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OPTION EVALUATION: BLACK – SCHOLES MODEL VERSUS IMPROVED POISSON MODEL IN OPTION PRICING

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Abstract

The aim of this study is to generate a model by equipping improved Poisson distribution discussed in [1],with some financial terms. And then generate a model for determining the prices of a European call option for n period model that approaches Black –Scholes model in option evaluation. For a non- dividend paying stock, it was found out that an improved Poisson model associated with financial terms approaches or approximates Black –Scholes for $\lambda = \frac{nA}{B}$, provided Δt is sufficiently small.

Keywords: Improved Poisson Black Scholes model, Cox-Ross-Rubinstein model and Option

1.0 Introduction

An option is a contract which gives the buyer right, but not the obligation to buy or sell an underlying asset or instrument at a specific strike price on a specified date, depending on the type of the option. The act of making the transaction is referred to as exercised the option .This study focuses on a particular type of derivative security known as European call option. Egege et al [1,2] recently introduced and show that an improved Poisson distribution can be used to evaluate call and put European option, which gives the same numerical result with CRR binomial for two period models as in [3]. as of the form

$$C = \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \binom{N}{x} e^{-\lambda} \frac{\lambda^x}{N^x} e^{\lambda} \left(\frac{B}{A+B}\right)^N fS(N) \qquad (1)$$

and

$$P = \frac{1}{e^{r\Delta t}} \sum_{x=0}^{N} \binom{N}{x} e^{-\lambda} \frac{\lambda^x}{N^x} e^{\lambda} \left(\frac{B}{A+B}\right)^N fS(N) \qquad (2)$$

for the call and put options with $fS(N)$, the payoff and $\lambda = \frac{nA}{B}$.

1.1 Option Pricing Model

In recent years, determining an option value or forecasting the price of an option has become popular in finances, this is because the financial markets have improved considerably, thus people can invest using various strategies or instrument to either reduce the risk of trading or investing or also maximize profit. It is also known that Black Scholes model,CRR Binomialmodel and Binomial model can play a role in evaluating the value of an option and also to strategized in other to reduce the risk of trading and investing.

1.2 Black Scholes Model

Black Scholes model was introduced by Fischer Black, Myron Scholes and Robert Merton in 1973.It was named after its co-author. This model has served as a breakthrough in the option markets and is widely used today. When the price process is conformed to be continuous ie prices change becomes smaller as time period get shorter Black –Scholes allows us to estimates the value of any option using small number of inputs.

1.3 Black –Scholes Assumption and Parameters

There are several assumptions for the Black-Scholes models

1. Stock pays no dividend
2. Option can only be exercised upon expiration

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3. Markets direction cannot be predicted , hence (Random walk)
4. No commission are charged in the transaction
5. Interest rate remains constant
6. Stock return are normally distributed, thus volatility is constant.

The value of a call option and put option in the Black-Sholes model can be written as a function of five variables.

1. The Current Stock S : This is the predetermined price at which the holder will exercise right.
2. The Time expiration t: This is the time duration or life to expiration of the option.
3. The risk – free interest rate r : which is the rate of investment on stock
4. The volatility of the stock price σ : which measure the uncertainty of movement in the markets

1.4 Black –Sholes Pricing Formula

Black –Sholes formula $(called\ Black - scholes\ merton)$ is a widely used model for option pricing. It is used to calculate the theoretical value of European Style option. Black- Scholes were studied and used by many authors, among the others.Nyustern [4] gave a Black -Scholes model for call and put value of the form

$$C = SN(d_1) - Ke^{-rt}N(d_2) \qquad (3)$$

where

$d_1 = \frac{In\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$and $d_2 = d_1 - \sigma\sqrt{t}$.

Chandra et al [5] gave a formula for black Scholes model for a non- dividend paying stock as

$$C_{(t)} = S_0 \text{Ⓘ} (d_1) - Ke^{-rt} \text{Ⓘ} (d_1), \qquad (4)$$

Where $d_1 = \frac{In\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{t}}$and $d_2 = d_1 - \sigma\sqrt{(T-t)}$.

Oduro et al [6] gave a formula for a Black Scholes model for the price at time 0 of a European call option on a non- dividend paying stock and a European put option on a non- dividend paying stock as

$$C = S_0N(d_1) - Ke^{-rt}N(d_1), \qquad (5)$$

where$d_1 = \frac{In\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$ and $d_2 = d_1 - \sigma\sqrt{(t)}$.

1.5 Assumptions for the Proposed Model

In what follows, we assume the following:

1. The initial values of the stock is S_0
2. At the end of the period , the price is either going up or down by a fixed factor $u = e^{\sigma\sqrt{\Delta t}}$ or go down by a factor $d = e^{-\sigma\sqrt{\Delta t}}$
3. The price of an option is dependent on the following
   a) The strike price K
   b) The expire time T
   c) The risk free rate r
   d) The underlying price S_0
   e) Volatility σ
4. $e^{\sigma\sqrt{\Delta t}} > e^{r\Delta t} > e^{-\sigma\sqrt{\Delta t}} > 0$
5. The stock pays no dividends
6. The continuous compounded interest rate r such that $B(0,T) = e^{r\Delta t}$
7. The length of each period Δt can be positive number
8. From market data for stock price one can estimate the stock price volatility σ per one time unit.$(typically\ one\ year)$
9. Set $n = \frac{t}{\Delta t}$.

2. Method

The tools for giving the result are generalized binomial distribution and financial terms.

The Generalized Binomial distribution in this study was first presented byDwass 1979. It is a discrete distribution that depends on four parameters $\hat{A}$, $\hat{B}$ N and α,where A and B are positive , n is a positive integer and α is an arbitrary real number ,satisfying $(n-1) \leq \hat{A} + \hat{B}$. And in[7,8] theDwass identity is of the form

$$x^{(i)} = x\,(x-\alpha)\ldots\ldots(x-(i-1)\alpha).$$

Let X be the generalized Binomial random variable. Then following [7],its probability function is of the form

$$P_X(x) = \binom{n}{x}\frac{[A(A-\alpha)\ldots.(A)-(x-1)\alpha][B(B-\alpha)..(B-(n-x-1)\alpha)]}{(A+B)(A+B-\alpha)\ldots.(A+B-(n-1)\alpha)}$$

$=\binom{n}{x}\frac{A^{(x)}B^{(n-x)}}{(A+B)^{(n)}}, x = 0, n.$ (6)

$$= \binom{n}{x}\frac{A^{(x)}B^{(n-x)}}{(A+B)^{(n)}}$$

$$= \binom{n}{x}\left(\frac{n\,A/_{A+B}}{n}\right)^{x}\left(\frac{B}{A+B}\right)^{n}\left(\frac{B}{A+B}\right)^{-x}$$

$$= \binom{n}{x}\left(\frac{n\,A/_{A+B}}{n}\right)^{x}\frac{\left(\frac{B}{A+B}\right)^{n}}{\left(\frac{B}{A+B}\right)^{x}}$$

$$= \binom{n}{x}\left(\frac{n\,A/_{A+B}}{n}\right)^{x}\frac{\left(\frac{B}{A+B}\right)^{n}}{\left(\frac{n\,A/_{A+B}}{\lambda}\right)^{x}}$$

$$= \binom{n}{x}\left(\frac{n^{x}\left(\frac{A}{A+B}\right)^{x}}{n^{x}}\right)\left(\frac{B}{A+B}\right)^{n}\times\frac{\lambda^{x}}{n^{x}\left(\frac{A}{A+B}\right)^{x}}.$$

The improved Poisson distribution applied in finances can be expressed a

$P_X(x) = \ n_{c_x}\frac{e^{-\lambda}\lambda^{x}e^{\lambda}}{n^{x}}\left(\frac{B}{A+B}\right)^{n},$ (7)

with mean and variance

$E(x) = \lambda e^{\lambda}\left(\frac{B}{A+B}\right)^{n}$ and $Var(x) = e^{\lambda}\left(\frac{B}{A+B}\right)^{n}\left[\lambda^{2}+\lambda-\lambda^{2}e^{\lambda}\left(\frac{B}{A+B}\right)^{n}\right].$

Equation (7) can be combine with financial terms to determine the call and put price of an option, provided the following is satisfy

I. $\sum_{x=0}^{n} n_{C_x}\frac{e^{-\lambda}\lambda^{x}e^{\lambda}}{N^{x}}\left(\frac{B}{A+B}\right)^{n} = 1$

II. $\left(\frac{B}{A+B}\right)^{N} > 0$

III. $1-\frac{B}{A+B} = \frac{A}{A+B}$

IV. $n > 0$

V. $\frac{A}{A+B} = \frac{A}{A+B}$ and $1-\frac{A}{A+B} = 1-\frac{A}{A+B}$

Lemma 1 : let $\frac{A}{A+B} = \frac{1+r\Delta t-d}{u-d}, e^{r\Delta t} \cong (1-r\Delta t)$ and $e^{r\Delta t} \cong (1+r\Delta t)$,then the following holds

1.$E\left[S_{\frac{A}{A+B}}\right] = S_t e^{r\Delta t}$

2.$u = e^{\sigma\sqrt{\Delta t}}$

Proof (1)

$$E\left[S_{\frac{A}{A+B}}\right] = \left[\frac{A}{A+B}S_t u + \left(1-\frac{A}{A+B}\right)S_t d\right] = \left(\left[\frac{1+r\Delta t-d}{u-d}\right]S_t u + \left[\frac{u-1+r\Delta t}{u-d}\right]S_t d\right)$$

$$\left[S_t\left[\frac{1+r\Delta t-d}{u-d}\right]u + \left[\frac{u-1+r\Delta t}{u-d}\right]d\right] = S_t\left[\frac{(1+r\Delta t)u-(1+r\Delta t)d}{u-d}\right] = S_t(1+r\Delta t)\left[\frac{u-d}{u-d}\right]$$

$$= S_t e^{r\Delta t}$$

Proof (2)

Given $Var\left[S_{\frac{A}{A+B}}\right] = E\left[S^{2}_{\frac{A}{A+B}}\right] - E\left[S_{\frac{A}{A+B}}\right]^{2}$ by lognormal property [1], we obtained

$Var\left[S_{\frac{A}{A+B}}\right] = S^{2}{}_{t}\sigma^{2}\Delta t$. (8)

$$S^{2}{}_{t}\sigma^{2}\Delta t = E\left[S^{2}_{\frac{A}{A+B}}\right] - E\left[S_{\frac{A}{A+B}}\right]^{2}$$

$$=\left[\frac{A}{A+B}S^{2}{}_{t}u^{2} + 1-\frac{A}{A+B}S^{2}{}_{t}d^{2}\right] - S^{2}{}_{t}\left[\left(\frac{A}{A+B}\right)^{2}u^{2} + 2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right)ud + \left(1-\frac{A}{A+B}\right)^{2}d^{2}\right].$$

Multiplying both side by $\frac{1}{S^{2}{}_{t}}$ given

$\sigma^2\Delta t = \left[\frac{A}{A+B}u^2 + \left(1-\frac{A}{A+B}\right)d^2\right] - \left[\left(\frac{A}{A+B}\right)^2 u^2 - 2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right)ud - \left(1-\frac{A}{A+B}\right)^2 d^2\right].$

Collecting like terms

$$\sigma^2\Delta t = \left(\frac{A}{A+B} - \left(\frac{A}{A+B}\right)^2\right)u^2 + \left(\left(1-\frac{A}{A+B}\right) - \left(1-\frac{A}{A+B}\right)^2\right)d^2 - 2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right)ud$$

$\sigma^2\Delta t = u^2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right) + \left(1-\frac{A}{A+B}\right)\left[1-\left(1-\frac{A}{A+B}\right)\right]d^2 - 2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right)ud$

$\sigma^2\Delta t = \frac{A}{A+B}\left(1-\frac{A}{A+B}\right)[u^2 - 2ud + d^2].$ (9)

$$\frac{A}{A+B}\left(1-\frac{A}{A+B}\right) = \frac{A}{A+B} - \left(\frac{A}{A+B}\right)^2 = \frac{1+r\Delta t-d}{u-d} - \left[\frac{1+r\Delta t-d}{u-d}\right]^2$$

$= \frac{1+r\Delta t-d}{u-d} - \frac{(1+r\Delta t)^2 - 2(1+r\Delta t)d + d^2}{(u-d)^2} = \frac{(1+r\Delta t)u - ud - (1+r\Delta t)d + d^2 - (1+r\Delta t)^2 + 2(1+r\Delta t)d - d^2}{(u-d)^2}.$

By definition

$$\frac{e^{r\Delta t}u - ud - e^{r\Delta t}d + d^2 - e^{2r\Delta t} + 2e^{r\Delta t}d - d^2}{(u-d)^2} = \frac{e^{r\Delta t}(u-d+2d) - ud - e^{2r\Delta t}}{(u-d)^2}$$

$$\frac{A}{A+B}\left(1-\frac{A}{A+B}\right) = \frac{e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}}{(u-d)^2}$$

Implies $\sigma^2\Delta t = \frac{e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}}{(u-d)^2}[u^2 - 2ud + d^2]$

$$\sigma^2\Delta t = \frac{e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}}{(u-d)^2}(u-d)^2$$

$\sigma^2\Delta t = e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}$

And letting $d = \frac{1}{u}$ and $e^{r\Delta t} \cong (1 - r\Delta t)$ and $e^{r\Delta t} \cong (1 + r\Delta t)$

$\sigma^2\Delta t = e^{r\Delta t}\left(u + \frac{1}{u}\right) - 1 - e^{2r\Delta t} = \sigma^2\Delta t + e^{2r\Delta t} + 1 = e^{r\Delta t}\left(u + \frac{1}{u}\right)$

$$u + \frac{1}{u} = \frac{\sigma^2\Delta t + e^{2r\Delta t} + 1}{e^{r\Delta t}} = e^{-r\Delta t}(\sigma^2\Delta t + e^{2r\Delta t}) + 1 = e^{-r\Delta t}\sigma^2\Delta t + e^{r\Delta t} + e^{r\Delta t}$$

$\frac{u^2+1}{u} = (1 - r\Delta t)\sigma^2\Delta t + (1 + r\Delta t) + (1 - r\Delta t)$

$= (1 - r\Delta t)\sigma^2\Delta t + 2$

$\frac{u^2+1}{u} = \sigma^2\Delta t + 2 - r\sigma^2(\Delta t)^2$ As $r\sigma^2(\Delta t)^2 \to 0$

$$\frac{u^2+1}{u} = \sigma^2\Delta t + 2$$

$u^2 + 1 = (\sigma^2\Delta t + 2)u$

$u^2 = (\sigma^2\Delta t + 2)u - 1$

$u^2 - (\sigma^2\Delta t + 2)u + 1 = 0by\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Where $a = 1, b = -\sigma^2\Delta t + 2, c = 1$

$u = \frac{\sigma^2\Delta t + 2 \pm \sqrt{(\sigma^2\Delta t + 2)^2 - 4}}{2} = \frac{\sigma^2\Delta t + 2\sqrt{\sigma^4\Delta t^2 + 4\sigma^2\Delta t + 4 - 4}}{2}$

Ignoring the higher power of $\sigma^4\Delta t^2 \to 0$

$$u = \frac{\sigma^{2\Delta t}}{2} + 1 \pm \frac{2\sigma\Delta t}{2} = \frac{\sigma^2\Delta t}{2} + 1 \pm \sigma\Delta t$$

Since σ^2 and Δt is always sufficiently small, it can it be ignored $\frac{\sigma^2\Delta t}{2}$

$u \cong 1 \pm \sigma\Delta t = e^{\sigma\sqrt{\Delta t}}$

Lemma 3: if $u > 1 + r\Delta t > d$ and $e^{-rn\Delta t} = \frac{1}{(1+r\Delta)^n}$and no arbitrary principle exist, then the following hold

1.$E_{\frac{A}{A+B}}S_{(2)} = e^{2r\Delta t}S_{(2)}$

2. $\sum_{x=1}^{3}\left[\frac{1+r\Delta t-d}{u-d}\right] = 1$

Proof (1)

For t=2 we define $\left(\frac{A}{A+B}\right)_1 = \left(\frac{A}{A+B}\right)^2, \left(\frac{A}{A+B}\right)_2 = 2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right)$ and $\left(\frac{A}{A+B}\right)_3 = \left(1-\frac{A}{A+B}\right)^2$

Then $E_{\frac{A}{A+\breve{B}}} = \left[\left(\frac{A}{A+B}\right)^2 S(2) + 2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right)S(2) + \left(1-\frac{A}{A+B}\right)S(2)\right] = \left(\frac{A}{A+B}\right)^2 S_{uu} + 2\frac{A}{A+A}\left(1-\frac{A}{A+B}\right)S_{ud} + \left(1-\frac{A}{A+B}\right)^2 S_{dd} = S\left[\left(\frac{A}{A+B}\right)^2 u^2 + 2\frac{A}{A+B}\left(1-\frac{A}{A+B}\right)du + \left(1-\frac{A}{A+B}\right)^2 d^2\right] = C\ \left[\frac{A}{A+B}u + \left(1-\frac{A}{A+B}\right)d\right]^2$

$= S\left[\frac{1+r\Delta t-d}{u-d}u + \left(1-\frac{u-1+r\Delta t}{u-d}\right)d\right]^2$

$= S\left[\frac{1+r\Delta tu-du+du-1+r\Delta td}{u-d}\right]^2 = S(1+r\Delta t)^2\left[\frac{u-d}{u-d}\right]^2$ by lemma 1

$E_{\frac{A}{A+B}}S_{(2)} = e^{2r\Delta t}S_{(2)}$

Proof (2)

Let $\frac{A}{A+B_1} = \left(\frac{1+r\Delta t-d}{u-d}\right)^2, \frac{A}{A+B_2} = 2\left(\frac{1+r\Delta t-d}{u-d}\right)\left(1-\frac{1+r\Delta t-d}{u-d}\right), \frac{A}{A+B_3} = \left(1-\frac{1+r\Delta t+d}{u-d}\right)^2$

$\sum_{x=1}^{3}\left[\frac{A}{A+B_1} + \frac{A}{A+B_2} + \frac{A}{A+B_3}\right] = \left(\frac{1+r\Delta t-d}{u-d}\right)^2 + 2\left(\frac{1+r\Delta t-d}{u-d}\right)\left(1-\frac{1+r\Delta t-d}{u-d}\right) + \left(1-\frac{1+r\Delta t+d}{u-d}\right)^2$

$$= \left(\frac{1+r\Delta t-d}{u-d}\right)^2 + 2\left(\frac{1+r\Delta t-d}{u-d}\right) - 2\left(\frac{1+r\Delta t}{u-d}\right)^2 + 1 - 2\left(\frac{1+r\Delta t-d}{u-d}\right) + \left(\frac{1+r\Delta t}{u-d}\right)^2$$

$= \left[\left(\frac{1+r\Delta t-d}{u-d}\right)^2 - \left(\frac{1+r\Delta t-d}{u-d}\right)^2 + 1\right] = 1$ Therefore

$$\sum_{x=1}^{3}\left[\frac{1+r\Delta t-d}{u-d}\right] = 1$$

Theorem 1. For $\lambda = \frac{nA}{B}$ t $\in [0\ n]$ let C be the cost of European call option, for $n \in \mathbb{N}$ and $E_{\frac{A}{A+B}}S_{(2)} = e^{2r\Delta t}C$, there exist $C = \frac{1}{e^{nr\Delta t}}\sum_{x=0}^{n}\frac{A^x B^{n-x}}{(A+B)^n}C_n(x)$ implies $C = \frac{1}{e^{nr\Delta t}}\sum_{x=0}^{n}\binom{n}{x}\frac{e^{-\lambda}\lambda^x e^{\lambda}}{n^x}\left(\frac{B}{A+B}\right)^n max[u^x d^{n-x}S-K,0]$

Where $x = 0,1 \ldots n$

Proof

If $E_{\frac{A}{A+B}}S_{(2)} = e^{2r\Delta t}C$ then $C = e^{2r\Delta t}E_{\frac{A}{A+B}}S_{(2)}$

$$C = e^{-2r\Delta t}\left[\left[\left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 f_{uu}S(2) + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)f_{ud}S(2) + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)f_{dd}S(2)\right]\right]$$

$$= \left(\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{uu} + 2\frac{\breve{A}}{\breve{A}+\breve{B}}\left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)C_{ud} + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)^2 C_{dd}$$

$C = e^{-2r\Delta t}\left[\frac{\breve{A}}{\breve{A}+\breve{B}}u + \left(1-\frac{\breve{A}}{\breve{A}+\breve{B}}\right)d\right]^2 f\big(S(2)\big)$by binomial expansion

$$C = e^{-2r\Delta t}\sum_{x=0}^{2}\binom{2}{x}\frac{A^x B^{n-2}}{(A+B)^N}f\big(S(2)\big) \qquad (10)$$

In general

$$C = \frac{1}{e^{r\Delta tn}}\sum_{x=0}^{n}\binom{n}{x}\frac{A^x B^{n-x}}{(A+B)^{(n-x)}}Max\left[u^x d^{n-x}S_{(0)}-K,0\right]$$

Where $n \in \mathbb{N}$and $\in \mathbb{N}$, and for $\lambda = \frac{nA}{B}$

$$= \frac{1}{e^{\,nr\Delta t}}\sum_{x=0}^{n}\left[\binom{n}{x}\left(\frac{n\,{}^{A}\!/_{A+B}}{n}\right)^x\left(\frac{A}{A+B}\right)^n\left(\frac{B}{A+B}\right)^{-x}\right]Max\left[u^x d^{n-x}S_{(0)}-K,0\right]$$

$$= \frac{1}{e^{nr\Delta t}}\sum_{x=0}^{n}\left[\binom{n}{x}\left(\frac{n\,{}^{A}\!/_{A+B}}{n}\right)^x\frac{\left(\frac{A}{A+B}\right)^n}{\left(\frac{B}{A+B}\right)^x}\right]Max\left[u^x d^{n-x}S_{(0)}-K,0\right]$$

$$= \frac{1}{e^{nr\Delta t}}\sum_{x=0}^{N}\left[\binom{n}{x}\left(\frac{n\,{}^{A}\!/_{\breve{A}+\breve{B}}}{n}\right)^x\frac{\left(\frac{B}{A+B}\right)^n}{\left(\frac{n\,{}^{A}\!/_{A+B}}{\lambda}\right)^x}\right]Max\left[u^x d^{n-x}S_{(0)}-K,0\right]$$

$$= \frac{1}{e^{nr\,\Delta t}} \sum_{x=0}^{n} \left[\binom{n}{x} \left(\frac{n^x \left(\frac{A}{A+B}\right)^x}{n^{x}0} \right) \left(\frac{B}{A+B}\right)^n \times \frac{\lambda^x}{n^x \left(\frac{A}{A+B}\right)^x} \right] Max\left[u^x d^{n-x} S_{(0)} - K, 0\right]$$

$$= \frac{1}{e^{nr\,\Delta t}} \sum_{x=0}^{n} \left[\binom{n}{x} \lambda^{x_0} \frac{n^x}{n^x} \left(\frac{\frac{A}{A+B}}{N \frac{A}{A+B}} \right)^x \left(\frac{B}{A+B}\right)^n \right] Max\left[u^x d^{n-x} S_{(0)} - K, 0\right]$$

$$= \frac{1}{e^{nr\,\Delta t}} \sum_{x=0}^{n} \left[\frac{n(n-1)(n-2)\ldots..(n-x+1)(n-x)..(2)(1)}{x(x-1)(x-2)\ldots..(2)(1).(n-x)..(2)(1)} \frac{\lambda^x e^{-\lambda}}{n^x} \left(\frac{B}{A+B}\right)^n e^{\lambda} \right] Max\left[u^x d^{n-x} S_{(0)} - K, 0\right]$$

$$= \frac{1}{e^{nr\,\Delta t}} \sum_{x=0}^{n} \left[\frac{n(n-1)(n-2)\ldots..(n-x+1)}{N^{x_0}} \frac{e^{-\lambda}\lambda^x}{x!} \left(\frac{\breve{B}}{\breve{A}+\breve{B}}\right)^n e^{\lambda} \right] Max\left[u^x d^{n-x} S_{(0)} - K, 0\right]$$

$$C = \frac{1}{e^{nr\,\Delta t}} \sum_{x=0}^{n} \binom{n}{x} \frac{e^{-\lambda}\lambda^x}{n^x} \left(\frac{B}{A+B}\right)^n e^{\lambda} \left[Max\left[u^x d^{n-x} S_{(0)} - K, 0\right] \right] \qquad (11)$$

3. Numerical Result

The following numerical results shows that improved Poisson model approaches Black –Scholes model.

Example 1. Let the annualized variance of logarithmic returns be $\sigma^2 = 0.1$, the interest rate is set to be $r = 0.1$ per annum .Suppose that the current stock price is $S_0 = \$50$ with strike price $k = \$53$ at maturity $T = 4$months. Assuming the length of each period$\Delta t = \frac{1}{12}$. Then Poisson model approaches Black –Scholes model.

Solution

$u = 1.0956\,, d = 0.9128, \frac{A}{A+B}\ 0.5224\ and\ \frac{B}{A+B} = 0.4776$

By Black –Scholes Model

$C = SN(d_1) - Ke^{-rt}N(d_2)$

$$d_1 = \frac{In\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = \frac{In\left(\frac{50}{53}\right) + \left(0.1 + \frac{0.1^2}{2}\right)0.3333}{0.3162\sqrt{0.3333}} = -0.045$$

And $d_2 = d_1 - \sigma\sqrt{(T)} = -0.045 - 0.1825 = -0.2275$

$N(d_1) = 0.4081$ and$N(d_2) = 0.4090$

$C = SN(d_1) - Ke^{-rt}N(d_2) = 50 \times 0.04081 - 0.9672 \times 0.4090 \times 53 = 24.0050 - 20.9660 = 3.0390 \approx \$3.0$

Improved Poisson Model

$$C = \frac{1}{e^{nr\,\Delta t}} \sum_{x=0}^{n} \binom{n}{x} \frac{e^{-\lambda}\lambda^x}{n^x} \left(\frac{B}{A+B}\right)^n e^{\lambda} \left[Max\left[u^x d^{n-x} S_{(0)} - K, 0\right] \right]$$

Given $\frac{A}{A+B}\ 0.5224\,, \frac{B}{A+B} = 0.4776\boldsymbol{\lambda} = 4.3752$ with the following pay of values

$C_{uuuu}\ 19.0407, C_{uuud}\ 7.0208, C_{uudd} = 0, C_{uddd} = 0$ and $C_{dddd} = 0$

$C = e^{-0.1\times4\times0.0833}\left[\binom{4}{0}e^{-4.3752} \times \frac{(4.3752)^0}{(4)^0} \times e^{4.3752} \times (0.4476)^4 \times 0 + \binom{4}{1}e^{-4.3752} \times \frac{(4.3752)^1}{(4)^1} e^{4.3752\times}(0.4476)^4 \times 0 + \right.$

42× e– 4.3752× 4.3752242× e4.3752× 0.44764× 0+43× e– 4.3752× 4.3752343× e4.3752× 0.44764× 7.0208× 44× e– 4.37 52× 4.3752444× e4.3752× 0.47764× 19.0407=e– 0.1× 4× 0.08331.9121+1.4181=3.2210 ≈*\$3.2*

Example 2: Assuming on March6,2001 CISCO system was trading at \$13.62 with a strike price of \$15 on July 2001. If the standard deviation for the stock is 1.556% weekly over the year and the annualized treasury bill rate corresponding to this option life is 4.63%. Then Poisson model approaches Black –Scholes model.

Solution

Black-School Model

$C = SN(d_1) - Ke^{-rt}N(d_2)$

$S = \$13.62, k = \$15.00, t = \frac{103}{365}, \sigma = 1.556\% \times 52 = 0.81, r = 0.0463$

$$d_1 = \frac{In\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = \frac{In\left(\frac{13.62}{15.00}\right) + \left(0.0463 + \frac{(0.82)^2}{2}\right)(0.2822)}{0.81\sqrt{0.2822}} = 0.0212$$

And $d_2 = d_1 - \sigma\sqrt{(T)} = 0.0212 - 0.81\sqrt{0.2822}$ = -0.4091
$N(d_1) = 0.5085$ And$N(d_2)$ = 0.3412
$C = SN(d_1) - Ke^{-rt}N(d_2) = 50(0.5085) - 15e^{-(0.0463)(0.2822)}(0.3412) =$ \$1.87.
Improve Poisson Model
Given the following pay off values
C_{uuu} 12.4663, C_{uud} 2.2072,$C_{udd} = 0$ and, $C_{ddd} = 0$
With λ=2.4555, $\frac{A}{A+B} = 0.4501, \frac{B}{A+B} = 0.5499, u = 1.2634$ *and* $d = 0.7915$

$$C = e^{-0.0463\times3\times0.0833}\left[\binom{3}{0}\times e^{-2.4555}\times\frac{(2.4555)^0}{(3)^0}\times e^{2.4555}\times(0.5499)^3\times 0+\binom{3}{1}\times e^{-2.4555}\times\frac{(2.4555)^1}{(3)^1}\times e^{2.4555}\right.$$
$$\times(0.5499)^3\times 0+\binom{3}{2}\times e^{-2.4555}\times\frac{(2.4555)^2}{(3)^2}\times e^{2.4555}\times(0.5499)^3\times 2.2072+\binom{3}{3}\times e^{-2.4555}$$
$$\left.\times\frac{(2.4555)^3}{(3)^3}\times e^{2.4555}\times(0.5499)^3\times 12.4663\right] = \$1.85$$

Example 3 Given $S = \$100, K = \$80, T = 0.33, R = 0.05$and $\sigma = 0.24, \Delta t = 0.0833$. Then the Poisson model are Black –Scholes model are;

Solution
Black Scholes Model
$C = SN(d_1) - Ke^{-rt}N(d_2)$=\$21.54
Improved Poisson Model

$$C = \frac{1}{e^{nr\Delta t}}\sum_{x=0}^{n}\binom{n}{x}\frac{e^{-\lambda}\lambda^x}{n^x}\left(\frac{B}{A+B}\right)^n e^{\lambda}\left[Max\left[u^x d^{n-x}S_{(0)} - K, 0\right]\right]$$

With the following payoff values
C_{uuuu} 51.9146, C_{uuud} 34.8545,$C_{uudd} = 20.0007, C_{uddd} = 7.0678$ and $C_{dddd} = 0$ and $\lambda = 4.2136$
We obtained
$C =$ \$21.56
Example 4 Let $S = 51$, $K = 50$ $T = 0.25, \sigma = 0.3$, $R = 0.08, \Delta t = 0.0833$. Then the Poisson model are Black –Scholes model are;

Solution
$C = SN(d_1) - Ke^{-rt}N(d_2)$
$C =$ \$4.1
Improved Poisson Model
With the following payoff values
C_{uuu} 16.1374, C_{udd} ,$C_{udd} = 5.6149$, *and* $C_{ddd} = 0$ and $\lambda = 4.2136$
$C =$ \$4.3

4. Discussion

Egege et al [1] recently introduced and show that an improved Poisson distribution can be used to evaluate call and put European option , which gives the same numerical result with CRR binomial for two period model. And it is a known fact that CRR binomial approaches Black –Scholes. This study has shown how well the improved Poisson distribution will approach or approximate Black –Scholes.
From example 1-4 ,we have shown that the difference in the calculation by two model of option price are very small, for Δt sufficiently small , thus this implies that improved Poisson distribution approachesBlack -Scholes there by giving a close value with Black –Scholes.

5. Conclusion

The improved Poisson distribution was first discussed and applied in finance in[1] shows that CRR binomial model gives the same numerical result with improved Poisson distribution model. it is also known that CRR is set to approach Black Scholes. Now the extension in this work shows that improved Poisson distribution model approximates Black – Scholes model.

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