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In honour of Prof. Ekhaguere at 70

Efficient portfolio selection of some assets in the Nigeria market

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Abstract. In any financial institution, an optimal portfolio of assets is correctly designed by some methods which include reliable mathematical programs called optimizers. Investors are often faced with the problem of selecting varying choices, as well as optimizing expected returns in the face of a high level of risk. This work therefore employs a quantitative approach to constructing portfolios by using a method of constrained optimization-the Lagrange multiplier method (LMM). The portfolio consisting of three assets namely: Stocks-Nigerian Stock Exchange Indices for a period of 10 years, Cash-Federal Government of Nigeria (FGN) 90 days Treasury Bills and Bonds-Federal Government of Nigeria (FGN) 12 year Bond was constructed. With the aid of a computational soft-package, the LMM was used to obtain an optimal solution for the minimization problem. The result obtained shows a higher expected return for both Cash and Bonds than Stocks in the period under study, which depicts the prevailing economy situation in Nigeria.

Keywords: optimal portfolio, Lagrange multiplier, option pricing, expected return.

1. Introduction

Optimality in any given investment is most times threatened in the face of high risk involved in the market. As a result of this, investors are often faced with the problem of selecting various types of assets in the market. This results to their finding means to averse the risk involved so they can combine portfolios that would make best yield of their expected returns; this is what the Portfolio theory in modern times is all about. The method of collecting different assets in their varying ratios as contained in a given portfolio so as to obtain the best yield of that given portfolio as compared to others is termed to be portfolio optimization. The main idea behind portfolio optimization is diversification. A strategy in portfolio optimization is such that when an asset's return decreases, another asset's return increases as a result of the correlation between the assets.

This paper would therefore use a quantitative approach to constructing portfolios by using a method of constrained optimization- Lagrange Multiplier to finding an optimal portfolio and to determine portfolios of some given assets that possess satisfactory features. The portfolio consisting of three assets namely: Stocks-Nigerian Stock Exchange Indices for a period of 10 years, Cash-Federal Government of Nigeria (FGN) 90 days Treasury Bills and Bonds-Federal Government of Nigeria (FGN 12 year Bond was constructed and used in this work. The remaining part of the paper is structured as follows: Related and existing literatures to the subject were extensively reviewed in section two. Section three dealt with methodology employed in this work while the fourth and fifth sections are on the analysis of data and discussion of results respectively.

2. Overview of portfolio optimization

Portfolio optimization is often said to be pioneered by Markowitz [1]. Markowitz described the Mean-Variance Model making an assumption that investors are risk averse. The model uses the

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mathematical concept of variance to quantify risk. However, it uses historical values for estimating correlations and covariance which neglects the fact of present circumstances. With respect to this, Chan et al. [2] described a method for forecasting covariance rather than relying on historical covariance.

Vercher et al. [3] proposed the use of fuzzy quantities in quantifying the uncertainty in the return of assets rather than the conventional use of probability and operation research as used by Markowitz [1].

Risk aversion is the behavior of investors when exposed to uncertainty, to attempt to reduce that uncertainty. Thomas et al. [4] presented a case in where the risk aversion of an investor is inversely proportional to his wealth rather than having a constant risk aversion. Hence, an analytic solution was provided where the equilibrium dollar value put into the risky asset is equivalent to the present value.

In attempt to finding the link between portfolio size and the risk involved, Kisaka [5] observed using the Nairobi Securities Exchange that portfolio risk reduced as the number of assets in the portfolio increased. It was also observed that beyond the optimal portfolio size the risk started rising again. Bekkers et al. [6] determined what asset classes add value to a traditional asset mix and to determine the optimal weights of all assets classes in the portfolio. Bekkers et al. [6] further suggests that adding real estate, commodities and high yield to the traditional asset mix delivers the most efficiency improving value for investors.

In further research, Offiong et al. [7] developed an optimal portfolio consisting of stocks from the Nigeria Stock Exchange. The difference between Offiong et al. [7] and this paper is the introduction of the varying risk aversion constant which gives the opportunity of selecting a portfolio based on an investor's risk appetite.

This works introduces the concept of risk aversion to selecting optimal portfolios in the Nigerian financial market comprising cash, stock and bond where the risk and return are based on the varying risk aversion constants. Also, stock prices can be modeled in terms of stochastic dynamics [8, 9].

3. Notes on existing model

We give attention to two properties of an asset namely: its mean rate of returns $\hat{m}_p(t)$ at time t and the volatility δ_p^2 of the returns with respect to the risk involved in a specified time interval [10]. The rate of return of the portfolio $\hat{r}$ at any time t is given by:

$$\hat{r}(t) \quad = \sum_{i=1}^{n} w_i \hat{r}_i \tag{1}$$

while the expected rate of return $\hat{m}_p$ of the portfolio is given by:

$$\hat{m}_p = E(\sum_{i=1}^{n} w_i \hat{r}_i) = \sum_{i=1}^{n} w_i \hat{m}_i \tag{2}$$

where the linearity property of expectations has been used. The volatility δ_p^2 of the returns is given by:

$$\begin{aligned} \delta_p^2 &= E(|\hat{r}_p - \hat{m}_p|^2) \\ &= E(|\sum_{i=1}^{n} w_i(\hat{r}_i - \hat{m}_i)|^2) \\ &= \sum_{i,j=1}^{n} w_i w_j \delta_{ij} \\ &= w^T V w. \end{aligned} \tag{3}$$

3.1 *Efficient frontier*

The efficient frontier is the curve that shows all efficient portfolios in a risk-return framework. An efficient portfolio is defined as the portfolio that maximizes the expected return for a given amount of risk (standard deviation), or the portfolios that minimizes the risk subject to a given expected return. Therefore, our optimization problem is:

$$\text{Minimize: } \delta_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \delta_{ij} \quad [w_1, ..., w_n]$$

subject to:

$$\sum_{i=1}^{n} w_i = 1$$

$$\sum_{i=1}^{n} w_i \hat{m}_i = \hat{m}_p.$$

We now solve the general unconstrained problem of optimizing portfolio asset allocation when the choices available are all linear combinations of a finite number of assets. We only treat the unconstrained problem in this paper. All values for the asset proportions are permitted, even those outside the range [0,1], which represent short-selling and leverage.

Let:
n = number of assets.
w_i = proportion of portfolio invested in asset i; $1 \leq i \leq n$.
w = column vector of proportions w_i.
α_i = expected instantaneous return of asset i; $1 \leq i \leq n$.
x = column vector of expected returns α_i.
$\hat{r}_{i,j}$ = covariance of asset i with asset j; $1 \leq i \leq n$ and $1 \leq j \leq n$.
V = n × n matrix of covariances $\rho_{i,j}$.
α_p = expected instantaneous return of portfolio.
δ_p = instantaneous standard deviation of portfolio.
A = iso-elastic coefficient of relative risk aversion.

By the Portfolio Choice Theorem, the problem is to maximize $f(w)$ where:

$$\begin{aligned} f(w) &= \hat{m}_p - \frac{1}{2}A\delta_p^2 \\ &= w'x - \frac{1}{2}Aw'Vw \\ &= \sum_{i=1}^{n} w_i\hat{m}_i - \frac{1}{2}A\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \hat{r}_{i,j} \end{aligned}$$

subject to the budget constraint:

$$\sum_{i=1}^{n} w_i = 1.$$

To deal with the budget constraint, we introduce a Lagrange multiplier λ and a new objective function $\hat{f}$ with no constraints:

$$\begin{aligned}\hat{f}(w,\lambda) &= f(w) + \lambda\Big(1 - \sum_{i=1}^{n} w_i\Big)\\ &= w'x - \frac{1}{2}Aw'Vw + \lambda\Big(1 - \sum_{i=1}^{n} w_i\Big)\\ &= \sum_{i=1}^{n} w_i\hat{m}_i - \frac{1}{2}A\sum_{i=1}^{n}\sum_{j=1}^{n} w_iw_j\hat{r}_{i,j} + \lambda\Big(1 - \sum_{i=1}^{n} w_i\Big).\end{aligned} \tag{4}$$

To solve the problem, we want to take the $n+1$ partial derivatives of $\hat{f}$ and set them equal to 0.

$$\begin{aligned}\frac{\partial \hat{f}}{\partial w_i} &= \hat{m}_i - A\sum_{j=1}^{n}\hat{r}_{i,j}w_j - \lambda = 0 \qquad (for 1 \leq i \leq n)\\ \frac{\partial \hat{f}}{\partial \lambda} &= 1 - \sum_{i=1}^{n} w_i = 0.\end{aligned} \tag{5}$$

Rewrite these equations as:

$$\sum_{j=1}^{n}\hat{r}_{i,j}w_j + \frac{\lambda}{A} = \frac{\hat{m}_i}{A} : \qquad (for 1 \leq i \leq n) \tag{6}$$

$$\sum_{i=1}^{n} w_i = 1. \tag{7}$$

Note that this is the budget constraint. This is a set of $n+1$ linear equations in $n+1$ unknowns which we can solve using linear algebra. Define vectors and matrices as follows:

$$\hat{V} = \begin{bmatrix}\hat{r}_{1,1} & \dots & \hat{r}_{1,n} & 1\\ \vdots & & \vdots & \vdots\\ \hat{r}_{n,1} & \dots & \hat{r}_{n,n} & 1\\ 1 & \dots & 1 & 0\end{bmatrix}$$

$$\hat{w} = \begin{bmatrix}w_1\\ \vdots\\ w_n\\ \frac{\lambda}{A}\end{bmatrix} \hat{x} = \begin{bmatrix}\hat{m}_1\\ \vdots\\ \hat{m}_n\\ 0\end{bmatrix} \hat{y} = \begin{bmatrix}0\\ \vdots\\ 0\\ 1.\end{bmatrix}$$

Then equations (3.6) and (3.7) become

$$\hat{V}\hat{w} = \frac{1}{A}\hat{x} + \hat{y}. \tag{8}$$

Let

$$\hat{c} = \hat{V^{-1}}\hat{x} \tag{9}$$

$$\hat{d} = \hat{V^{-1}}\hat{y}. \tag{10}$$

Our solution is:

$$\hat{w} = \frac{1}{A}\hat{c} + \hat{d}. \tag{11}$$

Note that the column vectors $\hat{c}$ and $\hat{d}$ depend only on the expected returns and covariances of the assets and are independent of the coefficient of relative risk-aversion A. Each optimal asset proportion is:

$$w_i = \frac{1}{A}c_i + d_i. \tag{12}$$

For an infinitely risk-averse investor with $A \to \infty$, the solution becomes simply $w_i = d_i$, and the resulting portfolio has minimum variance. The sign of c_i determines whether other investors (with $A < \infty$) have more or less than d_i invested in asset i, and whether as investors become more risk-averse the proportion of asset i increases ($c_i < 0$) or decreases ($c_i > 0$). The computation of the solution vectors $\hat{c}$ and $\hat{d}$ is done using Microsoft Excel.

4. Analysis and Discussion of Result

We begin by computing the expected return of the three assets using their historical values. Cash, Bond and Stock all have expected return of 8.51%, 12.12% and 4.27% respectively. Also, the standard deviation (which represents the risk) of the three assets are 0.037, 0.022 and 0.36 respectively for cash, bond and stocks. We see that stocks has the least return but the highest standard deviation. We then proceed to compute the covariances and correlation among the three assets.

Table 1. Expected Return and Risk

	cash	bond	stock	
sigma	0.03709	0.02226	0.3624	S.D of Return
alpha	8.51%	12.12%	4.27%	Expected Return

Table 2. Correlation

	cash	bond	stock
cash	1	0.7299	0.2492
bond	0.7299	1	0.1319
stock	0.2492	0.1319	1

Table 3. Covariance

	cash	bond	stock
cash	0.001375	0.000602	0.00335
bond	0.000602	0.000495	0.001064
stock	0.00335	0.001064	0.131357

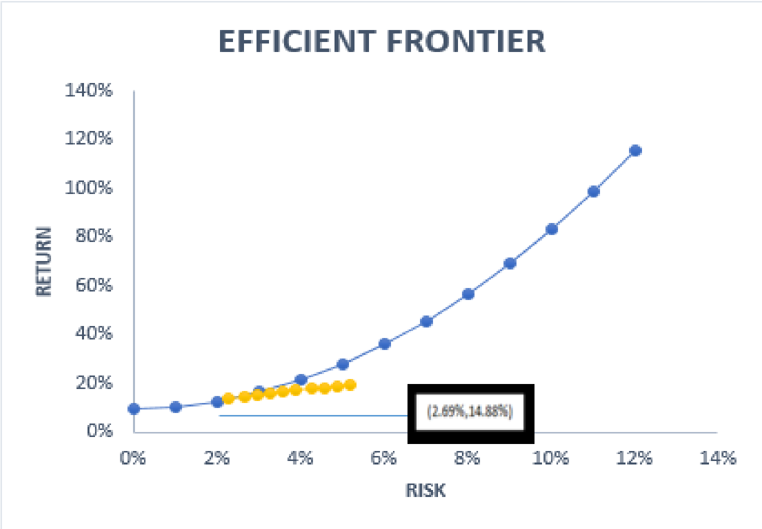

Fig. 1: Efficient portfolio-graphic

Table 2 shows the correlation among each assets. A popular strategy is to choose a portfolio which consists of various assets having both positive and negative correlations. A negative correlation implies that as one asset increases in value, another asset decreases in value and vice versa. In Table 1, we see all combinations of assets are positively correlated. This reflects the interplay of CBN interbank rate on both the fixed income assets(bond and treasury bill) and the stock market.

Table 3 shows the riskiness of each combination of assets. The idea here is to choose assets with minimum risk putting in mind the correlation among the assets.

The efficient frontier in Fig. 1 shows various efficient portfolios. The yellow line is the efficient frontier curve showing all efficient portfolios, while the blue line is the market line. The point where the efficient frontier curve touches the market line is the tangency portfolio showing the optimal portfolio and from the graph, this is $(2.69\%, 14.88\%)$. This means we choose a portfolio having a risk of 2.69% and return of 14.88%. This portfolio comprises -77% cash and 177% bond with nothing invested in stocks.The negative allocation for cash means that an average investor would go short on cash and invest in bonds.

5. Conclusion

This study shows that an optimal portfolio is the one with a return of 14.88% and risk of 2.69% and the allocation would be -77% cash and 177% bond with nothing invested in stocks. This shows the prevailing economic situation in Nigeria. Further research can be done on this study which could include portfolio re-balancing where an investor would know how to re-balance his portfolio after implementing the optimization. Lastly, this work does not show a specific strategy but helps an investors to be able to allocate his/her wealth to an asset depending on the investor's risk appetite.

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