_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$	Stability
1.0	$-0.7904498 \pm 0.9379974i$	$\pm 1.2206325i$	$0.7904498 \pm 0.9379974i$	U
0.5	$-0.6421885 \pm 0.8368136i$	$\pm 1.1934396i$	$0.6421885 \pm 0.8368136i$	U
0.12	$\pm 1.0909069i$	$\pm 0.7739479i$	$\pm 0.4592676i$	S

Table 8. Stability table for the equilibrium points as a function of q_2 for $\mu = 0.5, q_1 = -0.002$

q_2	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$	Stability
1.0	$-0.8093070 \pm 0.9519737i$	$\pm 1.2237024i$	$0.8093070 \pm 0.9519737i$	U
0.5	$-0.3438500 \pm 0.6899045i$	$\pm 1.1333707i$	$0.3438500 \pm 0.6899045i$	U
0.12	$\pm 1.0541033i$	$\pm 0.9192555i$	$\pm 1.0541033i$	S

Analysis of Tables 5, 6, 7 and 8 reveal that there is a limited value of the radiation factors for which the equilibrium points are stable. These results are in agreement with [21].

6 Discussion and conclusion

We studied the photogravitational Copenhagen restricted three-body problem in terms of its three dimensional dynamical properties. We have found that the equations of motion given in the literature allow the existence of out of plane equilibrium points. There are two out of plane equilibrium points that lie in the (x, z) plane in symmetrical positions with respect to the (x, y) plane. The effects of the radiation parameters involved on the positions of the out of plane points are shown in Tables 1—4 (as well as in Figs. 1—4). It is observed that they have influence on the positions of the out of plane equilibrium points. Furthermore, the system parameters is also seen to have significant effects on the topology of the zero velocity curves in the (x, z) plane (Figs.5 and 6). Finally, the stability of these points has been achieved numerically by determining the roots of

the characteristic equation (15). A numerical computation of the roots of (15) obtained in Tables 5—8 and other cases where the point exits show that there is a limited value of the radiation factors for which the equilibrium points are stable. These results are in agreement with [21]. We hope that our investigation and the corresponding outcomes to be useful in connection with planetary system formation, satellite motion, etc.
Finally, out of plane equilibrium points in the photogravitational Copenhagen restricted three-body problem with angular velocity we will present in a future article.

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