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COMPLETE GEOMETRIC GRAPH AND ASYMPTOTIC FORMULA

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Abstract

In a complete geometric digraph $\mathcal{K}_n$, the number of connected subgraphs (i.e points, lines, triangles and so on) is given by the generating function $P_n(t)$. In this paper, we give a normal approximation of the distribution of connected subgraphsof $\mathcal{K}_n$and also give an asymptotic formula for $P_n(t)$.

***Keywords*: Asymptotic formula, Normal distribution, Geometric graph.**

1. Introduction

In this paper, we continued with the line of research initiated in [1], [2], where we use the theory of geometric graph to classify the equations defining flag varieties and also degenerate flag varieties to toric varieties. The number of connected subgraphs in a complete geometric graph $\mathcal{K}_n$is given by a generating function $P_n(t)$. Our interest here is to investigate the asymptotic normality of the distribution of connected subgraphs of $\mathcal{K}_n$and give the asymptotic formula for $P_n(t)$.

A geometric graph $G = (V, E)$where $V$is a set of points in the plane and $E$is a set of line segments with endpoints in V. We assume that the points are in general position, i.e, no three points are collinear.

A complete geometric graph is a geometric graph in which any two points is joined by a line segment and is denoted by $\mathcal{K}_n$.n is the number of points and the number of lines of $\mathcal{K}_n$ is $\frac{n(n-1)}{2}$. In section 2, we give some background and relevant result on complete geometric graph. In section 3, we consider the asymptotic normality of the distribution of connected subgraphs of $\mathcal{K}_n$.In section 4, we give the asymptotic formula for $P_n(t)$.

2. Complete Geometric graph

In this section we give some background definitions and result on complete geometric graphs.

Definition 2.1. Let $\mathcal{K}_n$be a complete geometric graph with $n$points and let $\tau \subset [n]$. $x_\tau$is said to be a point if $|\tau| = 1$, a line if $|\tau| = 2$, a triangle if $|\tau| = 3$and so on.

Remark 2.2. All the $x_\tau{}'s$for which $|\tau| \geq 3$are empty, that is, they have no interior points.

Example 2.3.

i. For $n = 3$

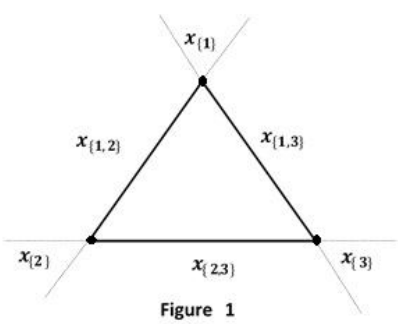

Figure 1

ii. . For $n = 4$

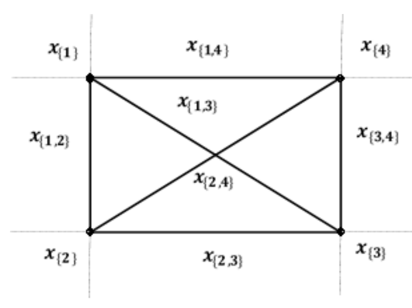

Figure 2

Figure 1 and 2 give the complete geometric digraph for $n = 3$and $n = 4$respectively.

In a complete geometric graph$\mathcal{K}_n$, let $\mathrm{F}_r = \{x_\tau : |\tau| = r, \tau \subseteq [n]\}$, F_1set of points, F_2set of lines and so on, We refer to $\mathrm{F}_r{}'s$as data in $\mathcal{K}_n$.

Theorem 2.4. [1] Given a complete geometric graph $\mathcal{K}_n$, then the cardinality of $\mathrm{F}_r$is given by the coefficient of

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$$P_n(t) = \sum_{r=1}^{n} \binom{n}{r} t^r$$

for $n \geq 3$.
Theorem 2.4 gives the cardinality of $F_r(\#F_r)$for $1 \leq r \leq n$in $\mathcal{K}_n$.Let the cardinality of $F_r(\#F_r)$ be d_r

Table 1. Statistics of d_r in $\mathcal{K}_n$

N	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
1	1									
2	2	1								
3	3	3	1							
4	4	6	4	1						
5	5	10	10	5	1					
6	6	15	20	15	6	1				
7	7	21	35	35	21	7	1			
8	8	28	56	70	56	28	8	1		
9	9	36	84	126	126	84	36	9	1	
10	10	45	120	210	252	210	120	45	10	1

3. Normal approximation of the number of connected subgraphs of $\mathcal{K}_n$

In this section, we give the normal approximation of the distribution of data in$\mathcal{K}_n$. See [3-6] for details on asymptotic normality.

3.1 Asymptotic Normality

$d_r$has a unimodal behavior which suggest that $d_r$may be asymptotically normal. This is studied by finding the generating function for the probability distribution of $d_r$for $m \leq n$.

$$S_n = \sum_{r=1}^{n} \binom{n}{r}$$
$$= 2^n - 1$$

where S_n is $\sum_{r=1}^{n}(d_r)$.
The generating function for the probability distribution is

$$E_n(t) = \frac{P_n(t)}{\sum_{r=1}^{n}\binom{n}{r}}$$
$$= \frac{P_n(t)}{2^n - 1}$$

For $1 \leq r \leq n$.
The moment generating function, $M_n(t)$ is calculated as follows:

$$M_n(t) = \frac{E_n(e^t)}{2^n - 1}$$
$$= \frac{\sum_{r=1}^{n}\binom{n}{r} e^{rt}}{2^n - 1}$$
$$= \frac{(1 + e^t)^n - 1}{2^n - 1}. \qquad (3.1)$$

The mean, $\mu = \frac{n2^{n-1}}{2^n - 1}$ and the variance,$\sigma^2 = \frac{n2^{n-1}}{2^n-1}\left(\frac{n+1}{2} - \frac{n2^{n-1}}{2^n-1}\right)$. The probability distribution function for $d_r$is

$$f(r) = \frac{\binom{n}{r}}{2^n - 1}. \qquad (3.2)$$

Figure 3 to 5 show the density for a normal random variable in broken line and the continuous line curve for probability distribution function for d_r for $n = 10$, $n = 20$ and $n = 50$. As the order of $\mathcal{K}_n$ increases, the continuous line curvemoves closer to the broken line curve. This implies that the approximation of $f(r)$ to normal distribution improves as the order of $\mathcal{K}_n$ increases.

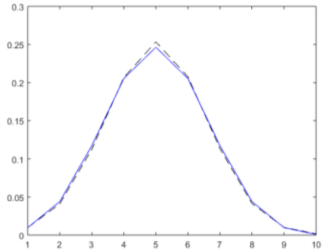

Figure 3. Comparison of the $f(r)$ to the normal density for $n = 10$

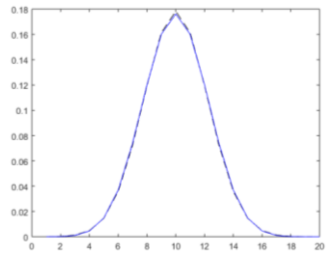

Figure 4. Comparison of the $f(r)$ to the normal density for $n = 20$

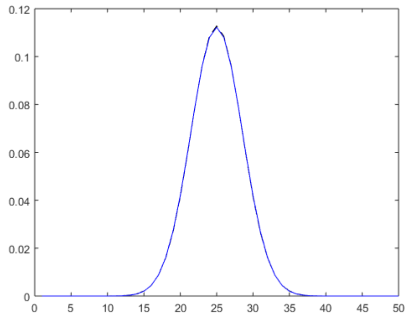

Figure 5. Comparison of the $f(r)$ to the normal density for $n = 50$

Figure 6 to 8 show the ratio of probability function, $f(r)$ to the estimate provided by normal distribution function. As the order of $\mathcal{K}_n$ increases, the curves tend to the shape of a cowboy hat and top of the hat get broader suggesting that the approximation improves as the order of $\mathcal{K}_n$ increases.

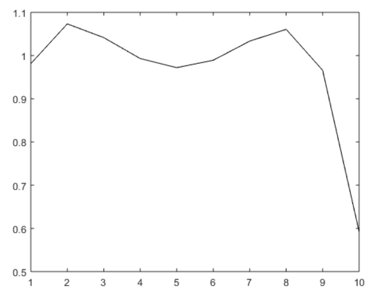

Figure 6. Ratio of probability function, $f(r)$ to the normal density for $n = 10$

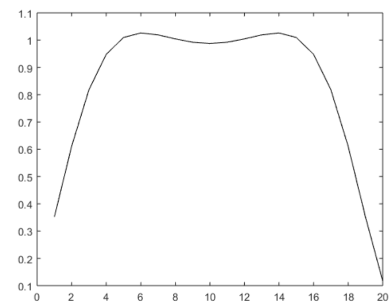

Figure 7. Ratio of probability function, $f(r)$ to the normal density for $n = 20$

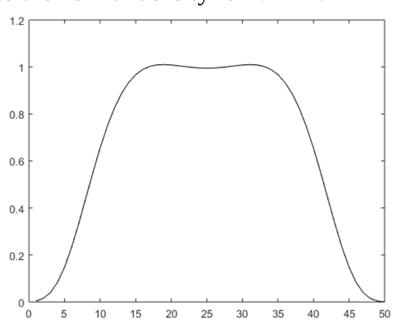

Figure 8. Ratio of probability function, $f(r)$ to the normal density for $n = 50$

4. Asymptotic Formula for $\mathbf{d}_r$.

We are interested in the sequence $\{H_{n+r}(n), n = 1, 2, \dots\}$. For $r \geq 1$, we have

$$H_{n+r}(n) = \binom{n+r}{n} = \frac{(n+r)!}{r!\,n!}. \qquad (4.1)$$

Equation (4.1) can be approximated using Stirling's approximation (see [7] for details).

$$n! = \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Then we have

$$H_{n+r}(n) = \frac{(n+r)!}{r!\,n!} = \frac{\sqrt{2\pi}(n+r)^{n+r+\frac{1}{2}}e^{-(n+r)}}{\sqrt{2\pi}r^{r+\frac{1}{2}}e^{-r} \times \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}} \times \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$= \frac{(n+r)^{n+r+\frac{1}{2}}}{r^{r+\frac{1}{2}} \times \sqrt{2\pi} n^{n+\frac{1}{2}}} \times \left(1 + O\left(\frac{1}{n}\right)\right)$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{n+r}{r}\right)^{r} \left(\frac{n+r}{n}\right)^{n} \left(\frac{n+r}{nr}\right)^{\frac{1}{2}} \times \left(1 + O\left(\frac{1}{n}\right)\right)$$
$$= \frac{n^r r^n}{\sqrt{2\pi}} \left(\frac{n+r}{nr}\right)^{n+r+\frac{1}{2}} \times \left(1 + O\left(\frac{1}{n}\right)\right).$$

Theorem 4.1.

$$H_{n+r}(n) = \frac{n^r r^n}{\sqrt{2\pi}} \left(\frac{n+r}{nr}\right)^{n+r+\frac{1}{2}} \times \left(1 + O\left(\frac{1}{n}\right)\right).$$

This provide an asymptotic estimate for the data in $\mathcal{K}_n$. Figure9 to 11 give comparison of the $(2^n - 1)$ times the normal density density (asterisk curve) with mean, $\mu = \frac{n2^{n-1}}{2^n-1}$ and the variance,$\sigma^2 = \frac{n2^{n-1}}{2^n-1}\left(\frac{n+1}{2} - \frac{n2^{n-1}}{2^n-1}\right)$, asymptotic estimate(broken line curve) and exact value of $H_n(r)$(continuous line curve).

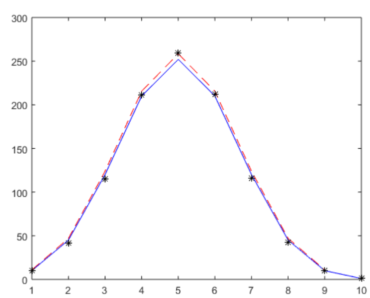

Figure 9. Comparison of normal density estimate to asymptotic estimate and actual $H_n(r)$ for $n = 10$.

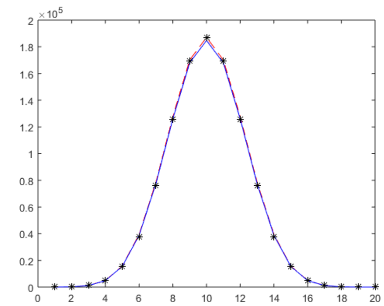

Figure 10. Comparison of normal density estimate to asymptotic estimate and actual $H_n(r)$ for $n = 20$.

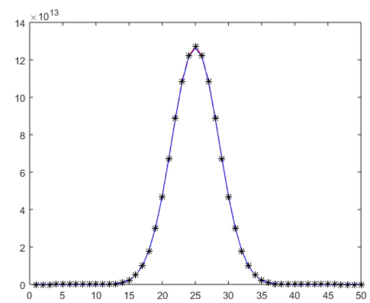

Figure 11. Comparison of normal density estimate to asymptotic estimate and actual $H_n(r)$ for $n = 50$.

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