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In honour of Prof. Ekhaguere at 70

The theorems of Radon and Helly on convex fuzzy sets

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Abstract. In Convex Geometry, the study of theorems of Radon, Helly and Carathéodory have played important roles and their generalizations have been studied from different points of view such as Convex Analysis, Optimization, Discrete Geometry. These theorems give excellent introduction to the theory of convexity. In this paper, we consider Radon's theorem and Helly's theorem; and extend them to the fuzzy case. In particular, we shall state and prove the fuzzy version of Radon's theorem and Helly's theorem.

Keywords: Fuzzy points, affinely independent sets, convex fuzzy sets, Radon's theorem, Helly's theorem

1. Introduction

The concept of convexity of fuzzy sets, which is a generalisation of the notion of classical convexity of sets, was first introduced in 1965 by Zadeh [16]. Several researchers have since then studied and developed various notions of convex fuzzy sets. Among them are Brown [2], Liaozu-Hua et al. [5], Lin [6], Liu [7], Lowen [8], Yang and Yang [13], Yang [14,15], Zhu [17], Al-Mayahi and Ali [1].

In 1991, Z. Feiyue [3], extended the classical theorem of Carathéodory about the generation of convex hulls from ordinary convex analysis to the fuzzy case. He did this using fuzzy points and fuzzy directions.

On the other hand, Y. Maruyama [10], studied some properties of (lattice) L valued-convex fuzzy sets. He obtained the L-fuzzy version of the five classical theorems in convex geometry, that is; Carathéodory's theorem, Radon's theorem, Finite and Infinite Helly's theorem, and Kakutani's fixed point theorem. He recorded that, these theorems apart from Kakutani's fixed point theorem are fundamental results in Combinatorial Convex Geometry and they characterize the dimension of a Euclidean spaces.

Motivated by the earlier work on fuzzy version of Carathéodory's theorem studied by Feiyue, we extend Radon's theorem and Helly's theorem to the fuzzy case using the notion of fuzzy points.

2. Preliminaries

The following definitions are from [3,8,11,16]

DEFINITION 2.1 *Let X be a nonempty set. A fuzzy set σ of the set X is a function $\sigma : X \longrightarrow \mathbb{I}$.*

DEFINITION 2.2 *Let σ be a fuzzy set of X, the t-level subset of σ, denoted by σ_t is made up of members whose membership function is at least t. That is;*

$$\sigma_t = \{x \in X : \sigma(x) \geq t\}.$$

DEFINITION 2.3 *Let σ and τ be two fuzzy sets of X. Then, $\forall\ x \in X$, $\sigma \subseteq \tau$ if $\sigma(x) \leq \tau(x)$.*

DEFINITION 2.4 *Let σ be a fuzzy set of X. Then, the support of σ, denoted by supp(σ) is defined*

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by

$$supp(\sigma) = \{x \in X : \sigma(x) > 0\}$$

A fuzzy set is said to be finite if its support is finite.

DEFINITION 2.5 *Let $a \in \mathbb{R}^d$, $\alpha \in (0,1]$ and X a nonempty set. Then, a fuzzy point, denoted by a_α is a fuzzy subset with membership function $\sigma_{a_\alpha}(x)$, $\forall\ x \in X$, defined by*

$$\sigma_{a_\alpha}(x) = \begin{cases} \alpha, & if\, x = a \\ 0, & if\, x \neq a \end{cases}$$

The point a is called the support of a_α and α its value.

Remark 1 Note that a fuzzy point is a fuzzy set with non-zero membership only at one point of the support space.

DEFINITION 2.6 *The fuzzy point a_α belongs to a fuzzy set $\sigma \in X$, denoted by $a_\alpha \in \sigma$ if $\alpha \leq \sigma(x)$, $\forall\ x \in X$.*

The set of all fuzzy points in $\mathbb{R}^d$ is denoted by $\tilde{\mathbb{R}}^d$.

Two fuzzy points a_α, $b_\beta \in \tilde{\mathbb{R}}^d$ are said to be equal, that is; $a_\alpha = b_\beta$ if $a = b$ and $\alpha = \beta$.

Let $a_\alpha, b_\beta \in \tilde{\mathbb{R}}^d$ and $\lambda \in \mathbb{R}$. Their sum $a_\alpha + b_\beta$ and scalar multiplication λa_α are defined by

$$a_\alpha + b_\beta = (a + b)_{\alpha \wedge \beta}$$

and

$$\lambda a_\alpha = (\lambda a)_\alpha$$

respectively.

If $\sigma, \tau \subset \tilde{\mathbb{R}}^d$ and $\lambda \in \mathbb{R}$,

$$\sigma + \tau = \{a_\alpha + b_\beta : a_\alpha \in \sigma,\ b_\beta \in \tau\}$$

and

$$\lambda\sigma = \{\lambda a_\alpha : a_\alpha \in \sigma\}.$$

Notation:Let $\mathcal{F}(\mathbb{R}^d)$ be the family of all fuzzy sets in $\mathbb{R}^d$. For $\sigma \in \mathcal{F}(\mathbb{R}^d)$ and $\tau \subset \tilde{\mathbb{R}}^d$, $\tau \sim \sigma$ or $\sigma \sim \tau$ denote τ is a pointwise representation of σ or σ is the fuzzy set represented by τ.

DEFINITION 2.7 *Let $\gamma \in \mathcal{F}(\mathbb{R}^d)$, a special pointwise representation of γ, denoted by $\tilde{\gamma}$ is defined by*

$$\tilde{\gamma} = \{a_\alpha : x \in \mathbb{R}^d,\ \ 0 \leq \alpha \leq \sigma_\gamma(x)\}.$$

Namely, $a_\alpha \in \tilde{\gamma}$ if and only if $a_\alpha \in \sigma$.

DEFINITION 2.8 *Let σ be a fuzzy set on $\mathbb{R}^d$ and $a^1_{\alpha_1}, \cdots, a^d_{\alpha_d}$ its fuzzy points, $\lambda \in \mathbb{R}$, $\sum\limits_{i=1}^{d} \lambda_i = 1$.*

Then, affine fuzzy combination of the fuzzy points $a^1_{\alpha_1}, \cdots, a^d_{\alpha_d}$ *is defined by* $\sum\limits_{i=1}^{d} \lambda_i a^i_{\alpha_i}$ *such that*

$$\sigma(\sum_{i=1}^{d} \lambda_i a^i_{\alpha_i}) \geq \sigma(a^1_{\alpha_1}) \wedge \cdots \wedge \sigma(a^d_{\alpha_d}).$$

DEFINITION 2.9 *A fuzzy set* $\sigma : \mathbb{R}^d \longrightarrow \mathbb{I}$ *is an affine fuzzy set if*

$$\sigma[\lambda x + (1-\lambda)y] \geq \sigma(x) \wedge \sigma(y), \quad \forall\ x, y \in \mathbb{R}^d,\ \lambda \in \mathbb{R}.$$

Affine fuzzy sets can also be defined in terms of affine fuzzy combination as follows:

DEFINITION 2.10 *Let* σ *be a fuzzy set on* $\mathbb{R}^d$ *and* $a^1_{\alpha_1}, \cdots, a^d_{\alpha_d}$ *its fuzzy points. Then,* σ *is called affine fuzzy set if for all* $a^1_{\alpha_1}, \cdots, a^d_{\alpha_d} \in \sigma$, $\lambda_1, \cdots, \lambda_d \in \mathbb{R}$ *such that*

$$\sigma(\sum_{i=1}^{d} \lambda_i a^i_{\alpha_i}) \geq \sigma(a^1_{\alpha_1}) \wedge \cdots \wedge \sigma(a^d_{\alpha_d})$$

and $\sum\limits_{i=1}^{d} \lambda_i = 1.$

DEFINITION 2.11 *The affine fuzzy hull of fuzzy set* σ, $aff(\sigma)_f$, *is the set of all of fuzzy points in* σ. *That is;*

$$aff(\sigma)_f = \{\sum_{i=1}^{d} \lambda_i a^i_{\alpha_i} : d \geq 1,\ a^i_{\alpha_i} \in \sigma,\ \lambda_i \in \mathbb{R},\ \sum_{i=1}^{d} \lambda_i = 1\}.$$

DEFINITION 2.12 *Let* σ *be a fuzzy set and* $a^1_{\alpha_1}, a^2_{\alpha_2}, \cdots, a^d_{\alpha_d}$ *its fuzzy points. Then, the fuzzy points are said to be affinely independent if* $\sum\limits_{i=1}^{d} \lambda_i a^i_{\alpha_i} = 0$ *and* $\sum\limits_{i=1}^{d} \lambda_i = 0$ *imply that* $\lambda_1 = \lambda_2 = \cdots = \lambda_d = 0$. *Otherwise, it is affinely dependent.*

DEFINITION 2.13 *Let* σ *be a fuzzy set on* $\mathbb{R}^d$, $a^1_{\alpha_1}, \cdots, a^d_{\alpha_d}$ *be its fuzzy points,* λ_i *are non negative,* $\sum\limits_{i=1}^{d} \lambda_i = 1$ *then*

$$a_\alpha = \sum_{i=1}^{d} \lambda_i a^i_{\alpha_i}$$

is called a fuzzy convex combination of the fuzzy points $a^i_{\alpha_i} \in \sigma$.

DEFINITION 2.14 *The fuzzy set* σ *on* $\mathbb{R}^d$ *is said to be a convex fuzzy set if;*

$$\sigma[\lambda a_\alpha + (1-\lambda) b_\beta] \geq \sigma(a_\alpha) \wedge \sigma(b_\beta)$$

for every $a_\alpha, b_\beta \in \sigma, \quad \alpha \in \mathbb{I}$.

Convex fuzzy sets can also be defined in terms of fuzzy convex combination as follows:

DEFINITION 2.15 *Let* σ *be a fuzzy set on* $\mathbb{R}^d$ *and* $a^1_{\alpha_1}, \cdots, a^d_{\alpha_d}$ *its fuzzy points. Then,* σ *is called a* *convex fuzzy set if for all* $a^1_{\alpha_1}, \cdots, a^d_{\alpha_d} \in \sigma,\ \lambda_1, \cdots, \lambda_d \in \mathbb{I}$ *such that*

$$\sigma(\sum_{i=1}^{d} \lambda_i a^i_{\alpha_i}) \geq \sigma(a^1_{\alpha_1}) \wedge \cdots \wedge \sigma(a^d_{\alpha_d})$$

and $\sum_{i=1}^{d} \lambda_i = 1$.

DEFINITION 2.16 *Let* σ *be a fuzzy set on* $\mathbb{R}^d$*. Then, the convex fuzzy hull of* σ *is defined as the set of all fuzzy convex combinations of fuzzy points in* σ*. That is;*

$$conv(\sigma)_f = \left\{ a^1_{\alpha_1}, \cdots, a^n_{\alpha_n} \in \sigma : \exists\ \lambda_i \in \mathbb{I},\ \sum_{i=1}^{p} \lambda_i = 1,\ a_\alpha = \sum_{i=1}^{p} \lambda_i a^i_{\alpha_i} \right\},$$

which is the smallest convex fuzzy set containing σ.

DEFINITION 2.17 *Let* S_o *be a nonempty set of points in* $\mathbb{R}^d$ *and* S_1 *a set of directions in* $\mathbb{R}^d$*. Then, the convex hull* $conv(S)$ *of* $S = S_0 \cup S_1$ *is defined as the smallest convex set* $C \in \mathbb{R}^d$ *such that* $S_0 \subset C$ *and* C *recedes in all directions in* S_1 *(i.e. for each* $y \in S_1,\quad x + \lambda y \in C \quad \forall\ x \in C,\ \lambda > 0$*).*

3. Results

In this section, we state and prove the fuzzy version of Radon's and Helly's theorems. We first state the classical version of the theorems and the fuzzy version of Carathéodory. The classical ones are as follows:

THEOREM 3.1 (Radon's theorem) *Any set* X *of* $d+2$ *points in* $\mathbb{R}^d$ *can be partitioned into two disjoint sets* $X_1, X_2 \subset X$ *whose convex hulls intersect. that is;* $conv(X_1) \cap conv(X_2) \neq \emptyset$.

THEOREM 3.2 (Helly's theorem) *For a family* $K_1, K_2, ..., K_n,\ n \geq d+1$ *of convex sets in* $\mathbb{R}^d$*, if every* $d+1$ *of the sets have a point in common, then all of the sets have a point in common.*

THEOREM 3.3 (Carathéodory's theorem) *If* Y *is a set of* n *points in* $\mathbb{R}^d$ *and* $y \in conv(Y)$*. Then, there is a subset* X *of* Y *consisting of at most* $(d+1)$*-points such that* $x \in conv(X)$*.*

While the fuzzy version of Caratheodory's theorem is as folows:

THEOREM 3.4 (Fuzzy Caratheodory's theorem) *Let* $A = A_0 \cup A_1$ *be a set of fuzzy points and fuzzy directions, and let* $\mathcal{C}(A, n+1)$ *denote the set of all convex combinations of* $n+1$ *or fewer elements in* A*. Then,* $conv(A) \backsim \mathcal{C}(A, n+1)$.

In order to prove the fuzzy version of Radon's theorem we need the following:

THEOREM 3.5 (11, Prop. 3.3) *For any fuzzy set of fuzzy points* $\sigma = \{a^1_{\alpha_1}, \cdots, a^m_{\alpha_m}\}$ *in* $\mathbb{R}^d$*, if* $\mid \sigma_t \mid \geq d+2$*, then the fuzzy points are affinely dependent.*

PROPOSITION 3.6 (Fuzzy Radon Theorem) *Let* σ *be a fuzzy set in* $\mathbb{R}^d$ *and* $a^1_{\alpha_1}, \cdots, a^n_{\alpha_n}$ *its fuzzy points. Then, any* t*-level subset of* σ *of at least* n *fuzzy points* ($\mid \sigma_t \mid \geq n$) *in* $\mathbb{R}^d$ *can be partitioned into two disjoint fuzzy subsets* τ_1 *and* τ_2 *such that* $\tau_1 \vee \tau_2 = \sigma,\ \ \tau_1 \wedge \tau_2 = 0$ *and* $conv(\tau_1)_f \wedge conv(\tau_2)_f \neq \emptyset$.

Proof. Consider a fuzzy set σ in $\mathbb{R}^d$ with fuzzy points $a^1_{\alpha_1}, \cdots, a^n_{\alpha_n}$. Since $\mid \sigma_t \mid = n \geq d+2$, then these fuzzy points are affinely dependent by Theorem 3.5. Thus, there exists real numbers $\lambda_1, \cdots, \lambda_n$ not all of them zero such that the linear combination of the fuzzy points $\sum_{i=1}^{n} \lambda_i a^i_{\alpha_i} = 0$

with $\sum_{i=1}^{n} \lambda_i = 0$. Define the fuzzy sets

$$\beta_1 = \begin{cases} t, \text{ if } \lambda > 0 \\ 0, \text{ otherwise} \end{cases}$$

$$\beta_2 = \begin{cases} t, \text{ if } \lambda < 0 \\ 0, \text{ otherwise} \end{cases}$$

set

$$\tau_1 = \begin{cases} a^k_{\alpha_k}, \text{ if } \beta_1 = t, \ k = 1, \cdots, i \\ 0, \text{ otherwise} \end{cases}$$

$$\tau_2 = \begin{cases} a^j_{\alpha_j}, \text{ if } \beta_2 = t, \ j = i+1, \cdots, n \\ 0, \text{ otherwise} \end{cases}$$

These fuzzy sets $\tau_1 \neq 0$ and $\tau_2 \neq 0$, $\tau_1 \vee \tau_2 = \sigma$, $\tau_1 \wedge \tau_2 = 0$. It remain to show that $conv(\tau_1)_f \wedge conv(\tau_2)_f \neq \emptyset$.

Thus,

$$0 = \sum_{i=1}^{n} \lambda_i a^i_{\alpha_i} = \sum_{k \in \beta_1} \lambda_k a^k_{\alpha_k} + \sum_{j \in \beta_2} \lambda_j a_j^{\alpha_j}$$

with

$$\sum_{k \in \beta_1} \lambda_k + \sum_{j \in \beta_2} \lambda_j = 0.$$

Using,

$$\sum_{k \in \beta_1} \lambda_k = - \sum_{j \in \beta_2} \lambda_j = \lambda, \ \lambda > 0$$

this implies

$$\sum_{k \in \beta_1} \frac{\lambda_k}{\lambda} = - \sum_{j \in \beta_2} \frac{\lambda_j}{\lambda} = 1.$$

Define

$$b_\beta = \sum_{k \in \beta_1} \frac{\lambda_k}{\lambda} a^k_{\alpha_k} = - \sum_{j \in \beta_2} \frac{\lambda_j}{\lambda} a^j_{\alpha_j}.$$

this implies $b_\beta = \sum_{k \in \beta_1} \frac{\lambda_k}{\lambda} a^k_{\alpha_k}$ and $b_\beta = - \sum_{j \in \beta_2} \frac{\lambda_j}{\lambda} a^j_{\alpha_j}$

with

$$\sum_{k\in\beta_1}\frac{\lambda_k}{\lambda} = 1 = -\sum_{j\in\beta_2}\frac{\lambda_j}{\lambda}.$$

Therefore, b_β is a convex combination of fuzzy points in τ_1 and τ_2. Hence,

$$conv(\tau_1)_f \wedge conv(\tau_2)_f \neq \emptyset.$$

■

DEFINITION 3.7 *The partitions in the fuzzy Radon theorem are called* fuzzy Radon partitions *and a fuzzy point in the intersection of their convex fuzzy hulls is called* fuzzy Radon point.

PROPOSITION 3.8 (Fuzzy Helly Theorem) *Let $\sigma := \{\sigma_1, \cdots, \sigma_n\} \in \mathbb{R}^d$, be convex fuzzy sets with $\mid \sigma \mid \geq d+1$. Suppose that the intersection of every $d+1$ of these sets in nonempty, then the intersection of all the convex fuzzy sets is nonempty.*

Proof. For a fixed d, we proceed by induction on $\mid \sigma \mid$. The case $n = d+1$ holds from the statement of the theorem, thus we suppose that $\mid \sigma \mid \geq d+1$. Consider convex fuzzy set $\sigma = \{\sigma_1, \cdots, \sigma_{n+1}\}$ satisfying the hypothesis, that is; any $d+1$ of the sets have a nonempty intersection. For $i = 1, \cdots, n+1$, let $a^i_{\alpha_i} \in \bigcap_{j\neq i}\sigma_j$ be fixed and consider the fuzzy points $a^1_{\alpha_1}, \cdots, a^n_{\alpha_n}$. Define the fuzzy set $\eta := \{a^1_{\alpha_1}, \cdots, a^{n+1}_{\alpha_{n+1}}\}$. By Proposition 3.6, there exists $\eta_1 := \{a^k_{\alpha_k} | i \in \beta_1\}$ and $\eta_2 := \{a^j_{\alpha_j} | i \in \beta_2\}$ such that: (i) $\eta_1 \wedge \eta_2 = 0$ (ii) $\eta_1 \vee \eta_2 = \sigma$ and (iii) $conv(\eta_1)_f \wedge conv(\eta_2)_f \neq \emptyset$.

Let

$$b_\beta \in conv(\eta_1)_f \wedge conv(\eta_2)_f.$$

that is;

$$b_\beta = \sum_{i\in\eta_1}\lambda_i a^i_{\alpha_i} + \sum_{i\in\eta_2}\lambda_i a^i_{\alpha_i}$$

with

$$\sum_{i\in\eta_1}\lambda_i + \sum_{i\in\eta_2}\lambda_i = 1$$

Claim:

$$\sigma(b_\beta) \geq \sigma(a^1_{\alpha_1}) \wedge \cdots \wedge \sigma(a^n_{\alpha_n}).$$

Consider some $i \in [n]$, then $i \notin \eta_1$ or $i \notin \eta_2$. Formally, each $a^j_{\alpha_j}$ with $j \in \eta_1$ lies in σ_i thus

$$b_\beta = \sum_{j\in\eta_1}\lambda_j a^j_{\alpha_j}$$

then we have

$$\sigma(b_\beta) \geq \sigma(\sum_{j\in\eta_1}\lambda_j a^j_{\alpha_j}) \geq \sigma(a^j_{\alpha_j})$$

since each fuzzy point $a^j_{\alpha_j}$ with $j \in \eta_1$ lies in σ_i thus

$$b_\beta = \sum_{j \in \eta_1} \lambda_j a^j_{\alpha_j} \subseteq \sigma_i.$$

similarly, for $i \notin \eta_2$, each $a^j_{\alpha_j}$ with $j \in \eta_2$ lies in σ_i thus

$$b_\beta = \sum_{j \in \eta_2} \lambda_j a^j_{\alpha_j}.$$

then we have

$$\sigma(b_\beta) \geq \sigma(\sum_{j \in \eta_2} \lambda_j a^j_{\alpha_j}) \geq \sigma(a^j_{\alpha_j})$$

since each fuzzy point $a^j_{\alpha_j}$ with $j \in \eta_2$ lies in σ_i thus

$$b_\beta = \sum_{j \in \eta_2} \lambda_j a^j_{\alpha_j} \subseteq \sigma_i.$$

Therefore,

$$b_\beta = \sum_{i \in \eta_1} \lambda_i a^i_{\alpha_i} + \sum_{i \in \eta_2} \lambda_i a^i_{\alpha_i} = \sum_{i=1}^{n} \lambda_i a^i_{\alpha_i}.$$

Hence,

$$\sigma(b_\beta) = \sigma(\sum_{i=1}^{n} \lambda_i a^i_{\alpha_i}) \geq \sigma(a^1_{\alpha_1}) \wedge \cdots \wedge \sigma(a^n_{\alpha_n}).$$

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References

[1] N. R. Al-Mayahi and S. M. Ali, Affine Fuzzy Sets, *Journal of Qadisiyah Computer Science and Mathematics*, 2017, pp. 1-6.

[2] J. G. Brown, A note on fuzzy sets, *inform. and control*, **18** (1971) 32 - 39.

[3] Z. Feiyue, The recession cones and Carathéodory theorem *Fuzzy sets and Systems* **44** (1991) 57-69.

[4] A. K. Katsaras and D. B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, *J. Math. Anal. Appl.* **58** (1977) 135-146.

[5] Liaozu-Hua, Haocui-Yum and Yaoshi-Yan: R-generalised fuzzy convex sets, *Procedia Engineering* **29**, 2012, pp. 2230-2234.

[6] R. Lin, Note on Fuzzy sets, *Yugoslav Journal of Operations Research* **24 (2)**, 2014, pp. 299-303.

[7] Y. M. Liu, Some Properties of Convex Fuzzy Sets, *J. Math. Anal. Appl.* **111** (1985) 119-129.

[8] R. Lowen, Convex fuzzy sets, *Fuzzy sets and Systems* **3** (1980) 291-310.

[9] H. Martini, N. M. Nam and A. Robinson, On the Equivalence of the Theorems of Helly, Radon and Carathéodory via Convex Anlysis, *Journal of Convex Analysis*, 22(2), (2015), 591-601.

[10] Y. Maruyama, Lattice-Valued Fuzzy Convex Geometry, **1641**, (2009) 22-37.

[11] T. O. Sangodapo and Ajayi, On Affine and Convex Fuzzy Sets using Fuzzy Points *MAYFEB Journal of Mathematics* **3** 2017, 13 - 19.

[12] M. D. Weiss, Fixed point, separation and induced topologies for fuzzy sets, *J. Math. Anal. Appl.* **58** (1975) 142-150.

[13] X. M Yang and F. M. Yang, A Property on Convex Fuzzy Sets, *Fuzzy sets and Systems* **126(2)**, 2002, pp. 269-271.

[14] X. Yang, A Note on Convex Fuzzy Sets, *Fuzzy sets and Systems* **53** (1993) 117-118.

[15] X. Yang, Some Properties of Convex Fuzzy Sets, *Fuzzy sets and Systems* **72** (1995) 129-132.

[16] L. A. Zadeh, Fuzzy Sets, *Inform and Control* **8** (1965) 338-353.
[17] Y. Zhu, Convex hull and Closure of a Fuzzy Sets, *Journal of Intelligent and Fuzzy Systems* **16** (2005) 67-73.