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FREE VIBRATIONAL ANALYSIS OF BEAM WITH DIFFERENT SUPPORT CONDITIONS

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Abstract

This paper presents an analysis of free vibrations of a cantilever beam and simply supported beam using series solution. It was found that the Deflection of beam increases as the length of the beam increases for a cantilever beam but decreases for the case of a simply supported beam. The response amplitude of a cantilever beam is greater than that of a simply supported beam.

1. Introduction

Vibration occurs when a system is displaced from a position of stable equilibrium. The system tends to return to this equilibrium position under the action of restoring forces (such as the elastic forces, as for a mass attached to a spring, or gravitational forces, as for a simple pendulum). The system keeps moving back and forth across its position of equilibrium. A system is a combination of elements intended to act together to accomplish an objective. For example, an automobile is a system whose elements are the wheels, suspension, car body, and so forth [1].

The vibration analysis for structures is a very important field in engineering and computational mechanics. These dynamic problems are classically described by a partial differential equation associated with a set of boundary conditions. The analysis of free vibration of beam has been a topic of interest for well over a century [2-5].

Free vibration equation of the beam on partially elastic foundation including only bending moment effect was analytically solved [6-7] while the eigenvalues for free vibration of beam-column systems on elastic foundation were obtained using a numerical approach [8]. The separation of variables technique was used to obtain the free vibration circular frequencies of piles partially embedded in soils [9-10]. In addition, differential transform method (DTM) has been proposed to solve eigenvalue problems for free and transverse vibration problems of a rotating twisted Timoshenko beam under axial loading [11-13]. Furthermore, the DTM was also used to find the nondimensional natural frequencies of tapered cantilever Bernoulli-Euler beam [14-15]. Free vibration equations for one end clamped and other end simply supported beam on elastic foundation were solved by using the DTM for various axial loads acting on the beam [16]. Meanwhile, both the variational iteration method (VIM) and homotopy perturbation method (HPM) were used to solve the free vibration equations of beam on elastic foundation for support conditions of one end clamped, and other end simply supported, both ends clamped and both ends simply supported considering various case studies [17-18]. The beam on elastic foundation was investigated for these three different support conditions considering various the values of the ratio of axial load acting on the beam to Euler buckling load.

Recently, there have been also other studies which are helpful to better understand dynamic behavior of both infinite beams resting on elastic foundation [19] and tapered column with pinned ends embedded in Winkler-Pasternak elastic foundation. In this study, comprehensive analysis of free vibration of Euler-Bernoulli beam, using practical technique for determining the response of beams with different boundary conditions. The analytical results are in excellent agreement with the results of the particular problem solved using the VIM method and the HPM method available in the literature [20]

2. Mathematical formulation

Considering the free vibration of a beam of finite length L , the differential equation for the free vibration of a beam when the beam is of constant flexural rigidity EI is given as

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+\rho A\frac{\partial^2 W(x,t)}{\partial t^2}=0 \qquad (1)$$

Where:

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$E =$ Coefficent of elasticity

$I =$ is the moment of inertia of the cross section

$\rho =$ Density of the mass

$A =$ Surface area of the beam cross section

$t =$ Time coordinate

$x =$ Spatial coordinate

$\rho A =$ Mass per unit length

$W(x,t) =$ is the deflection of the beam

The initial conditions are:

$$W(x,0) = w_0(x) \quad (2)$$

$$\frac{\partial W(x,0)}{\partial t} = v_0(x) = 0 \quad (3)$$

2.1 Method of Solution

The above equation is a homogeneous equation. To solve the equation of motion above, we use the separation of variable method.

We first assume a trial solution of the form

$W(x,t) = F(x)G(t)$

where

$F(x)$ is a function of x only

$G(t)$ is a function of t only

We have $W = FG$

$$\frac{\partial W}{\partial x} = F^{'}G, \quad \frac{\partial^2 W}{\partial x^2} = F^{''}G \quad \frac{\partial^3 W}{\partial x^3} = F^{'''}G \quad \frac{\partial^4 W}{\partial x^4} = F^{iv}G$$

$$\frac{\partial W}{\partial t} = FG^{'} \qquad \frac{\partial^2 W}{\partial t^2} = FG^{''}$$

Let $\gamma^2 = \frac{EI}{\rho A}$ The above equation of motion can be written as

$$F^{iv}G + \frac{1}{\gamma^2}FG^{''} = 0$$

$$F^{iv}G = -\frac{1}{\gamma^2}FG^{''}$$

which can be transposed into $\frac{F^{iv}}{F} = -\frac{1}{\gamma^2}\frac{G^{''}}{G}$ Denoting the arbitrary constant by K, then we have

$$\frac{F^{iv}}{F} = K \quad (4)$$

and

$$-\frac{1}{\gamma^2}\frac{G^{''}}{G} = K \quad (5)$$

2.2 Solution of the Time Domain Function

By letting $K = \lambda^4$, we have equation (5) to be

$$G^{''} + \gamma^2\lambda^4 G = 0 \quad (6)$$

The auxillary equation is therefore,

$$m^2 + \gamma^2\lambda^4 = 0$$

so that

$$m = \pm i\gamma\lambda^2$$

So, the general solution to equation (6) is

$$G_n(t) = I_n \cos\gamma\lambda_n^2 t + J_n \sin\gamma\lambda_n^2 t \quad (7)$$

Since

$$W_n(x,t) = F_n(x)\cdot G_n(t)$$

Substituting (7) into the above equation we have,

$$W_n(x,t) = F_n(x)\left(I_n \cos \gamma\lambda_n^2 t + J_n \sin \gamma\lambda_n^2 t\right) \quad (8)$$

Differentiating (8) with respect to t, we have

$$\frac{\partial W_n(x,t)}{\partial t} = F_n(x)\cdot \gamma\lambda^2\left(-I_n \sin \gamma\lambda_n^2 t + J_n \cos \gamma\lambda_n^2 t\right) \quad (9)$$

Substituting the expression for W into initial conditions of (2)

$$W_n(x,0) = w_0(x) = F_n(x)\cdot I_n \quad (10)$$

$$\frac{\partial W_n(x,0)}{\partial t} = 0 = F_n(x)\cdot J_n$$

which implies that $J_n = 0$, hence we have

$$W(x,t) = \sum_{n=1}^{\infty} W_n(x,t) = \sum_{n=1}^{\infty} F_n(x)\left(I_n \cos \gamma\lambda^2 t\right)$$

Using the Fourier series technique to determine the coefficient I_n. From (10)

$$w_0(x) = F_n(x)\cdot I_n$$

$$I_n = \frac{2}{L}\int_0^L w_0(x)F_n(x)dx$$

$$W(x,t) = \sum_{n=1}^{\infty} F_n(x)\left(\frac{2}{L}\int_0^L w_0(x)F_n(x)dx\left(\cos\gamma\lambda^2 t\right)\right)$$

$$W(x,t) = \sum_{n=1}^{\infty} F_n(x)\cos\gamma\lambda^2 t\left(\frac{2}{L}\int_0^L w_0(x)F_n(x)dx\right) \quad (11)$$

such that the time domain function is

$$G(t) = \sum_{n=1}^{\infty} \cos\gamma\lambda^2 t\left(\frac{2}{L}\int_0^L w_0(x)F_n(x)dx\right)$$

2.3 Solution of the Spatial Function

Here, we shall consider the solution of the spatial function and then apply the boundary conditions considering two cases of beam, the simply supported beam and the cantilever beam. From equation (4), with $K = \lambda^4$

$$F^{iv} = \lambda^4 F$$

$$F^{iv} - \lambda^4 F = 0$$

The auxilliary equation is therefore,

$$m^4 - \lambda^4 = 0$$

So that

$$m = \pm\lambda \text{ or } m = \pm i\lambda$$

The general solution now becomes

$$F_n(x) = A_n \sin\lambda_n x + B_n \cos\lambda_n x + C_n \sinh\lambda_n x + D_n \cosh\lambda_n x \quad (12)$$

Equation (11) now becomes

$$W(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\lambda_n x + B_n \cos\lambda_n x + C_n \sinh\lambda_n x + D_n \cosh\lambda_n x\right)\cos\gamma\lambda^2 t \quad (13)$$

$$\left(\frac{2}{L}\int_0^L w_0(x)\left(A_n \sin\lambda_n x + B_n \cos\lambda_n x + C_n \sinh lambda_n x + D_n \cosh\lambda_n x\right)dx\right)$$

$$F_n(t) = A_n \sin\lambda_n x + B_n \cos\lambda_n x + C_n \sinh\lambda_n x + D_n \cosh\lambda_n x$$

$$\frac{dF_n(x)}{dx} = \lambda\left(A_n \cos\lambda_n x - B_n \sin\lambda_n x + C_n \cosh\lambda_n x + D_n \sinh\lambda_n x\right)$$

$$\frac{d^2F_n(x)}{dx^2} = \lambda^2\left(-A_n \sin\lambda_n x - B_n \cos\lambda_n x + C_n \sinh\lambda_n x + D_n \cosh\lambda_n x\right)$$

$$\frac{d^3F_n(x)}{dx^3} = \lambda^3\left(-A_n \cos\lambda_n x + B_n \sin\lambda_n x + C_n \cosh\lambda_n x + D_n \sinh\lambda_n x\right)$$

2.3.1 Simply Supported Beam

For a simply supported beam, we have that the displacement and the bending moments ar zero at both ends, this translate into the following boundary conditions

$$W(0,t) = 0 = W(L,t) \quad (14)$$

$$\frac{\partial^2 W(0,t)}{\partial x^2} = 0 = \frac{\partial^2 W(L,t)}{\partial x^2} \quad (15)$$

Substituting the expression W in the boundary conditions

$$F(0) = 0 = B + D$$

$$\frac{d^2F(0)}{dx^2} = 0 = -B + D$$

$$F(L) = 0 = A\sin\lambda L + B\cos\lambda L + C\sinh\lambda L + D\cosh\lambda L \quad (16)$$

$$\frac{d^2F(L)}{dx^2} = 0 = -A\sin\lambda L - B\cos\lambda L + C\sinh\lambda L + D\cosh\lambda L \quad (17)$$

From the above equation it implies that $B = D = 0$ so that equations (16) and (17) becomes

$$A\sin\lambda L + C\sinh\lambda L = 0 \quad (18)$$

$$-A\sin\lambda L + C\sinh\lambda L = 0 \quad (19)$$

Subtracting equation (19) from (19)

$$-2A\sin\lambda L = 0$$

$$A \neq 0$$

$$\sin\lambda L = \sin n\pi$$

$$\lambda = \frac{n\pi}{L}$$

Therefore,

$$F_n(x) = \sin\frac{n\pi}{L}x \quad (20)$$

Equation (11) now becomes

$$W(x,t) = \sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{L}x\right)\cos\gamma\lambda^2 t\left(\frac{2}{L}\int_0^L w_0(x)\left(\sin\frac{n\pi}{L}x\right)dx\right) \quad (21)$$

Equation (21) is the free vibration of a simply supported beam.

2.3.2 Cantilever Beam

For a cantilever beam, we have that the displacement and slope are zero at the free end. While at the other end (the fixed end), bending moment and shear force is zero which translate into the following boundary conditions

$$W(0,t) = 0 = \frac{\partial W(0,t)}{\partial x} \quad (22)$$

$$\frac{\partial^2 W(L,t)}{\partial x^2} = 0 = \frac{\partial^3 W(L,t)}{\partial x^3} \quad (23)$$

Substituting the expression W in the boundary conditions

$$F(0) = 0 = B + D \quad (24)$$

$$\frac{dF(0)}{dx} = 0 = A + C \quad (25)$$

$$\frac{d^2F(L)}{dx^2} = 0 = -A\sin\lambda L - B\cos\lambda L + C\sinh\lambda L + D\cosh\lambda L \quad (26)$$

$$\frac{d^3F(L)}{dx^3} = 0 = -A\cos\lambda L + B\sin\lambda L + C\cosh\lambda L + D\sinh\lambda L \quad (27)$$

From (24) $B + D = 0$ which implies that $D = -B$

From (25) $A + C = 0$ which implies that $C = -A$

Substituting $D = -B$ and $C = -A$ into equations (26) and (27)

$$-A\sin\lambda L - B\cos\lambda L - A\sinh\lambda L - B\cosh\lambda L = 0$$

$$-A(\sin\lambda L + \sinh\lambda L) - B(\cos\lambda L + \cosh\lambda L) = 0 \quad (28)$$

$$-A\cos\lambda L + B\sin\lambda L - A\cosh\lambda L - B\sinh\lambda L = 0$$

$$-A(\cos\lambda L + \cosh\lambda L) + B(\sin\lambda L - \sinh\lambda L) = 0 \quad (29)$$

Arranging equations (28) and (29) in matrix form

$$\begin{bmatrix} -(\sin\lambda L + \sinh\lambda L) & -(\cos\lambda L + \cosh\lambda L) \\ -(\cos\lambda L + \cosh\lambda L) & (\sin\lambda L + \sinh\lambda L) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Finding the determinant of M where

$$M = \begin{bmatrix} -(\sin\lambda L + \sinh\lambda L) & -(\cos\lambda L + \cosh\lambda L) \\ -(\cos\lambda L + \cosh\lambda L) & (\sin\lambda L + \sinh\lambda L) \end{bmatrix}$$

We have

$$1+\cos\lambda L\cosh\lambda L=0 \qquad (30)$$

From equations (28) and (29)

$$-A_n\sin\lambda L-B_n\cos\lambda L-A_n\sinh\lambda L-B_n\cosh\lambda L=0$$

hence,

$$A_n=-\frac{\cos\lambda L+\cosh\lambda L}{\sin\lambda L+\sinh\lambda L}B_n \qquad (31)$$

using equation (31), the mode function becomes

$$X_n(x)\quad=B_n\left[\frac{-(\cos\lambda L+\cosh\lambda L)(\sin\lambda x-\sinh\lambda x)}{\sin\lambda L+\sinh\lambda L}+\frac{(\cos\lambda x-\cosh\lambda x)(\sin\lambda L+\sinh\lambda L)}{\sin\lambda L+\sinh\lambda L}\right]$$

$B_n\neq 0$ let

$$K_n(x)=-(cos\lambda L+\cosh\lambda L)(\sin\lambda x-\sinh\lambda x)+(\cos\lambda x-\cosh\lambda x)(\sin\lambda L+\sinh\lambda L)$$

$$X_n(x)=\frac{K_n(x)}{\sin\lambda L+\sinh\lambda L}$$

so that (13) becomes

$$W(x,t)=\sum_{n=1}^{\infty}\frac{2K_n(x)\cos\gamma\lambda^2t\int_0^1K_n(x)w_0(x)dx}{(\sin\lambda L+\sinh\lambda L)^2} \qquad (32)$$

where

$$K_n(x)=-(cos\lambda L+\cosh\lambda L)(\sin\lambda x-\sinh\lambda x)+(\cos\lambda x-\cosh\lambda x)(\sin\lambda L+\sinh\lambda L)$$

λ_n are the successive positive roots of the equation $1+\cos\lambda L\cosh\lambda L=0$

Equation (32) is the free vibration of a cantilever beam.

3. Results and Discussion

Beam dimension and specification:

The beam was made of steel $E=2.10\times10^{11}N$

Length $(L)=10m$

Density of the mass $(\rho)=7800kg/m^3$

Surface area of the beam cross section $A=0.01\times0.01m^2$

Moment of Inertia $I=8.33\times10^{-17}m^4$

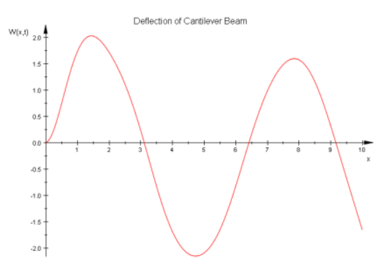

Figure 1. The deflection of Cantilever Beam

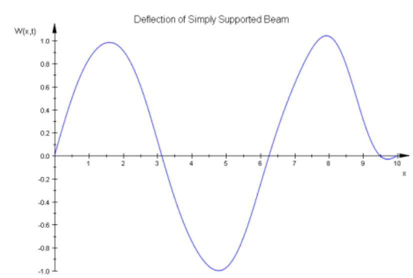

Figure 2. The deflection of Simply Supported Beam

Figure 1 and Figure 2 shows the deflection of a cantilever beam and a Simply Supported beam respectively for mode number up to 10.

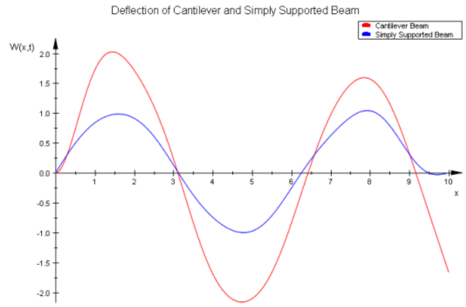

Figure 3. Comparison between the deflection of Cantilever and Simply Supported Beam

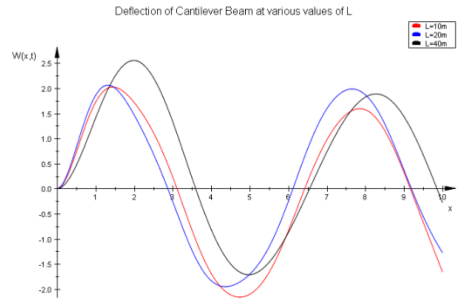

Figure 4. The deflection of Cantilever Beam at various values of L

Figure 3 show the comparison between the deflection of a cantilever beam and a Simply Supported beam, It is found that the response amplitude of cantilever is greater than that of a simply supported beam.

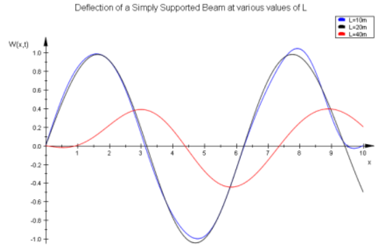

Figure 5. The deflection of Simply Supported Beam at various values of L

Figures 4 and 5 show the deflection of beam for $L = 10m, 20m, 40m$ of a cantilever beam and a simply supported beam respectively. It is found that the response amplitude of the beam increases as the length of the beam increases for a cantilever beam while itdecreases for a simply supported beam.

4. Conclusion

We have looked at the free vibration analysis for a simply supported beam and a cantilever beam. The deflection for various values of the length of the beam was considered for each of the beam and was plotted against x using a computer program (MATLAB).

It can be concluded from Figure 1-5 that the Deflection of beam increases as the length of the beam increases for a cantilever beam but decreases for the case of a simply supported beam. The response amplitude of a cantilever beam is greater than that of a simply supported beam.

We recommend further that the research be carried out for which the beams are transverse by moving loads, taking the damping effects and shear deformation into consideration.

5. REFERENCES

[1] Catal H. H. (2002): Free vibration of partially supported piles with the effects of bending moment, axial and shear force, Engineering Structures, vol. 24, no. 12, pp. 1615-1622.

[2] Catal S. (2008): Solution of free vibration equations of beam on elastic soil by using differential transform method. Applied Mathematical Modelling, vol. 32, no. 9, pp. 1744-1757.

[3] Chen C.K. and Ho S.H. (1996): Application of differential transformation to eigenvalue problems. Applied Mathematics and Computation, vol. 79, no. 2-3, pp. 173-188.

[4] Chen C. K. and Ho S. H. (1999): Transverse vibration of a rotating twisted Timoshenko beams under axial loading using differential transform. International Journal of Mechanical Sciences, vol. 41, no. 11, pp. 1339-1356.

[5] Civalek O. and Ozturk B. (2010): Free vibration analysis of tapered Â¨ beam-column with pinned ends embedded in Winkler-Pasternak elastic foundation. Geomechanics and Engineering, vol. 2, no. 1, pp. 45-56.

[6] Coskun S. B., Atay M. T. and Ozturk B.(2011): Transverse vibration Â¨ analysis of euler-bernoulli beams using analytical approximate techniques. In Advances in Vibration Analysis Research, chapter 1, pp. 1-22, InTech, Vienna, Austria.

[7] Doyle P. F. and Pavlovic M. N.(1982): Vibration of beams on partial elastic foundations. Earthquake Engineering and Structural Dynamics, vol. 10, no. 5, pp. 663-674.

[8] Ozdemir O. and Kaya M. O. (2006): Flapwisebending vibration analysis of a rotating tapered cantilever Bernoulli-Euler beam by differential transform method. Journal of Sound and Vibration, vol. 289, no. 1-2, pp. 413-420.

[9] Ozturk B. (2009): Free vibration analysis of beam on elastic foundation by the variational iteration method. International Journal of Nonlinear Sciences and Numerical Simulation, vol. 10, no. 10, pp. 1255-1262.

[10] Ozturk B. and Coskun S. B. (2011): The homotopy perturbation method for free vibration analysis of beam on elastic foundation Structural Engineering and Mechanics. An International Journal, vol. 37, no. 4.

[11] Ozturk B., Coskun S. B., Koc M. Z. and Atay M. T. (2010): Homotopy perturbation method for free vibration analysis of beams on elastic foundations. IOP Conference Series, vol. 10, no. 1.

[12] Raftoyiannis I. G., Avraam T. P. and Michaltsos G. T. (2010): Dynamic behavior of infinite beams resting on elastic foundation under the action of moving loads. Structural Engineering and Mechanics. An International Journal, vol. 35, no. 3.

[13] Sioma A. (2013): The rope wear analysis with the use of 3D vision system, Control Engineering, 60, 5, 48-49

[14] Snamina J., SapiÅ„ski B., Romaszko M. (2012a): Vibration parameter analysis of a sandwich cantilever beams with multiple MR fluid segments, Engineering Modeling, 43, 247-254

[15] Snamina J., SapiÅ„ski B., WszoÅ‚ek W., Romaszko M. (2012b): Investigation on vibrations of a cantilever beam with magnetorheological fluid by using the acoustic signal, ActaPhysicaPolonica A, 121, 1-A

[16] Sriram P., Hanagud S., Craig J.I. (1992): Mode shape measurement using a scanning laser Doppler vibrometer, The International Journal of Analytical and Experimental Modal Analysis, 7, 3, 169-178

[17] Trucco E., Verri A. (1998): Introductory Techniques for 3D Computer, Vision, Prentice- -Hall

[18] Tuma J. and Cheng F. (1983): Theory and Problems of Dynamic Structural Analysis, Schaumâ€™s Outline Series, McGraw-Hill, New York, NY, USA.

[19] Van der Auweraer H., Steinbichler H., Haberstok C., Freymann R., Storer D.(2002): Integration of pulsed-laser ESPI with spatial domain modal analysis, Shock and Vibration, 9, 1/2, 29-42

[20] West H. H. and Mafi M.(1984): Eigenvalues for beam-columns on elastic supports. Journal of Structural Engineering, vol. 110, no. 6, pp. 1305-1320.