Transactions of the Nigerian Association of Mathematical Physics
Volume 7, (March, 2018), pp15 – 22
© Trans. of NAMP

HESITANT FUZZY MAPPING ON B-METRIC SPACE AND FIXED POINT THEOREMS

E.K. OSAWARU[1] and H. Akewe[2]

Department of Mathematics, University of Benin, Nigeria[1]
Department of Mathematics, University of Lagos, Nigeria[2]

Abstract

We extend the Hesitant fuzzy mapping in a previous work to Hesitant fuzzy mapping on b-metric space and prove some fixed point results. Our results generalizes fixed point results of fuzzy mapping in the sense of Heilpern and hesitant fuzzy mapping in literature.

1. Introduction

Banach [1], Kannan [2], Chatterjea [3], and Zamfirescu [4] proved fixed points of contractive self-maps on metric space under some conditions. Also Nadler [5] defined multivalued mappings as a generalization of single-valued maps and extended the contractive maps of [1] and others to contractive multivalued maps and proved some fixed point theorems on the map. Heilpern [6] defined fuzzy mapping and proved some fixed points results by utilizing the idea of Nadler and others on fixed points of multivalued maps. Recently, Osawaru in [7] defined hesitant fuzzy mapping and proved the existence of fixed point of the generalized hesitant fuzzy mapping which is the hesitant fuzzy set version of the fixed point result of fuzzy mapping of Sahani and Bose [8] in the sense of heilpern [6]. The results in [7] generalizes results of [6], [8] and other extensions of them in literature.

In this work, we extend results in [7] by introducing hesitant fuzzy mapping on b-metric space and prove some fixed point results for some generalized hesitant fuzzy maps on b-metric space. The results generalizes results in [6,7,8] and related results for fuzzy mapping in the Heilpern sense.

We recall the definition of concepts and statements of results needed in the sequel.

Definition 1.1[9]
Let X be any non-empty set. A map $d: X \times X \rightarrow \mathbb{R}$ is called a b-metric if for any real number $s \geq 1$ we have that
[1.1a] $d(x, y) \geq 0$
[1.1b] $d(x, y) = d(y, x)$
[1.1c] $d(x, y) \leq s[d(x, z) + d(z, y)]$
for all $x, y, z \in X$
Thus, the pair (X, d) denotes a b-metric space with coefficient s. Note that (X, d) is a metric space if $s = 1$

Torra[10] defined hesitant fuzzy set and developed hesitant fuzzy logic. As a set-valued fuzzy set, the hesitant fuzzy set generalizes the fuzzy set concept pioneered by L.A.Zadeh.

Definition 1.2 Hesitant Fuzzy Set[10]
Let X be any non-empty set and S be a family of all subsets of the interval $[0,1]$. A hesitant fuzzy set on X is a fuzzy set on X characterized by the map $h: X \rightarrow S$ such that $h(x) \in S$. A hesitant fuzzy map reduces to fuzzy map when h is single-valued for all $x \in X$.

In this work, we denote by $H(X)$, a collection of hesitant fuzzy set on X.

Next we recall the definition of the relation on the collection of all subsets of [0,1]

Definition 1.3 Relation "$\leq$" on S [7]
Let $A, B \in S$. Then $A \leq B$ iff $A^- \leq B^-$ and $A^+ \leq B^+$, where $A^- = min\{a\}$ and $A^+ = max\{a\}$ for all $a \in A$.

Remarks 1.4
[1]For any $A, B \in S, A \leq B$ and $B \leq A$ implies that $A = B$, where equality indicates that the $A^- = B^-$ and $A^+ = B^+$.
[2]For any $A \in S$, $A = \{0\}$ iff $A^+ = 0$
[3]For any $A, B \in S, A < B$ implies that $A^- < B^-$ and $A^+ < B^+$ and neither $A^- = B^-$ nor $A^+ = B^+$ holds
[4]If $A \in S$ is a singleton set, then $A^- = A^+$

Correspondence Author: Osawaru E.K., Email: Kelly.osawaru@uniben.edu, Tel: +2348075421301, +2348036011788 (HA)

Xia and Xu [11] defined a relation on hesitant fuzzy membership values by comparing their scores. They defined a score of a hesitant membership value $A \in S$ as

$$s(A) = \frac{1}{n(A)} \sum_{a \in A} a$$

where$n(A)$ denotes the cardinality of A

A relation on the set of hesitant fuzzy membership values was defined in [13] as $A > B$ if $s(A) > s(B)$ and $A \neq B$ (A is indifferent to B) if $s(A) = s(B)$ for all $A, B \in S$.

However, it is remarked in Lia et al. [12] that the relation fails for some special cases. To resolve the issue therefore, Chen et al. [13] defined the concept of deviation degree. The deviation degree of a hesitant fuzzy membership value $A \in S$ is given as

$$d(A) = \sqrt{\frac{1}{n(A)} \sum_{a \in A} (a - s(A))^2}$$

and defined a comparison on sets of hesitant fuzzy membership values as

$A < B$ if$s(A) < s(B)$ or if $s(A) = s(B)$ and $d(A) > d(B)$

$A = B$ if$s(A) = s(B)$ and $d(A) = d(B)$

$A > B$ if$s(A) = s(B)$ and $d(A) < d(B)$

In this work, we shall adopt the definition of relation on hesitant fuzzy membership values in [13] instead of the definition 1.3 above.

Definition 1.5 α set-level sets of a hesitant fuzzy set[7]

Let h be a hesitant fuzzy set. Then the set $h_\alpha = \{x \in X: \{0\} < h(x) \leq \alpha\}$ for any $\alpha \in S$ and $h_0 = C(\{x \in X: h(x) > \{0\}\})$ with $\alpha = \{0\} \in S$ is called an α set-level set of a hesitant fuzzy set, where $C(B)$ means the closure of B

Definition 1.6 Hesitant fuzzy approximate quantity[7]

A hesitant fuzzy subset h of X is a hesitant fuzzy approximate quantity iff its α set-level set is a compact convex subset of X for each $\alpha \in S$ and $sup_{x \in X} \{h(x)^+\} = \{1\}$.

Definition 1.7 α set-space[7]

Let $W(X) \subset H(X)$ be a collection of hesitant fuzzy approximate quantities of X, $h, k \in W(X)$ and $\alpha \in S$. Then the α set-space of h and k is defined as $p_\alpha(h,k) = inf_{x \in h_\alpha, y \in k_\alpha} d(x,y)$ and

$p(h,k) = sup_\alpha\, p_\alpha\,(h,k)$

Definition 1.8 α set-distance [7]

Let $h, k \in W(X)$and $\alpha \in S$. Then the α set-distance of h and k is defined as

$D_\alpha\,(h,k) = HD(h_\alpha, k_\alpha)$

Where HD denotes the Hausdorff distance.

Let $h, k \in W(X)$ and $\alpha \in S$. Then the distance between h and k is defined as

$D(h,k) = sup_\alpha\, D_\alpha(h,k)$

Definition 1.9[7]

Let $h, k \in W(X)$. Then a hesitant fuzzy approximate quantity h is said to be more accurate than k denoted $h \subset k$ iff $h(x) \leq k(x)$ for each $x \in X$.

Definition 1.10 Hesitant Fuzzy Mapping[7]

Let (Y, d) be a metric space and $W(Y)$ a subcollection of hesitant fuzzy approximate quantities of $H(Y)$. Then the mapping

$H_F: X \to W(Y)$

such that $H_F(x) \in W(Y)$ for each $x \in X$, X any nonempty set is called hesitant fuzzy mapping.

Definition 1.11 Generalized Contraction Hesitant Fuzzy Map[7]

Let (X, d) be a metric space and $W(X)$ a sub collection of hesitant approximate quantities of the collection $H(X)$ of hesitant fuzzy sets of X. Then the pair of hesitant fuzzy maps $H_{F1}, H_{F2}: X \to W(X)$

such that

$$D(H_{F1}(x), H_{F2}(y)) \leq a_1\, p(x, H_{F1}(x)) + a_2\, p(y, H_{F2}(y)) + a_3\, p(y, H_{F1}(x)) + a_4 p\big(x, H_{F2}(y)\big) + a_5 d(x,y) \quad \ldots\ldots\ldots\ldots \ (1)$$

for any $x, y \in X$,where $\sum_{i=1}^{5} a_{\{i\}} < 1$and $a_1 = a_2$ or $a_3 = a_4$ $(a_{\{i\}} \in \mathbb{R}^+)$.

is called the hesitant fuzzy generalized contraction mapping.

Theorem 1.12 [7]

Let (X, d) be a metric space and $H_{F1}, H_{F2}: X \to W(X)$ hesitant maps such that equation (1) holds. Then there exist $x^* \in X$ such that $x^* \subset H_{F1}x^*$ and $x^* \subset H_{F2}x^*$) hold.

2. Main Result

In this section, we redefine some of the concepts above in the context of hesitant fuzzy mapping on a b-metric space.

Definition 2.1 set-level sets of a hesitant fuzzy set of a b-metric space

Let (X, d) be a b-metric space with coefficient s and h be a hesitant fuzzy set on X. Then the set$h_\alpha = x \in X: \{0\} < h(x) \geq \alpha$ } for any $\alpha \in S$and $h_0 = C(x \in X: h(x) > \{0\})$ with $\alpha = \{0\} \in S$ is called an α set-level set of a hesitant fuzzy set, where $C(B)$ means the closure of B

Example 2.2: Let $(\mathbb{Z}, d)$ be a b-metric space with d defined as $d(x, y) =| x - y |^2$ for all $x, y \in \mathbb{Z}$. Suppose $h: X \to S$ is a hesitant fuzzy map of $X = \{1 \leq x \leq 5\} \subset \mathbb{Z}$ such that $h(x) = \{\frac{1}{s} \in [0,1], s \text{ } a \text{ } multiple \text{ } of \text{ } x, s \leq 10\}$

So

$h(1) = \{1,0.5,0.33,0.25, 0.2,0.17,0.14, 0.13, 0.11, 0.1\}$
$h(2) = \{0.5,0.25,0.17,0.13, 0.1\}$
$h(3) = \{0.33,0.17,0.11\}$
$h(4) = \{0.25, 0.13\}$
$h(5) = \{ 0.2, 0.1\}$

Then

$$s(h(1)) = \frac{1}{n(h(1))} \sum_{a \in h(1)} a$$

$$= \frac{1}{10}(1 + 0.5 + 0.33 + 0.25 + 0.2 + 0.17 + 0.14 + 0.13 + 0.11 + 0.1) = 0.29$$

$$s(h(2)) = \frac{1}{5}(0.5 + 0.25 + 0.17 + 0.13 + 0.1) = 0.23$$

$$s(h(3)) = \frac{1}{3}(0.33 + 0.17 + 0.11) = 0.20$$

$$s(h(4)) = \frac{1}{2}(0.25 + 0.13) = 0.19$$

$$s(h(5)) = \frac{1}{2}(0.2, 0.1) = 0.15$$

and

$$d(h(1)) = \sqrt{\frac{1}{h(1)} \sum_{a \in h(1)} \left(a - s\left(h(1)\right)\right)^2}$$

$$d(h(2)) = \sqrt{\frac{1}{h(2)} \sum_{a \in h(2)} \left(a - s\left(h(2)\right)\right)^2}$$

$$d(h(3)) = \sqrt{\frac{1}{h(3)} \sum_{a \in h(3)} \left(a - s\left(h(3)\right)\right)^2}$$

$$d(h(4)) = \sqrt{\frac{1}{h(4)} \sum_{a \in h(4)} \left(a - s\left(h(4)\right)\right)^2}$$

$$d(h(5)) = \sqrt{\frac{1}{h(5)} \sum_{a \in h(5)} \left(a - s\left(h(5)\right)\right)^2}$$

Let $\alpha = \{0.13,0.23\}$, then $s(\alpha) = \frac{1}{n(\alpha)} \sum_{a \in \alpha} a = \frac{1}{2}(0.13 + 0.23) = 0.18$ and

$$d(\alpha) = \sqrt{\frac{1}{\alpha} \sum_{a \in \alpha} \left(a - s(\alpha)\right)^2}$$

Thus, $h_\alpha = \{\}$

Definition 2.3: Hesitant Fuzzy Mapping on b-metric space

Let Y be any nonempty set, (X, d) a b-metric space and $W(X)$ a subcollection of hesitant fuzzy approximate quantities of $H(X)$. Then the mapping

$H_F: Y \to W(X)$

such that $H_F(y) \in W(X)$ for each $y \in Y$, is called hesitant fuzzy mapping on b-metric space.

Example 2.4: Suppose in the example below, we have that there is a hesitant fuzzy map g on X also such that $g(x) = \{\frac{1}{s} : s \text{ is a factor of } x\}$, then $H(X) = \{h, g\}$. Then
$g(1) = \{\emptyset\}$
$g(2) = \{0.5\}$
$g(3) = \{0.33\}$
$g(4) = \{0.5, 0.25\}$
$g(5) = \{0.2\}$
Also, suppose each $h_{\{\alpha\}}, g_{\{\alpha\}}$is compact and convex for each $\alpha \in S$, the $W(X) = H(X)$. Let us defined a mapping $H_F: \{2,3\} \rightarrow W(X)$ such that $H_F(2) = h$ and $H_F(3) = g$.
Then H_F is a hesitant fuzzy map on {2,3}

Definition 2.5: Contraction Hesitant Fuzzy Map on b-metric space
Let (X, d) be a b-metric space with coefficient s. Then the hesitant fuzzy map $H_F: X \rightarrow W(X)$ on b-metric space is said to be a contraction hesitant fuzzy map if

$$D\big(H_F(x), H_F(y)\big) \leq ad(x, y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

for any $x, y \in X$, where $a \in (0, \frac{1}{s})$.

Definition 2.6: Generalized Contraction Hesitant Fuzzy Map on b-metric space
Let (X, d) be a b-metric space with coefficient s and $W(X)$ a sub collection of hesitant approximate quantities of the collection $H(X)$ of hesitant fuzzy sets of X. Then the pair of hesitant fuzzy maps on a b-metric space $H_{F1}, H_{F2}: X \rightarrow W(X)$ such that

$$D(H_{F1}(x), H_{F2}(y)) \leq \frac{1}{s}\,[\,a_1\ p(x, H_{F1}(x)) + a_2\, p(y, H_{F2}(y)) + a_3\, p(y, H_{F1}(x)) + a_4\, p(x, H_{F2}(y))] + a_5 d(x, y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (3)$$

for any $x, y \in X$, where$a_{\{1\}} + a_2 + s(a_3 + a_4) + a_5 < 1$, and $a_1 = a_2$ or $a_3 = a_4$ ($a_{\{i\}} \in \mathbb{R}^+$)is called the hesitant fuzzy generalized contraction mapping on a b-metric space.
Next, we prove generalized results of lemma 2.3.1 - 3 of [7].

Lemma 2.7:
Let X be a b-metric space $x \in X, h \in W(X)$ and $\{x\}$ a hesitant fuzzy set whose hesitant membership function equals a hesitant characteristic function of the set $\{x\}$. If $\{x\}$ is a subset of h then $p_{\{\alpha\}}(x, h) = 0$ for each $\alpha \in S$.

Proof:
If $\{x\}$ is a subset of h then $x \in h_\alpha$ for each $\alpha \in S$ since h is an approximate quantity. So $p_\alpha(x, h) = inf_{y \in h_\alpha} d(x, y) = 0$

Lemma 2.8:
Let (X, d) be a b-metric space with coefficient s then $p_\alpha(x, h) \leq s[d(x, y) + p_\alpha(y, h)]$
for any $x, y \in X$

Proof:

$$p_\alpha(x, h) = inf_{z \in h_\alpha} d(x, z)$$
$$\leq inf_{z \in h_\alpha} s[d(x, y) + d(y, z)]$$
$$\leq s[inf_{z \in h_\alpha} d(x, y) + inf_{z \in h_\alpha} d(y, z)]$$
$$= s[d(x, y) + p_\alpha(y, h)]$$

The proof is complete.

Lemma 2.9:
Let (X, d) be a b-metric space. If $\{x_0\}$ is a subset of h ($h \in W(X)$) then for each $k \in W(X)$ we have that
$p_\alpha(x_0, k) \leq D_\alpha(h, k)$

Proof:

$$p_\alpha(x_0, k) = \inf_{y \in k_\alpha} d(x, y)$$
$$\leq sup_{\{x \in h_\alpha\}} inf_{y \in k_\alpha} d(x, y)$$
$$\leq D_\alpha(h, k)$$

Lemma 2.10:
Let (X, d) be a complete b-metric space with coefficient $s$and $h \in W(X)$. Then
$p_{\{\alpha\}}(x, h) \leq sd(x, y)$if $\{y\} \subset h$.

Proof:
By lemma 2.8 above, for any $x, y \in X$ we have that
$p_\alpha(x,h) \leq s[d(x,y) + p_\alpha(y,h)]$

Since $y \in h$ then lemma 2.7, $p_\alpha(y,h) = 0$ so that
$p_\alpha \ (x,h) \leq sd(x,y)$

The proof is complete.

Theorem 2.11:
Let (X,d) be a complete b-metric space with coefficient s and $H_{F1}, H_{F2}: X \to W(X)$ hesitant maps such that equation (3) holds. Then there exist $x^* \in X$ such that $\{x^*\} \subset H_{F1}(x^*)$ and $\{x^*\} \subset H_{F2}(x^*)$ hold.

Proof:
Let $x_0 \in X$then $H_{F1}(x_0) \in W(X)$. Let also $\{x_1\} \subset H_{F1}(x_0)$, then there is $x_2 \in X$ such that $\{x_2\} \subset H_{F2}(x_1) \in W(X)$. So $d(x_1,x_2) \leq D_1\,(H_{F1}\,(x_0), H_{F2}\,(x_1))$. Also there is $x_3 \in X$ such that $\{x_3\} \subset H_{F2}(x_2) \in W(X)$. So
Let (X,d) be a complete b-metric space with coefficient s and $H_{F1}, H_{F2}: X \to W(X)$ hesitant maps such that equation (3) holds. Then there exist $x^* \in X$ such that $\{x^*\} \subset H_{F1}(x^*)$ and $\{x^*\} \subset H_{F2}(x^*)$ hold.

Proof:
Let $x_0 \in X$then $H_{F1}(x_0) \in W(X)$. Let also $\{x_1\} \subset H_{F1}(x_0)$, then there is $x_2 \in X$ such that $\{x_2\} \subset H_{F2}(x_1) \in W(X)$. So $d(x_1,x_2) \leq D_1\,(H_{F1}\,(x_0), H_{F2}\,(x_1))$. Also there is $x_3 \in X$ such that $\{x_3\} \subset H_{F2}(x_2) \in W(X)$. So$d(x_2,x_3) \leq D_1\,(H_{F2}\,(x_1), H_{F1}\,(x_2))$
Continuing, we have that there is $x_n \in X$ such that
$\{x_{\{2n+1\}}\} \subset H_{F1}(x_{\{2n\}})\ and\ \{x_{\{2n+2\}}\} \subset H_{F2}\,(x_{\{2n+1\}})$
So that

$d(x_{\{2n+1\}}, x_{\{2n+2\}}) \leq D_1\left(H_{F1}(x_{\{2n\}}), H_{F2}(x_{\{2n+1\}})\right) \ldots \ldots \ldots \ldots \ldots (4)$

and

$d(x_{\{2n+2\}}, x_{\{2n+3\}}) \leq D_1\,(H_{F1}\left(x_{\{2n+2\}}, H_{F2}(x_{\{2n+1\}})\right) \ldots \ldots \ldots \ldots .(5)$

Letting $n = 0$ then equation (4) gives

$d(x_1, x_{\{2\}}) \leq D_1\,(H_{F1}\,(x_0), H_{F2}\,(x_1))$
$\qquad \leq D(H_{F1}(x_0), H_{F2}\,(x_1))$
$\leq \frac{1}{s}\,[a_1\,p(x_0, H_{F1}\,(x_0)) + a_2\,p(x_1, H_{F2}\,(x_1)) + a_3\,p(x_1, H_{F1}\,(x_0))$
$+a_4\,p(x_0, H_{F2}(x_1)] + a_5\,d(x_0,x_1)$
$\leq a_1\,d(x_0,x_1) + a_2\,d(x_1,x_2) + a_3\,d(x_1,x_1) + a_4\,d(x_0,x_2) + a_5\,d(x_0,x_1)$
$\leq a_1\,d(x_0,x_1) + a_2\,d(x_1,x_2) + a_4\,s[d(x_0,x_1) + d(x_1,x_2)] + a_5\,d(x_0,x_1)$
$\leq a_1\,d(x_0,x_1) + a_2\,d(x_1,x_2) + a_4sd(x_0,x_1) + a_4sd(x_1,x_2) + a_5\,d(x_0,x_1)$
$\leq (a_2 + a_4s)d(x_1,x_2) + (a_1 + a_4s + a_5)d(x_0,x_1)$
$\leq \frac{(a_1 + a_4s + a_5)}{(1 - a_2 - a_4s)}d(x_0,x_1)$
Put $t = \frac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}$
Then $d(x_1,x_2) \leq td(x_0,x_1)$
But $a_1 + a_2 + s(a_3 + a_4) + a_5 < 1$ giving $a_1 + sa_3 + a_5 < 1 - a_2 - a_4s$ so that if $a_3 \geq a_4$, then we have that $0 < t < 1$
Letting $n = 0$ then equation (5) gives
$d(x_2, x_{\{3\}}) \leq D_1\,(H_{F1}x_2), H_{F2}x_1)$
$\leq D(H_{F1}x_2, H_{F2}x_1)$
$\leq \frac{1}{s}\,[\,a_1\,p(x_2, H_{F1}\,(x_2)) + a_2\,p(x_1, H_{F2}\,(x_1)) + a_3\,p(x_1, H_{F1}\,(x_2)) + a_4\,p(x_2, H_{F2}\,(x_1)] + a_5\,d(x_1,x_2)$
$\leq a_1\,d(x_2,x_3) + a_2\,d(x_1,x_2) + a_3\,d(x_1,x_3) + a_4\,d(x_2,x_2)a_5\,d(x_1,x_2)$
$\leq a_1\,d(x_2,x_3) + a_2\,d(x_1,x_2) + a_3\,s[d(x_1,x_2) + d(x_2,x_3)] + a_5\,d(x_1,x_2)$
$\leq a_1\,d(x_2,x_3) + a_2\,d(x_1,x_2) + a_3sd(x_1,x_2) + a_3sd(x_2,x_3) + a_5\,d(x_1,x_2)$
$\leq (a_1 + a_3s)d(x_2,x_3) + (a_2 + a_3s + a_5)d(x_1,x_2)$
$\leq \frac{(a_2 + a_3s + a_5)}{(1 - a_1 - a_3s)}\,d(x_1,x_2)$

$\leq \dfrac{(a_2+a_3\,s+a_5)}{(1-a_1-a_3s)}\dfrac{(a_1+a_4\,s+a_5)}{(1-a_2-a_4s)}d(x_0,x_1)$

Put $f=\frac{(a_2+a_3\,s+a_5)}{(1-a_1-a_3s)}\frac{(a_1+a_4\,s+a_5)}{(1-a_2-a_4s)}$

Then $d(x_2,x_{\{3\}})\ \leq\ tfd(x_0,x_1)$

But $a_1+a_2+s(a_3+a_4)+a_5<1$ giving $a_1+sa_4+a_5<\ 1-a_2\ -a_3s$ so that if $a_4\ \geq a_3$, then we have that $0<f<1$. Then $a_3=a_4$ and $0<tf<1$ for both cases

Letting $n=1$ then equation (4) gives

$d(x_3,x_4)\ \leq\ D_1\ (H_{F1}\ (x_2),H_{F2}\ (x_3))$

$\leq\ D(H_{F1}\ (x_2)\ ,H_{F2}\ (x_3\ ))$

$\leq\ \dfrac{1}{s}\ [a_1\ p(x_2,H_{F1}(x_2))+\ a_2\ p(x_3,H_{F2}\ (x_3\ ))+a_3\ p\big(x_3,H_{F1}(x_2)\big)+a_4\ p(x_2,H_{F2}\ (x_3\ )]+\ a_5\ d(x_2,x_3)$

$\leq\ a_1\ d(x_2,x_3)+\ a_2\ d(x_3,x_4)+\ a_3\ d(x_3,x_3)+a_4\ d(x_2,x_4)+a_5\ d(x_2,x_3)$

$\leq a_1\ d(x_2,x_3)+a_2\ d(x_3,x_4)+a_4\ s[d(x_2,x_3)+d(x_3,x_4)]+a_5\ d(x_2,x_3)$

$\leq a_1\ d(x_2,x_3)+a_2\ d(x_3,x_4)+a_4sd(x_2,x_3)+a_4sd(x_3,x_4)+a_5\ d(x_2,x_3)$

$\leq\ (a_1+a_4s+a_5)d(x_2,x_3)\ +(a_2+a_4s)d(x_3,x_4)$

$\leq\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\dfrac{(a_2+a_3s+a_5)}{(1-a_1-a_3s)}\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}d(x_0x_1)$

Then $d(x_3,x_4)\leq\ (t^2)fd(x_0,x_1)$

If $a_3=a_4s$ then we have that $0<t^2f<1$

Continuing in this manner we have that for any $n\in\ \mathbb{N}$ and n even

$d(x_1,x_n\ )\leq\ s[d(x_1,x_2)+d(x_2,x_3\ )+d(x_3,x_4)+\cdots\ +d(x_{\{n-1\}},x_n)]$

$\leq\ \ s[(\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}+(\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\dfrac{(a_2+a_3s+a_5)}{(1-a_1-a_3s)}$

$+(\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\ (\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}+\cdots+(\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\ (\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\ \ldots\ )d(x_0,x_1)]$

$\leq\ s[\ t+tf+t^2f+(tf)^2+t^3f+\cdots\ (tf)^{\{n-1\}}]d(x_0,x_1)]$

If n is odd then we have that

$d(x_1,x_n)\leq s[d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)+\cdots\ +d(x_{\{n-1\}},x_n)]$

$\leq\ \ s[(\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}+\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\dfrac{(a_2+a_3s+a_5)}{(1-a_1-a_3s)}$

$+\ \dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}+\cdots\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\dfrac{(a_1+a_4s+a_5)}{(1-a_2-a_4s)}\ \ldots\ )d(x_0,x_1)]$

$\leq\ s[\ t+tf+t^2f+(tf)^2+t^3f+\cdots\ (tf)^{\{n-1\}}t]d(x_0,x_1)]$

Next, we show that any sequence $\{x_n\}$ in X is Cauchy. Let $k,l\in\ \ \mathbb{N}$

$d(x_{\{l\}},x_k)\leq\ \ s[d(x_{\{l\}},x_{\{l+1\}})+d(x_{\{l+1\}},x_{\{l+2\}})+\cdots\ +d(x_{\{k-1\}},x_k)]$

$\leq\ s^2[(tf)^{\{l-1\}}+(tf)^{\{l\}}+(tf)^{\{l+1\}}\ \ldots\ +(tf)^{\{k-2\}}]d(x_0,x_1)$………………(6)

and if k is odd we have that

$d(x_{\{l\}},x_k)\leq\ \ s[d(x_{\{l\}},x_{\{l+1\}})+d(x_{\{l+1\}},x_{\{l+2\}})+\cdots\ +d(x_{\{k-1\}},x_k)]$

$\leq\ s^2[(tf)^{\{l-1\}}t+(tf)^{\{l\}}t+(tf)^{\{l+1\}}t\ \ldots+(tf)^{\{k-2\}}t]d(x_0,x_1)$……………(7)

Since $\sum_{i=1}^{5}a_{\{i\}}<\frac{1}{s}$, then $(tf)^{\{\frac{n}{2}\}}t[1+(tf)^{\{\frac{1}{2}\}}\}+(tf)<\frac{1}{s^2}$ for each n. Thus as $n\to\infty$ we have that the RHS of the inequality above tends to 0. The same is true if n is even. Therefore the sequence $\{x_n\}$ in X is a Cauchy sequence. Thus there exist $x^*\in\ X$ such that $\{x_n\}\to\ x^*$ as $\ n\to\infty$ since (X,d) is a complete space. Now

$p_0\ (x^*,H_{F2}\ (x^*\ ))\leq\ \ s[d(x^*,x_{\{2n+1\}})+H_{F2}(x_{\{2n+1\}},H_{F1}\ (x^*\ ))]$

$\leq\ \ s[d(x^*,x_{\{2n+1\}}\ )+D\ (x_{\{2n\}},H_{F2}\ (x^*\ ))]$

So

$p_0\big(x^*,H_{F2}(x^*)\big)\leq\ \ s\left[d\big(x^*,x_{\{2n+1\}}\big)+D\ \left(x_{\{2n\}},H_{F2}(x^*)\right)\right]\ldots\ldots\ldots\ldots(8)$

But

$D\ \left(x_{\{2n\}},H_{F2}(x^*)\right)\leq\dfrac{1}{s}[\ a_1\ p(x_{\{2n\}},H_{F1}\ (x_{\{2n\}}))+\ a_2\ p(x^*,H_{F2}\ (x^*\ )$

$+a_3\ p(x^*,H_{F1}\ (x_{\{2n\}}\ )+a_4\ p(x_{\{2n\}},H_{F2}\ (x^*\ ))]+a_5d(x_{\{2n\}},x^*)$

$\leq a_1\, d\left(x_{\{2n\}}, x_{\{2n+1\}}\right) + a_2\, d\left(x^*, x_{\{2n+1\}}\right)$
$+a_2 D\left(H_{F1}\left(x_{\{2n\}}\right), H_{F2}(x^*)\right) + a_3 d\left(x^*, x_{\{2n+1\}}\right) \quad + a_4 d\left(x_{\{2n\}}, x_{\{2n+1\}}\right) + a_4 D\left(H_{F1}\left(x_{\{2n\}}\right), H_{F2}(x^*)\right)$
$+ a_5\, d\left(x_{\{2n\}}, x^*\right) \quad \leq (a_1 + a_4) d(x_{\{2n\}}, x_{\{2n+1\}}) + (a_2 + a_3) d(x^*, x_{\{2n+1\}})$
$+(a_2 + a_4) D(H_{F1}(x_{\{2n\}}, H_{F2}(x^*)) + a_5\; d(x^*, x_{\{2n\}})$
$\leq \frac{(a_1 + a_4)}{(1 - a_2 - a_4)}\, d\left(x_{\{2n\}}, x_{\{2n+1\}}\right) + \frac{(a_2 + a_3)}{(1 - a_2 - a_4)} d(x^*, x_{\{2n+1\}})$
$+ \frac{a_5}{(1 - a_2 - a_4)} d(x^*, x_{\{2n\}})$
Thus equation (8) becomes $p_0\left(x^*, H_{F2}(x^*)\right) \leq \quad s[d(x^*, x_{\{2n+1\}}) + \frac{(a_1+a_4)}{(1-a_2-a_4)}\, d(x_{\{2n\}}, x_{\{2n+1\}}) + \frac{(a_2+a_3)}{(1-a_2-a_4)} d(x^*, x_{\{2n+1\}})$
$+ \frac{a_5}{(1 - a_2 - a_4)} d(x^*, x_{\{2n\}})] \leq s[\frac{(a_1 + a_4)}{(1 - a_2 - a_4)} d(x_{\{2n\}}, x_{\{2n+1\}}) + \frac{(1 - a_4 + a_3)}{(1 - a_2 - a_4)} d(x^*, x_{\{2n+1\}})$
$+ \frac{a_5}{(1 - a_{2s} - a_{4s})} d(x^*, x_{\{2n\}})] \rightarrow 0 \quad as \quad n \quad \rightarrow \infty$
Therefore $\{x^*\} \subset H_{F2}(x^*)$ by Lemma 2.7.
Similarly,
we can show that $\{x^*\} \subset H_{F1}(x^*)$ and the proof complete.

Corollary 2.12:
Let (X, d) be a complete b-metric space and $H_F: X \rightarrow W(X)$ hesitant map such that $D(H_F(x), H_F(y)) \leq ad(x, y)$ for any $x, y \in X$, where $a < \frac{1}{s}$. Then there exist a unique $x^* \in X$ such that $\{x^*\} \subset H_F(x^*)$ holds.

Proof:
We have our desired result $\{x^*\} \subset H_F(x^*)$ if we put $H_F = H_{F1} = H_{F2}$, $a_i = 0$ for $i = 1,2, \ldots ,4$ and $a_5 = a$ in theorem 2.11 above. Next, we prove that x^* is unique. Suppose there is another point y^* such that $\{y^*\} \subset H_F(y^*)$ holds also. Then
$d(x^*, y^*) \leq D_1(H_F(x^*), H_F(y^*))$
$\leq D(H_F(x^*), H_F(y^*))$
$\leq ad(x^*, y^*)$
a contradiction. Therefore $x^* = y^*$. The proof is complete.

Corollary 2.13:
Let (X, d) be a complete metric space and $H_{F1}, H_{F2}: X \rightarrow W(X)$ hesitant selfmaps such that
$D(H_{F1}(x), H_{F2}(y)) \leq a_1\, p(x, H_{F1}(x)) + a_2\, p(y, H_{F2}(y) + a_3\, p(y, H_{F1}(x)$
$+ a_4\, p(x, H_{F2}(y)) + a_5\, d(x, y)$
for any $x, y \in X$,where $\sum_{i=1}^{5} a\_i$<1,and $a_1 = a_2$ or $a_3 = a_4$ $(a_{\{i\}} \in \mathbb{R})$.
called the hesitant fuzzy generalized contraction mapping on a metric space. Then there exist $x^* \in X$ such that $\{x^*\} \subset H_{F1}(x^*)$ and $\{x^*\} \subset H_{F2}(x^*)$ hold.
Proof: The proof follows from Theorem 2.11 for $s = 1$. The proof is complete.

Corollary 2.14:
Let (X, d) be a complete metric space and $H_F: X \rightarrow W(X)$ a hesitant selfmap such that equation (2) holds. Then there exist $x^* \in X$ such that $\{x^*\} \subset H_F(x^*)$ holds.
Proof: The proof follows from Theorem 2.11 for $s = 1$ and $H_F = H_{F1} = H_{F2}$, $a_i = 0$ for $i = 1,2, \ldots ,4$ and $a_5 = a$. The proof is complete.

3. CONCLUSION
This work is the extension of hesitant fuzzy mapping and fixed point theorems of the maps defined on a b-metric space. The ordering defined on hesitant fuzzy membership values in ths work generalizes that of [7] . Thus our results extends results of S. Heipern[6] and other extensions and generalizations of it.

4. REFRENCES
[1] S. Banach, Sur les operation dans les ensembles abstaits et leur application aux equations integrals, Fund. Math. 3(1922) 133-181.
[2] Kannan, Some results on fixed points,Bull.Calcutta Math. Soc. 60 (1968) 71-76.
[3] S. K. Chatterjea, Fixed-point theorems, C.R. Acad. Bulgare Sci. 25 (1972) 727-730.
[4] T. Zamferiscu, Fixed point theorems in metric spaces, Arch. Math. (Basel)23 (1972) 292-298.

[5] S. B. Nadler, Multivalued contraction mapping, Pacific. J. Math. 30 (1969), 475–488.
[6] S. Heilpern, Fuzzy Mappings and Fixed Point Theorem, Journal of Mathematical Analysis and Applications 83, 566-569(1981).
[7] K. E. Osawaru, Hesitant Fuzzy Mapping and Fixed Point Theorems, Journal of the Nigerian Association of Mathematical Physics, Vol.37 (2016) pp 31-36.
[8] R. K. Bose, D. Sahani, Fuzzy mappings and Fixed point theorems, Fuzzy Sets and Systems 21 (1987) 53–58.
[9] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum 9 (1) (2004) 43-53.
[10] V. Torra, Hesitant Fuzzy Sets, International Journal of Intelligent Systems, 2010.
[11] Xia, M.M., Xu, Z.S.: Studies on the aggregation of intuitionistic fuzzy and hesitant fuzzy information, Technical Report (2011)
[12] Liao, H.C., Xu, Z.S., Xia, M.M.: Multiplicative consistency on hesitant fuzzy preference relation and the application on group decision making, International Journal of Information Technology and Decision Making (2013), doi:10.1142/S0219622014500035
[13] Chen, N., Xu, Z.S., Xia, M.M.: Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis, Applied Mathematical Modelling 37, 2197–2211 (2013a)
[14] T. Som, R. Mukherjee, Some Fixed point theorems for fuzzy mappings, Fuzzy Sets and Systems 33 (1989) 213–219
[15] J. Y. Park, J.U. Jeong, Fixed point theorems for fuzzy mappings, Fuzzy Sets and Systems 33 (1997) 111–116