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In honour of Prof. Ekhaguere at 70

3-step block hybrid linear multistep methods for solution of special second order ordinary differential equations

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Abstract. This paper presents set of two Implicit Hybrid Block Methods which are derived through multistep collocation method using power series as a basis function for generating solution of special second order ordinary differential equations. The derived continuous forms which are evaluated at some grids and off-grid points of collocation and interpolation to form the block hybrid methods for step number k=3. The discrete schemes obtained possess uniformly high order and found to be zero-stable, consistent and hence convergent. Some numerical examples are given to test the accuracy and efficiency advantages. The results of our evaluation show that our methods outperform reviewed work.

Keywords: Linear Multistep Method (LMM), hybrid, block, implicit K-step, convergence, error constant.

1. Introduction

Numerical methods are becoming more important in applications of engineering, science and social science due to the difficulties experienced in obtaining the analytical solution. Consider the special second order ordinary differential equation of the type

$$y^{''} = f\left(x, y\right), y\left(a\right) = y_0, y^{'}\left(a\right) = y_0^{'}, \tag{1.1}$$

Numerical methods need to be developed for such problem and many researchers that have worked extensively in this area see Lambert [1], Yahaya and Adegboye [7], to mention but few. The main aim of this research paper is to develop hybrid block linear multistep method when k=3 with one off-grid point at both collocation and interpolation respectively that can be used to solve special second order ordinary differential equations.

Definition 1.1 Hybrid method
Hybrid method was as a result of the desire to increase the order without increasing the step number and without reducing the stability interval. Therefore, a k-step hybrid method is defined as:

$$\sum_{j=o}^{k} \alpha_j \, y_{n+j} = h^2 \left[\sum_{j=0}^{k} \beta_j \, f_{n+j} + \beta_v \, f_{n+v} \right] \tag{1.2}$$

Whereα_k=1, just to remove arbitrariness, $\alpha_0 \, and \, \beta_0$are not both zero and. $v \notin [0, 1, 2, ..., k \; \left[A^{(0)}\right]^{-1}$ which is the off grid function of evaluation see Lambert [1].

Definition 1.2 Linear Multistep Method (LMM)
Linear multistep method is the computational method which is used to determine the sequence [y_n] and it is a linear relationship between y_{n+j} and f_{n+j}, j = 0, 1, 2, . . . , k. See [1].

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A linear k-step method of order two is mathematically expressed as:

$$\sum_{j=0}^{k} \alpha_j \, y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j \, f_{n+j} \tag{1.3}$$

where $\alpha_k + \beta_k \neq 0$. Where $\beta_k = 0$ is an explicit scheme and $\beta_k \neq 0$is an implicit scheme see Lambert [1].

Definition 1.3 Order and error constant
The linear multistep method of type (1.2) is said to be of order p if $C_0 = C_1 = \ldots C_{p+1} = 0$, but C_{p+2}? 0 and C_{p+2} is called the error constant, see Lambert [1]

Definition 1.4 Convergences
The necessary and sufficient conditions for linear multistep method of type (1.2) is said to be convergent if and only if it is consistent and zero-stable.

Definition 1.5 Stability regions
The stability region of linear multistep method is part of the complex plane where the method when applied to the test equation$y'' = \lambda^2 y$ is absolute stable whose resultant finite difference equation has characteristics equation $\pi(z, r) = \rho(r) - z^2\sigma(r)$, $z = i\lambda h$, see Awoyemi [3].

2. Derivation of the Methods

The methods are derived for the special second order ordinary differential equation base on the multistep collocation. The general power series is used as our basis function to produce an appropriate solution to (1.3) as follows

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j \, x^j \tag{2.1}$$

and

$$y''(x) = \sum_{j=2}^{t+m-1} j(j-1) \, \alpha_j \, x^{(j-2)} \tag{2.2}$$

where $\alpha_{js}$the parameters to be determined, t and m are the points of interpolation and collocation respectively. This process leads to (t+m-1) non-linear system of equations with (t+m-1) unknown coefficients, which are to be determined by the use of maple 13 mathematical software.

I. Derivation of the hybrid block method when k=3 with one off-grid point at collocation.
Using equations (2.1) and (2.2) with t=2, m=5. The degree of our polynomial is (t+m-1). Equations (2.1) was interpolated at x=$x_{n+j, j=0,1}$ and (2.2) collocated at x=$x_{n+j, j=0,1,\frac{6}{5},2,3}$which gives the following non-linear system of equation of the form as follows

$$\sum_{j=0}^{t+m-1} \alpha_j \, x_{n+i}^{j} = y_{n+i}, i = 0, 1 \tag{2.3}$$

$$\sum_{j=2}^{t+m-1} j(j-2) \, \alpha_j \, x_{n+i}^{(j-2)} = f_{n+i}, i = 0, 1, \frac{6}{5}, 2, 3 \tag{2.4}$$

With maple 13 software, we obtain the continuous formulation of equations (2.3) and (2.4) as follows

$$y(x) = \left(\frac{x_{n+1}}{h} - \frac{x}{h}\right) y_n + \left(\frac{-x_{n+1}+h}{h} + \frac{x}{h}\right) y_{n+1} +$$

$$h^2 \left[\left(-\frac{1}{72}\ \frac{(-x_{n+1}h)\, x_{n+1}\left(2x_{n+1}h^2 + 6x_{n+1}^2 h + 3x_{n+1}^3 + 2h^3\right)}{h^3}\right.\right.$$

$$+\frac{1}{72}\frac{\left(12x_{n+1}^2\ h^2 -\ 12x_{n+1}^3 h - 15x_{n+1}^4 + 2h^4\right) x}{h^3}$$

$$-\frac{1}{12}\frac{x_{n+1}\left(2h^2 - 3x_{n+1}h - 5x_{n+1}^2\right) x^2}{h^3} + \frac{1}{36}\frac{\left(2h^2 - 6x_{n+1}h - 15x_{n+1}^2\right) x^3}{h^3}$$

$$+\frac{1}{24}\frac{(h + 5x_{n+1})\, x^4}{h^3} - \frac{1}{24}\frac{x^5}{h^3}\,] f_n + [$$

$$-\frac{25}{504}\frac{(-x_{n+1}h)\, x_{n+1}\left(-3x_{n+1}^3 + 7x_{n+1}h^2 - 3x_{n+1}^2 h + 7h^3\right)}{h^3}$$

$$+\frac{25}{504}\frac{\left(15x_{n+1}^4 - 30x_{n+1}^2 h^2 + 7h^4\right) x}{h^3}$$

$$+\frac{125}{84}\frac{(-x_{n+1}h)\, x_{n+1}\,(x_{n+1}h)\, x^2}{h^3} - \frac{125}{252}\frac{\left(h^2 - 3x_{n+1}^2\right) x^3}{h^3}$$

$$-\frac{125}{168}\frac{x_{n+1}x^4}{h^3} + \frac{25}{168}\frac{x^5}{h^3}\,)\, f_{n+\frac{3}{5}} +$$

$$\left(-\frac{1}{24}\frac{(-x_{n+1} + h)\, x_{n+1}\left(3x_{n+1}^3 + x_{n+1}^2 h - 9x_{n+1}h^2 + 3h^3\right)}{h^3}\right.$$

$$+\frac{1}{24}\frac{\left(30x_{n+1}^2 h^2 + 8x_{n+1}^3 h - 15x_{n+1}^4 - 24x_{n+1}h^3 + 3h^4\right) x}{h^3}$$

$$+\frac{1}{4}\frac{(-x_{n+1} + h)\left(2h^2 -\ 3x_{n+1}\ h - 5x_{n+1}^2\right) x^2}{h^3}$$

$$\left.+\frac{1}{12}\frac{\left(4x_{n+1}h - 15x_{n+1}^2 + 5h^2\right) x^3}{h^3} - \frac{1}{24}\frac{(2h - 15x_{n+1})\, x^4}{h^3} - \frac{1}{8}\ \frac{x^5}{h^3}\right) f_{n+1} +$$

$$\left(-\frac{1}{168}\,\frac{\left(-x_{n+1}+h\right)x_{n+1}^{3}\left(-3x_{n+1}+4h\right)}{h^{3}}\right.$$

$$+\frac{1}{168}\frac{x_{n+1}^{2}\left(15x_{n+1}^{2}-28x_{n+1}\,h+12h^{2}\right)x}{h^{3}}$$

$$-\frac{1}{28}\frac{\left(-x_{n+1}+h\right)x_{n+1}\left(-5x_{n+1}+2h\right)x^{2}}{h^{3}}$$

$$\left.+\frac{1}{84}\frac{\left(15x_{n+1}^{2}-14x_{n+1}\,h+2h^{2}\right)x^{3}}{h^{3}}+\ \frac{1}{168}\frac{\left(-15x_{n+1}+7h\right)x^{4}}{h^{3}}+\ \frac{1}{56}\,\frac{x^{5}}{h^{3}}\right)\,f_{n+2}] \tag{2.5}$$

when equation (2.5) is evaluated at x=x_{n+j} where $j=\frac{6}{5},2,3$ and its first derivative also evaluated at $x=x_n$ gives the following set of discrete schemes that form the first hybrid block method when k=3 with one off-grid point at collocation.

$$\left.\begin{array}{l}
y_{n+3}-3y_{n+1}+2y_n=\frac{h^2}{2}[\frac{1}{4}\,f_n+5f_{n+1}-\frac{125}{72}\,f_{n+\frac{6}{5}}+\frac{19}{8}\,f_{n+2}+\frac{1}{9}\,f_{n+3}]\\
y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{12}[\frac{5}{6}\,f_n+13f_{n+1}-\frac{125}{36}\,f_{n+\frac{6}{5}}+\frac{7}{4}\,f_{n+2}-\frac{1}{9}\,f_{n+3}]\\
y_{n+\frac{6}{5}}-\frac{6}{5}\,y_{n+1}+\frac{1}{5}\,y_n=\frac{h^2}{500}[\frac{1651}{250}\,f_n+\frac{15601}{125}\,f_{n+1}-\frac{2821}{36}\,f_{n+\frac{6}{5}}+\frac{3811}{500}\,f_{n+2}-\frac{757}{1125}\,f_{n+3}]\\
hz_0-y_{n+1}+y_n=\frac{h^2}{12}[-\frac{53}{20}\,f_n-\frac{143}{10}\,f_{n+1}+\frac{875}{72}\,f_{n+\frac{6}{5}}-\frac{53}{40}\,f_{n+2}+\frac{11}{90}\,f_{n+3}]
\end{array}\right\} \tag{2.6}$$

Equation (2.6) having uniform order five (5) with error constant as follows

$$\left(\frac{1}{750},\frac{11}{6000},\frac{28373}{93750000},\frac{617}{252000}\right)^{T}.$$

II. Derivation of the second block method when k=3 with one off-grid point at interpolation as follows: equation (2.1) was interpolated at x=$x_{n+j}, j=0,1,\frac{6}{5}$ and equation (2.2) collocated at $x=x_{n+j}, j=0,1,2,3$ which gives the system of nonlinear equations of the form as follows:

$$\sum_{j=0}^{t+m-1}\alpha_j\,x_{n+u}^{j}=y_{n+u}, u=0,1,\frac{6}{5} \tag{2.7}$$

$$\sum_{j=2}^{t+m-1}j(j-2)\,\alpha_j\,x_{n+v}^{(j-2)}=f_{n+v}, v=0,1,2,3 \tag{2.8}$$

Adopting the previous procedure in the first block to generate the continuous formula and this continuous formula is evaluated at x=$x_{n+j}, j=2,3$. Its second derivative is evaluated at $x=x_{n+\frac{6}{5}}$ and first derivative is evaluated at $x=x_n$ gives the second hybrid block method when k=3 with one off-grid point at interpolation as follows:

$$\left.\begin{aligned}
&y_{n+3}-\tfrac{15625}{2821}\,y_{n+\frac{6}{5}}+\tfrac{10287}{2821}\,y_{n+1}+\tfrac{2517}{2821}\,y_n=\tfrac{h^2}{62}[\tfrac{45}{14}\,f_n+\tfrac{12609}{182}\,f_{n+1}+\tfrac{12447}{182}\,f_{n+2}+\tfrac{711}{182}\,f_{n+3}]\\
&y_{n+2}-\tfrac{15625}{8463}\,y_{n+\frac{6}{5}}+\tfrac{608}{2821}\,y_{n+1}+\tfrac{5338}{8463}\,y_n=\tfrac{h^2}{7}[\tfrac{88}{279}\,f_n+\tfrac{1756}{403}\,f_{n+1}+\tfrac{332}{403}\,f_{n+2}-\tfrac{172}{3627}\,f_{n+3}]\\
&\tfrac{18000}{2821}\,y_{n+\frac{6}{5}}-\tfrac{21600}{2821}\,y_{n+1}+\tfrac{3600}{2821}\,y_n=\tfrac{h^2}{875}[\tfrac{2286}{31}\,f_n+\tfrac{561636}{403}\,f_{n+1}-875f_{n+\frac{6}{5}}+\tfrac{34299}{403}\,f_{n+2}-\tfrac{3028}{403}\,f_{n+3}]\\
&hz_0+\tfrac{15625}{2418}\,y_{n+\frac{6}{5}}-\tfrac{3528}{403}\,y_{n+1}+\tfrac{5543}{2418}\,y_n=\tfrac{h^2}{5}[-\tfrac{21}{31}\,f_n+\tfrac{849}{403}\,f_{n+1}-\tfrac{24}{403}\,f_{n+2}+\tfrac{3}{403}\,f_{n+3}]
\end{aligned}\right\} \tag{2.9}$$

Equation (2.9) having uniform order five (5), with error constant as follows

$$\left(\frac{387}{1128400},\frac{809}{634725},\frac{85119}{44078125},\frac{139}{282100}\right)^T$$

3. Further analysis

The convergence analysis of all the block hybrid methods are determined using Fatunla (1991) approach which state that the block method is presented as a single block r-point multistep method as follows

$$A^{(0)}Y_m=\sum_{i=1}^{k}A^{(i)}Y_{m-i}+h^2\sum_{i=0}^{k}B^{(i)}F_{m-i} \tag{3.1}$$

where h is fixed mesh size within a block, $A^{(i)}$, $B^{(i)}$, $i=0\,(1)\,k$ are r x r matrix coefficients and $A^{(0)}$ is r by r identity matrix, Y_m,Y_{m-i}, F_m and $F_{m-i}$are vectors of numerical estimates.

The method in (2.6) is presented in matrix form as

$$\begin{bmatrix}-1&0&0&0\\-\frac{6}{5}&1&0&0\\-2&0&1&0\\-3&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{6}{5}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&0&-1\\0&0&0&-\frac{1}{5}\\0&0&0&-1\\0&0&0&-2\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-\frac{9}{5}}\\y_{n-1}\\y_n\end{bmatrix}+h^2\left[\begin{bmatrix}-\frac{143}{120}&\frac{875}{864}&-\frac{53}{480}&\frac{11}{1080}\\\frac{15601}{62500}&-\frac{2821}{18000}&\frac{3811}{250000}&-\frac{757}{562500}\\\frac{13}{12}&-\frac{125}{432}&\frac{7}{48}&-\frac{1}{108}\\\frac{5}{2}&-\frac{125}{144}&\frac{19}{16}&\frac{1}{8}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{6}{5}}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\right.$$

$$\begin{bmatrix}0&0&0&-\frac{53}{240}\\0&0&0&\frac{1651}{125000}\\0&0&0&\frac{5}{72}\\0&0&0&\frac{1}{8}\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-\frac{9}{5}}\\f_{n-1}\\f_n\end{bmatrix} \tag{3.2}$$

We normalize the above block method (3.2) by multiplying matrices $A^{(0)},A^1,B^{(0)}\,and\,B^1$, with inverse of $A^{(0)}$ to obtain the below method as follow

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{6}{5}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-\frac{9}{5}}\\y_{n-1}\\y_n\end{bmatrix}+h^2\left[\begin{bmatrix}\frac{143}{120}&-\frac{875}{864}&\frac{53}{480}&-\frac{11}{1080}\\\frac{26244}{15625}&-\frac{343}{250}&\frac{4617}{31250}&-\frac{212}{15625}\\\frac{52}{15}&-\frac{125}{54}&\frac{11}{30}&-\frac{4}{135}\\\frac{243}{40}&-\frac{125}{32}&\frac{243}{160}&\frac{1}{40}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{6}{5}}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\right.$$

$$\begin{bmatrix} 0\,0\,0 & -\frac{53}{240} \\ 0\,0\,0 & \frac{4347}{15625} \\ 0\,0\,0 & \frac{23}{45} \\ 0\,0\,0 & \frac{63}{80} \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-\frac{9}{5}} \\ f_{n-1} \\ f_n \end{bmatrix} \tag{3.3}$$

The block method (3.3) is the normalized form of the above schemes for the block hybrid method (2.6). Consider the first characteristics polynomial of the block method (2.6) defined as $\rho(\lambda) = \det\left[\lambda \mathrm{I} - A_1^1\right]$ see [1]. Substituting the values of $\lambda \mathrm{I}$ and A_1^1in the function above, gives

$$\left| \lambda \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} - \begin{bmatrix} 0\,0\,0\,1 \\ 0\,0\,0\,1 \\ 0\,0\,0\,1 \\ 0\,0\,0\,1 \end{bmatrix} \right| = \begin{vmatrix} \lambda\,0\,0 & -1 \\ 0\,\lambda\,0 & -1 \\ 0\,0\,\lambda & -1 \\ 0\,0\,0 & \lambda-1 \end{vmatrix}.$$

Solving the above determinant yield the following solution

$$\rho(\lambda) = \lambda^3(\lambda - 1) = 0 \tag{3.4}$$

Since, $\lambda_1 = \lambda_2 = \lambda_3 = 0, \lambda_4 = 1$. That is to say the block method is zero-stable and consistent and its order $(5,5,5,5)^T > 1$, as stated by Henrici [6] and Fatunla [4], hence the block method is convergent. The same analysis holds for the second block methods (2.9). Thus they are zero-stable, consistent and convergent.

3.1 *Region of Absolute Stability (RAS)*

The regions of absolute stability of all the block methods derived are determine by reformulating them as general linear method expressed as follows

$$\begin{bmatrix} Y \\ \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & U \\ \\ B & V \end{bmatrix} \begin{bmatrix} h^2 f(Y) \\ \\ Y_{i-1} \end{bmatrix} \begin{bmatrix} Y \\ \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & U \\ \\ B & V \end{bmatrix} \begin{bmatrix} h^2 f(Y) \\ \\ Y_{i-1} \end{bmatrix} \tag{3.5}$$

$i = 1, 2, ..., N$. Applying (3.5) to the test equation $y'' = \lambda^2 y$. its lead to a recursion of form:

$$M(z) := V + zB\,(I - zA)^{-1}\,U, \tag{3.6}$$

where $z = \lambda h$, equation (3.6) is the stability matrix and the stability function is

$$\rho(\eta, z) = \det[\eta I - M(z)] \tag{3.7}$$

Computing the stability function gives the stability polynomial of the methods which is plotted to produce the required absolute stability region of the method. To plot the absolute stability region of equation (2.6) is expressed in the form of equation (3.5) and the values of the matrices A, B, U and V are substituted into equations (3.6) and (3.7), with the aids of maple software, we obtained the characteristics polynomial and the stability function. These values of the stability function and characteristics polynomial are used in matlab programme to obtain the region of absolute stability as shown below in Figure 1.

The same analysis holds for the second block methods (2.9), as shown in Figure 2.

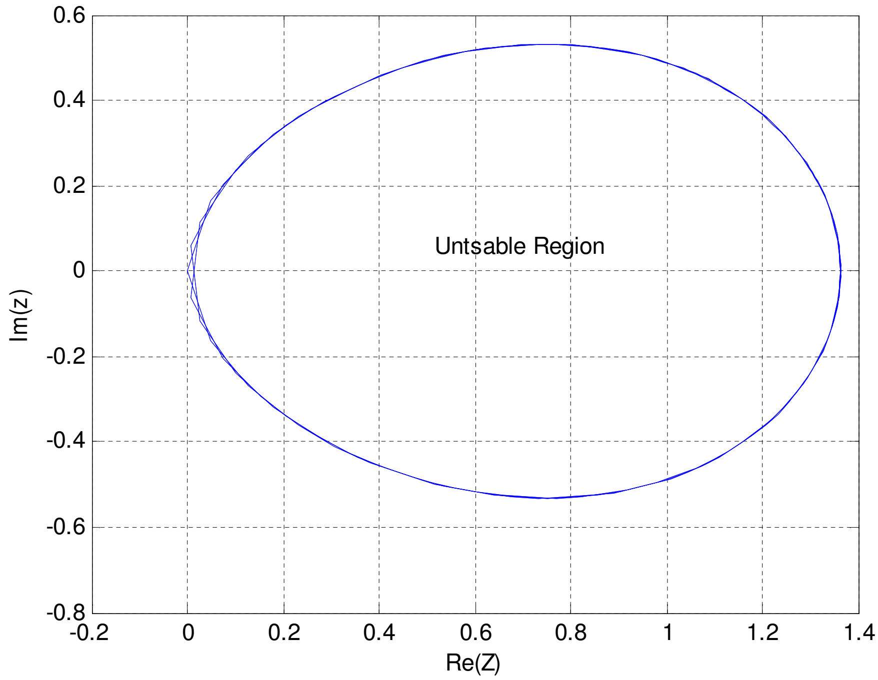

Figure 1. Region of absolute stability when k=3 with one off-grid point at collocation and the block method is A-stable

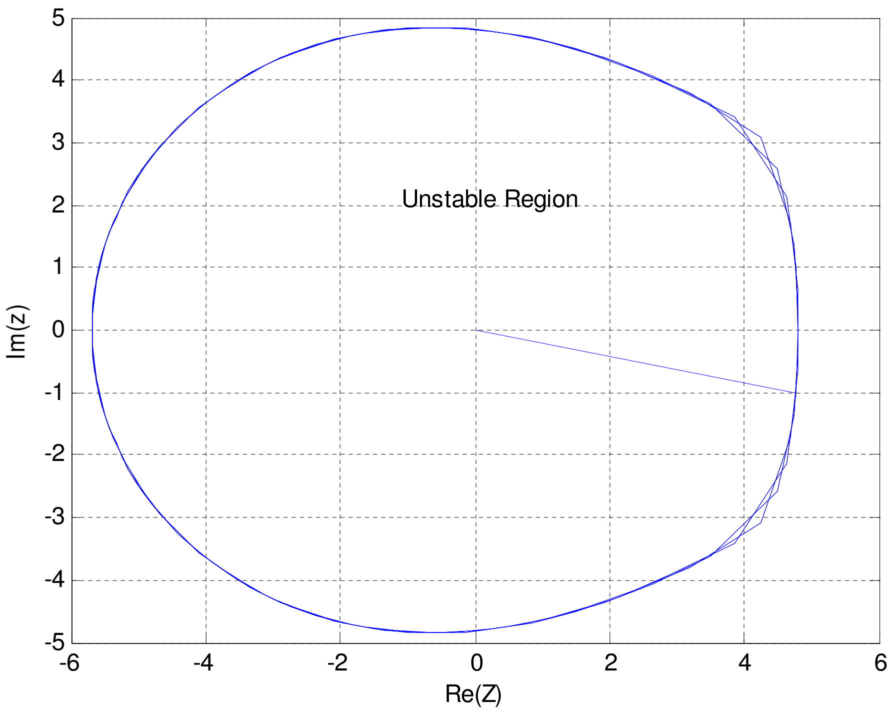

Figure 2. The block method is A (α)-stable

4. Numerical experiment

This section deals with numerical experiment by considering the derived discrete schemes in block form for solution of stiff and non-stiff differential equations of second order ordinary differential equation for case k=3.

Problem 1: Consider the problem solved by Yahaya and Muhammad (2016)

$$y'' = -y; y(0) = 1, y'(0) = 1,$$

$h = 0.1, 0.1 \leq x \leq 0.4$. Exact solution:$y(x) = Sinx + Cosx$.

Problem 2: We consider the stiff differential equation

$$y'' = 2\,y^3; y(1) = 1, y'(1) = -1,$$

$h = 0.1, 0.1 \leq x \leq 0.4$. Exact solution: $y(x) = \frac{1}{x}$.

Table 1: Comparison of Errors when k=3 at Collocation for Problem 1

x	Exact Solution	Yahaya and Muhammad (2016)	Error of Proposed Method
0.1	1.094837582	1.47E-07	1.000000000E-09
0.2	1.178735909	1.99E-07	3.000000000E-09
0.3	1.250856696	4E-09	1.000000000E-09
0.4	1.310479336	4.6E-08	9.000000000E-08

Table 2: Comparison of Errors when k=3 at Interpolation for Problem 1

x	Exact Solution	Yahaya and Muhammad (2016)	Error of Proposed Method
0.1	1.094837582	1.47E-07	0.000000000E-00
0.2	1.178735909	1.99E-07	1.000000000E-09
0.3	1.250856696	4E-09	2.000000000E-09
0.4	1.310479336	4.6E-08	1.000000000E-09

Table 3: Results for the derived method (2.6) at Collocation with One Point Off-Grid when k=3 for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091412	5.03E-07
0.2	0.833333333	0.833334723	1.39E-06
0.3	0.769230769	0.769232741	1.972E-06
0.4	0.714285714	0.714482411	1.96697E-04

Table 4: Results for the derived method (2.9) at Interpolation with One Point Off-Grid when k=3 for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091410	5.01E-07
0.2	0.833333333	0.833334721	1.388E-06
0.3	0.769230769	0.769232737	1.968E-06
0.4	0.714285714	0.714023971	2.61743E-04

5. Conclusion

All the derived hybrid block methods developed for the step number k=3 can be used for the solution of special second order ordinary differential equation of type (1.1). The derived methods were implemented in block mode which have the advantages of being self-stating, uniformly of order five (5) respectively, and do not need predictors. The stability domains of the methods are presented in figures 1 and 2. Maple13 and Matlab 2013 software packages were employed to generate the schemes and results. Also, each of the new block methods displayed its superiority over Yahaya and Muhammad (2016).

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