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In honour of Prof. Ekhaguere at 70

Convergence of a finite element solution for a nonlinear parabolic equation with discontinuous coefficient

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Abstract. Solution of a nonlinear parabolic interface problem with Finite Element-Backward Difference Scheme (FE-BDS) is presented. The convergence of the scheme on a two-dimensional convex polygonal domain is analyzed. Error estimates of optimal order in the $L^2(0,T;L^2(\Omega))$-norm and $L^2(0,T;H^1(\Omega))$-norm are determined for spatially discrete scheme. A fully discrete scheme based on 2-step BDS is analyzed. Numerical experiment is presented to support the theoretical result. It is assumed that the interface could be fitted exactly.

Keywords: interface, semi-discrete, fully discrete, optimal error estimates

1. Introduction

Time evolution equations (which in some cases lead to parabolic PDEs) are considered to study and understand the dynamics of nature. The most well-known linear parabolic PDE is the heat equation. However, the heat equation has some limitations which could be addressed with the nonlinear generalizations of the heat equation [5]. Nonlinear PDEs appear for example in non-Newtonian fluids, glaceology, rheology, nonlinear elasticity, flow through a porous medium, and image processing [5]. The problem becomes an interface problem when more than one material medium with different properties such as the conductivities, diffusion constants, are involved.

Parabolic interface problems are frequently encountered in scientific computing and industrial applications. However, the solutions of interface problems may have higher regularities in each individual material region than in the entire physical domain because of the discontinuities across the interface [2,4]. Thus, achieving higher order accuracy may be difficult using the classical method, hence there is need to find the solution to the problem by variational formulation.

Babuska [2] studied finite element approximation to elliptic interface problems on smooth domains with a smooth interface and formulated the problem as an equivalent minimization problem. For more works on Linear elliptic interface problems, see [3,6,11,12,16].

Using backward Euler time discretization, Chen and Zou [4] studied the convergence of fully discrete solution to the exact solution using fitted FEM. They proved suboptimal error estimates in $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$ norms when global regularity of the solution is low. Sinha et al [20] proposed and analyzed an unfitted finite element discretization for both elliptic and parabolic problems with discontinuous coefficients. An optimal order error estimate in the H^1-norm and almost optimal order error estimate in the L^2-norm were derived for elliptic interface problems. An extension to parabolic interface problems was also discussed and estimates in $L^2(H^1)$-norm and $L^2(L^2)$-norm were derived for the spatially discrete scheme. A fully discrete scheme based on the backward Euler method was analyzed and an optimal order error estimates in $L^2(H^1)$-norm was derived.

Sinha and Deka [21] studied the FEMs for second order semilinear elliptic and parabolic interface problems in two-dimensional convex polygonal domain. The approximation theory of Brezzi-Rappaz-Raviart was used to obtain an optimal error estimate in the H^1-norm for semilinear elliptic problems and linear theory of interface problems was used to obtain a similar estimate for semilinear parabolic problems. They assumed that the mesh can be fitted exactly to the arbitrary interface which might not be so in practice.

Deka et al [7] improved on the works of [4,19] and also confirmed the optimal error estimates in

$L^2(0,T;L^2(\Omega))$-norm. Optimal error estimates in the $L^2(L^2)$ and $L^2(H^1)$ norms were established for linear semi discrete scheme and a similar error estimates was also extended semilinear interface problems.

The finite element approximation of nonlinear elliptic interface problems were discussed by [10,13,24] and recently [14]. Chaoxia Yang [23] studied the convergence of the finite element solution of a nonlinear parabolic interface problem with a linear source term. She focused on the fully discrete approximation and used a linearized 2-step backward difference scheme for the time discretization while piecewise linear interpolation was used to approximate the interface. With the assumption that the coefficient $\sigma(u)$ is positive and smooth with respect to $u \in \mathbb{R}$ but not continuous across the interface, the author proved a convergence rate of almost optimal order in the L^2-norm. Her mathematical analysis was carried out using body fitted triangulation, error splitting technique, and some projection operators under certain regularity conditions that guaranteed a unique solution.

In this work, we consider a nonlinear parabolic interface problem with nonlinear source term and obtain optimal order of convergence rates for spatially discrete scheme in $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$ norms. Time discretization is done using 2-step backward difference scheme and optimal order of convergence is obtained when the interface could be fitted exactly with the spatial descretization. In our study, the linear theories of interface and non-interface problems, Sobolev imbedding inequality were used. Other tools used in this paper are approximation properties of linear interpolation and projection operators.

We use the standard notations for Sobolev spaces and norms in this paper. For $m \geq 0$ and real p with $1 \leq p \leq \infty$, we use $W^{m,p}$ to denote Sobolev space of order m. For the case $p = 2$, we write $W^{m,p} = H^m$. $H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$, which is also the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^m(\Omega)$. We use the definition and notation in [1] when m is negative or fractional.

For a given Banach space B, we define

$$W^{m,p}(0,T;B) = \left\{ \begin{array}{ll} u(t) \in B \text{ for a.e } t \in (0,T) \text{ and } \displaystyle\sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt < 0 & 1 \leq p < \infty \\ u(t) \in B \text{ for a.e } t \in (0,T) \text{ and } \displaystyle\sum_{i=0}^{m} \text{ ess } \sup_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B < 0 & p = \infty \end{array} \right\}$$

equipped with the norms

$$\|u\|_{W^{m,p}(0,T;B)} = \left\{ \begin{array}{ll} \left[\displaystyle\sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt \right]^{1/p} & 1 \leq p < \infty \\ \displaystyle\sum_{i=0}^{m} \text{ ess } \sup_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B & p = \infty \end{array} \right\}$$

We write $L^2(0,T;B) = W^{0,2}(0,T;B)$ and $H^m(0,T;B) = W^{m,2}(0,T;B)$.
We shall need the following spaces

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norms

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} \quad \forall\, v \in X$$

Throughout this paper, C is a generic constant which is independent of the mesh parameters h and k.

The remaining part of the paper is organized as follows: we define the nonlinear interface problem in section two, describe the FE discretization and state some existing results in section three. In section four, we obtain optimal error estimates for the semi-discrete and fully discrete schemes. We verify our error estimates with numerical examples in section five and conclude in section six.

2. The nonlinear parabolic interface problem

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \in \Omega$ be an open domain with smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \setminus \bar{\Omega}_1$ be another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$. We consider the parabolic interface problem

$$u_t - \nabla \cdot (a(x,u)\nabla u) = f(x,u) \quad \text{in} \quad \Omega \times (0,T] \tag{2.1}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x) \text{ in } \Omega \\ u(x,t) = 0 \qquad \text{on } \partial\Omega \times [0,T] \end{cases} \tag{2.2}$$

and interface conditions

$$\begin{cases} [u]_\Gamma = 0 \\ \left[a(x,u)\dfrac{\partial u}{\partial n}\right]_\Gamma = g(x,t) \end{cases} \tag{2.3}$$

where $0 < T < \infty$, the symbol $[u]$ is a jump of a quantity u across the interface Γ and n is the unit outward normal to the boundary $\partial\Omega_i$, $(i = 1,2)$.

The interface conditions are defined as the difference of the limiting values from each side of the interface ie

$$[u]_{m\in\Gamma} := \lim_{x\to m^+} u_2(x,t) - \lim_{x\to m^-} u_1(x,t)$$

and

$$\left[a(x,u)\frac{\partial u}{\partial n}\right]_{m\in\Gamma} := \left[\lim_{x\to m^+} a_2\nabla u_2(x,t) - \lim_{x\to m^-} a_1\nabla u_1(x,t)\right]\cdot n$$

The coefficient function $a(x,u)$ is assumed piecewise across Γ ie $a(x,u) = a_i(x,u)$ for $u \in \mathbb{R}$ and $x \in \Omega$, $i = 1,2$.

This kind of problems arises in various branches of material science, biochemistry, multiphase flow etc., often when two or more different materials are involved with different conductivities or densities.

Assumption 2.1

A_1 Ω is a bounded convex polygonal domain in $\mathbb{R}^2$, the interface $\Gamma \in \Omega$ and the boundary $\partial\Omega$ are piecewise smooth, Lipschitz continuous and 1-dimensional.

A_2 The functions $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable and bounded with respect to their first variable $x \in \Omega$ and continuously differentiable with respect to their second variable $\eta \in \mathbb{R}$. $g(x,t) \in L^2(0,T;H^2(\Gamma)) \cap L^2(0,T;H^{1/2}(\Gamma))$.

A_3 Functions a and f satisfy

$$0 < \mu_1 \leq a(x,u) \leq \mu_2, \qquad \left|\frac{\partial a}{\partial \xi}(x,\xi)\right| + \left|\frac{\partial f}{\partial \xi}(x,\xi)\right| \leq \mu_3 \,,$$

for $u \in \mathbb{R}$, $x \in \Omega$ with positive constants μ_1 and μ_2 independent of (x,ξ).

Due to the low regularity of the solution across the interface, sufficient conditions for classical solvability of (3.1)-(3.3) are not required in this paper. However, a suitable weak form will turn to be relevant in our context. The weak form is:

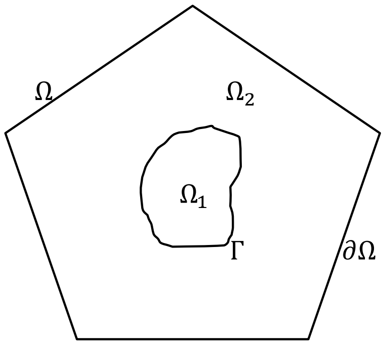

Figure 1. A polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ with interface Γ

Find $u(t) \in H_0^1(\Omega)$, $t \in (0, T]$ such that

$$(u_t, v) + A(u : u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall\, v(t) \in H_0^1(\Omega),\ t \in (0, T] \tag{2.4}$$

where

$$(\phi, \psi) = \int_\Omega \phi\psi\, dx \qquad A(\xi : \phi, \psi) = \int_\Omega a(x, \xi)\nabla\phi \cdot \nabla\psi\, dx \qquad \langle \phi, \psi\rangle_\Gamma = \int_\Gamma \phi\psi\, d\Gamma$$

We recall that for $u \in H^1(\Omega)$, the boundary value of u (ie $u_{|\partial\Omega}$) is defined on $H^{1/2}(\partial\Omega)$ the trace space of $H^1(\Omega)$. Similarly, the trace space on the interface Γ is $H^{1/2}(\Gamma)$. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is continuous and satisfies the embedding

$$\|z\|_{H^{1/2}(\partial\Omega)} \le C\|z\|_{H^1(\Omega)} \quad \forall\, z \in H^1(\Omega)$$

See Adams [1] and Evans [8] for more information on trace operator. It is known that $u_t \in L^2(0, T; H^{-1}(\Omega))$ (cf Evans [8]) and $g \in L^2(0, T; H^{1/2}(\Gamma) \cap H^2(\Gamma))$ (cf Ladyzhenskaya [15] and Chen et al [4]). For (2.1) – (2.3), we have the following regularity estimates (cf [17]):

Lemma 2.2 Suppose that the conditions of Assumption 2.1 are satisfied for every $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g \in L^2(0, T; H^{1/2}(\Gamma))$, there exists a constant C depending on μ_1, μ_2, μ_3, T and Ω such that

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega))} + \|u_t\|_{L^2(0,T;H^{-1}(\Omega))} \le C\left(\|g\|_{L^2(0,T;H^{1/2}(\Gamma))} + \|u_0\|_{L^2(\Omega)}\right) \tag{2.5}$$

and

$$\|u\|_{L^2(0,T;X)} \le C\left(\|g\|_{L^2(0,T;H^{1/2}(\Gamma))} + \|u_0\|_{L^2(\Omega)}\right) \quad \text{for } u(t) \in X \cap H_0^1(\Omega) \tag{2.6}$$

3. Finite element discretization and some auxiliary results

$\mathcal{T}_h$ denotes a partition of Ω into disjoint triangles K (called elements) such that no vertex of any triangle lies on the interior or side of another triangle.

Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{T}_h^\star$ denote the set of all elements whose edges lie on the interface Γ;

$$\mathcal{T}_h^\star = \{K \in \mathcal{T}_h : \bar{K} \cap \Gamma \ne \phi\}$$

$K \in \mathcal{T}_h^\star$ is called an interface element and we write $\Omega_h^\star = \bigcup_{K\in\mathcal{T}_h^\star} K$. The triangulation $\mathcal{T}_h$ of the domain Ω satisfies the following conditions

(i) $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$

(ii) If $\bar{K}_1, \bar{K}_2 \in \mathcal{T}_h$ and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.

(iii) Each $K \in \mathcal{T}_h$ is either in Ω_1 or Ω_2, and has at most one edge lying on Γ.

(iv) For each element $K \in \mathcal{T}_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles respectively. It is assumed that, for some fixed $h_0 > 0$, there exists two positive constants C_0 and C_1, independent of h, such that

$$C_0 r_K \leq h \leq C_1 \bar{r}_K \quad \forall\, h \in (0, h_0)$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise linear functions on $\mathcal{T}_h$ vanishing (in the sense of trace) on $\partial\Omega$.

The FE solution $u_h(x,t) \in S_h$ is represented as

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x)\ ,$$

where each basis function ϕ_j, $(j = 1, 2, \ldots, N_h)$ is a pyramid function with unit height. For the approximation $\hat{g}(t)$, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ be the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h. See [4,22] for the construction of such finite element spaces.

We present the analysis and computation for the case where the spatial discretisation can be fitted exactly to the interface. This could be achieved with the use of interface elements with curved edges along the interface.

Let $\pi_h : C(\bar{\Omega}) \to S_h$ be the Lagrange interpolation operator corresponding to the space S_h. The standard interpolation theory can not be applied due to the low regularity of the solution across the interface.

We recall some existing results which will be used in our analysis. See [4,7,17] for proofs

Lemma 3.1 Let $\Omega_h^\star$ be the union of all interface elements, $\pi_h : C(\Omega) \to S_h$ be the interpolation operator, and $g \in H^2(\Gamma)$, we have

$$\|v - \pi_h v\|_{H^m(\Omega)} \leq Ch^{2-m}\|v\|_X \qquad \forall\, v \in X, \quad m = 0, 1 \tag{3.1}$$

$$\|v\|_{H^1(\Omega_h^\star)} \leq Ch^{1/2}\|v\|_X \quad \forall\, v \in X \tag{3.2}$$

$$|\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}| \leq Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_h^\star)} \qquad \forall\, v_h \in S_h \tag{3.3}$$

$$\begin{aligned} |A(\xi : \nu_h, \omega_h) - A_h(\psi : \nu_h, \omega_h)| &\leq \mu_3\|\nabla\nu_h\|_{L^\infty(\Omega)}\|\xi - \psi\|_{L^2(\Omega)}\|\omega_h\|_{H^1(\Omega)} \\ &\quad + Ch\|\nu_h\|_{H^1(\Omega_h^\star)}\|\omega_h\|_{H^1(\Omega_h^\star)} \end{aligned} \tag{3.4}$$

4. Error estimates

This section is devoted to the analysis of the error estimates of the nonlinear parabolic interface problem. Optimal order error estimates are analysed in $L^2(0,T;H^1(\Omega))$-norm for spatially discrete scheme and $L^2(0,T;L^2(\Omega))$-norm for both spatially and fully discrete schemes. The finite element analysis of nonlinear non-interface problems are contained in Thomee [22] and references therein.

4.1 *Spatially discrete approximation*

We may pose the semidiscrete problem as: find $u_h : [0,T] \to S_h$ such that $u_h(0) = u_{h,0}$ and satisfies

$$(u_{h,t}, v_h) + A_h(u_h : u_h, v_h) = (f(x, u_h), v_h)_h + \langle g_h, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h,\ \text{a.e } t \in [0,T] \tag{4.1}$$

where $A_h(\xi:\phi,\psi): H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ and $(f(x,u_h),v_h)_h:\mathbb{R}\times H^1(\Omega)\to\mathbb{R}$ are defined as

$$A_h(\xi:\phi,\psi)=\sum_{K\in\mathcal{T}_h}\int_K a(x,\xi)\nabla\phi\cdot\nabla\psi\,dx\,,$$

$$(f(x,u_h),\phi)_h=\sum_{K\in\mathcal{T}_h}\int_K f(x,u_h)\phi\,dx\quad\forall\,\phi,\psi\in H^1(\Omega),\ t\in[0,T]$$

$A_h(\xi:\phi,\psi): H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ and $(f(x,u_h),\phi)_h:\mathbb{R}\times H^1(\Omega)\to\mathbb{R}$ are the discrete versions of $A(\xi:\phi,\psi): H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ and $(f(x,u),\phi):\mathbb{R}\times H^1(\Omega)\to\mathbb{R}$ respectively. These are obtained numerically by using well known quadrature schemes.

The existence of a unique solution to (4.1) follows the standard theory of Ordinary Differential Equations (see [22] for details). With u_h expressed as $u_h(x,t)=\sum_{j=1}^{N_h}\alpha_j(t)\phi_j(x)$ ($\alpha_j(t):[0,T]\to\mathbb{R}$) in (4.1), this results to a system of nonlinear ODEs. The assumptions on $a(x,u), f(x,u)$ and $g(x,t)$ guarantee a unique bounded solution for $t\in[0,T]$.

It is easy to see that u_h in (4.1) satisfies the a priori estimate (2.5)

$$\|u_h\|_{L^\infty(0,T;L^2(\Omega))}+\|u_h\|_{L^2(0,T;H^1(\Omega))}+\|u_{h,t}\|_{L^2(0,T;H^{-1}(\Omega))}\le C\left(\|g\|_{L^2(0,T;H^{1/2}(\Gamma))}+\|u_0\|_{L^2(\Omega)}\right)\quad(4.2)$$

Below are the main results concerning the convergence of the semi-discrete solution to the exact solution in the $L^2(0,T;H^1(\Omega))$-norm and $L^2(0,T;L^2(\Omega))$-norm respectively:

Theorem 4.1 Suppose that the conditions of Assumption 2.1 are satisfied for every $a:\Omega\times\mathbb{R}\to\mathbb{R}$, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g\in L^2(0,T;H^2(\Gamma))$ and let u and u_h be the solutions of (2.4) and (4.1) respectively, then for $u_0\in H_0^1(\Omega)$ and $\gamma=\gamma(\mu_1,\mu_3)$, there exists a positive constant C, independent of h, such that

$$\|u-u_h\|_{L^2(0,T;H^1(\Omega))}\ \le\ Ch\left\{\|u_0\|_X+\left(\int_0^T\exp(-\gamma t)\left(\|g\|^2_{H^2(\Gamma)}+\|u\|_X^2+\|u_t\|_X^2\right)\,dt\right)^{1/2}\right\}$$

Theorem 4.2 Suppose that the conditions of Assumption 2.1 are satisfied for every $a:\Omega\times\mathbb{R}\to\mathbb{R}$, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g\in L^2(0,T;H^2(\Gamma))$ and let u and u_h be the solutions of (2.4) and (4.1) respectively, then for $u_0\in H_0^1(\Omega)$ there exists a positive constant C, independent of h, such that

$$\|u-u_h\|_{L^2(0,T;L^2(\Omega))}\ \le\ Ch^2\left\{\|u_0\|_X+\|u\|_{L^\infty(0,T;X)}+\left(\int_0^T\exp(-\gamma t)\left(\|g\|^2_{H^2(\Gamma)}+\|u\|_X^2+\right.\right.\right.$$
$$\left.\|u_t\|_X^2\right)\,dt^{1/2}.$$

We shall prove the two theorems using the elliptic projection defined below

Let $P_h: X\cap H^1(\Omega)\to S_h$ be the elliptic projection of the exact solution u in S_h defined by

$$A_h(u:P_h\nu,\phi)=A(u:\nu,\phi)\quad\forall\,\phi\in S_h,\ t\in[0,T]\tag{4.3}$$

It is easy to see from (4.3) that there exist a constant $C>0$ such that

$$\|P_h\nu\|_{H^1(\Omega)}\le C\|\nu\|_{H^1(\Omega)}\qquad\forall\,\nu\in H^1(\Omega)\tag{4.4}$$

For this projection, we have

Lemma 4.3 Let u be a smooth function in $\Omega\times T$ and $a=a(x,u)$ satisfies Assumption 2.1. Assume that $u \in X \cap H_0^1$ and let $P_h u$ be defined as in (4.3), then

$$\|P_h u - u\|_{L^2(\Omega)} + h\|P_h u - u\|_{H^1(\Omega)} \le Ch^2\|u\|_X \tag{4.5}$$

Proof Following [17], we have

$$\|P_h u - u\|_{H^1(\Omega)} \le Ch\|u\|_X \tag{4.6}$$

Now consider the dual problem

$$A(u : \psi, \phi) = (P_h u - u, \phi) \quad \forall\, \phi \in H_0^1(\Omega) \tag{4.7}$$

It follows from a similar argument of [22,pg 233] that

$$\|\psi\|_X \le C\|P_h u - u\|_{L^2(\Omega)} \tag{4.8}$$

From (4.7)

$$\begin{aligned}
\|P_h u - u\|^2_{L^2(\Omega)} &= A(u : P_h u - u, \psi) \\
&= A(u : P_h u - u, \psi - \phi) + A(u : P_h u - u, \phi) \quad \phi \in S_h \\
&\le C\|P_h u - u\|_{H^1(\Omega)}\|\psi - \phi\|_{H^1(\Omega)} + |A(u : P_h u, \phi) - A_h(u : P_h u, \phi)|
\end{aligned}$$

Using (3.1) and (4.6) with $\phi = \pi_h\psi$ we obtain

$$\|P_h u - u\|^2_{L^2(\Omega)} \le Ch^2\|u\|_X\|\psi\|_X + |A(u : P_h u, \pi_h\psi) - A_h(u : P_h u, \pi_h\psi)|$$

It follows from (3.4), (3.2), (4.4) and the fact that $\|\pi_h\psi\| \le C\|\psi\|$, that

$$\|P_h u - u\|^2_{L^2(\Omega)} \le Ch^2\|u\|_X\|\psi\|_X \tag{4.9}$$

(4.5) follows from (4.6), (4.8) and (4.9). □

Lemma 4.4 Let u be a smooth function in $\Omega \times T$ and $a = a(x, u)$ satisfies Assumption 2.1. Assume that $u \in X \cap H_0^1$ and let $P_h u$ be defined as in (4.3), then

$$\|(P_h u - u)_t\|_{L^2(\Omega)} + h\|(P_h u - u)_t\|_{H^1(\Omega)} \le Ch^2(\|u\|_X + \|u_t\|_X) \tag{4.10}$$

Proof Let $\xi = P_h u - u$, and assume that a_t is uniformly bounded. Following the argument of [22], we have

$$\begin{aligned}
\rho\|\xi_t\|^2_{H^1(\Omega)} &\le A(u : \xi_t, \xi_t) \\
&= A(u : \xi_t, \phi - u_t) + A(u : \xi_t, (P_h u)_t - \phi) \\
&= A(u : \xi_t, \phi - u_t) + \int_\Omega \left[\frac{\partial}{\partial t}(a\nabla\xi) - \frac{\partial a}{\partial t}\nabla\xi\right] \cdot \nabla((P_h u)_t - \phi)\, dx \\
&\le \|\xi_t\|_{H^1(\Omega)}\|\phi - u_t\|_{H^1(\Omega)} + \|\xi\|_{H^1(\Omega)}\|(P_h u)_t - \phi\|_{H^1(\Omega)}
\end{aligned}$$

Take $\phi = \pi_h u_t$. Using (3.1), (4.5) and Young's inequality, we obtain

$$\|(P_h u - u)_t\|_{H^1(\Omega)} \le Ch(\|u\|_X + \|u_t\|_X) \tag{4.11}$$

Following the duality argument (4.7) – (4.9), it is easy to see that

$$\|(P_hu-u)_t\|_{L^2(\Omega)} \le Ch^2(\|u\|_X+\|u_t\|_X) \tag{4.12}$$

(4.10) follows from (4.11) and (4.12). □

Proof of Theorem 4.1 Subtract (4.1) from (2.4)

$$(u_t-u_{h,t},v_h)+A(u:u,v_h)=A_h(u_h:u_h,v_h)+(f(x,u),v_h)-(f(x,u_h),v_h)_h+\langle g,v_h\rangle_\Gamma-\langle g_h,v_h\rangle_{\Gamma_h}$$

$\forall\, v_h\in S_h$. Let $e(t)=u-u_h$, $v_h=P_hu-u_h$ and use (4.3)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{L^2(\Omega)}+A_h(u_h:e(t),e(t)) &= (u_{h,t}-u_t,P_hu-u)+A_h(u_h:e(t),u-P_hu)+A_h(u_h:u,\\ &\quad P_hu-u_h)-A_h(u:P_hu,P_hu-u_h)+(f(x,u),P_hu-u_h)\\ &\quad -(f(x,u_h),P_hu-u_h)_h+\langle g,P_hu-u_h\rangle_\Gamma-\\ &\quad \langle g_h,P_hu-u_h\rangle_{\Gamma_h}\le B_1+B_2+B_3+B_4+B_5\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}B_1&=|(u_t-u_{h,t},P_hu-u)| \qquad B_2=|A_h(u_h:e(t),u-P_hu)|\\ B_3&=|A_h(u_h:u,P_hu-u_h)-A_h(u:P_hu,P_hu-u_h)|\\ B_4&=|(f(x,u),P_hu-u_h)-(f(x,u_h),P_hu-u_h)_h|\\ B_5&=|\langle g,P_hu-u_h\rangle_\Gamma-\langle g_h,P_hu-u_h\rangle_{\Gamma_h}|\end{aligned}$$

For B_1, we have

$$\begin{aligned}B_1&=|\frac{d}{dt}(e(t),P_hu-u)-(e(t),(P_hu-u)_t)|\\ &\le\frac{1}{4}\frac{d}{dt}\|e(t)\|^2_{L^2(\Omega)}+\frac{d}{dt}\|P_hu-u\|^2_{L^2(\Omega)}+\frac{1}{4\varepsilon}\|e(t)\|^2_{L^2(\Omega)}+\varepsilon\|(P_hu-u)_t\|^2_{L^2(\Omega)}\\ &\le\frac{1}{4}\frac{d}{dt}\|e(t)\|^2_{L^2(\Omega)}+\frac{1}{4\varepsilon}\|e(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\|P_hu-u\|^2_{L^2(\Omega)}+C(\varepsilon)\|(P_hu-u)_t\|^2_{L^2(\Omega)}\end{aligned} \tag{4.14}$$

$$\begin{aligned}B_2&\le\|e(t)\|_{H^1(\Omega)}\|u-P_hu\|_{H^1(\Omega)}\\ &\le\frac{1}{4\varepsilon}\|e(t)\|^2_{H^1(\Omega)}+\varepsilon\|P_hu-u\|^2_{H^1(\Omega)}\end{aligned} \tag{4.15}$$

For B_3, we obtain

$$B_3\ \le\ \sum_{K\in\mathcal{T}_h}\int_K(\mu_2|\nabla(u-P_hu)\cdot\nabla(P_hu-u_h)|+\mu_3|e(t)||\nabla P_hu\cdot\nabla(P_hu-u_h)|)$$

By Holder's and Young's inequalities with the fact that ∇P_hu is constant on $K\in\mathcal{T}_h$, we obtain

$$B_3\ \le\ C(\mu_2,\mu_3,\varepsilon)\|P_hu-u\|^2_{H^1(\Omega)}+\frac{1}{2\varepsilon}\|e(t)\|^2_{H^1(\Omega)} \tag{4.16}$$

$$
\begin{aligned}
B_4 &\leq |(f(x,u), P_h u - u_h) - (f(x,u), P_h u - u_h)_h| + |(f(x,u) - f(x,u_h), P_h u - u_h)_h| \\
&\leq Ch\|u\|_{H^1(\Omega_h^\star)}\|P_h u - u_h\|_{H^1(\Omega_h^\star)} + \mu_3\|e(t)\|_{L^2(\Omega)}\|P_h u - u_h\|_{L^2(\Omega)} \qquad (4.17)\\
&\leq C(\varepsilon)h^3\|u\|_X^2 + \frac{1}{2}\|P_h u - u\|^2_{H^1(\Omega)} + \frac{1}{2\varepsilon}\|e(t)\|^2_{H^1(\Omega)} + C(\varepsilon)\mu_3^2\|e(t)\|^2_{L^2(\Omega)} \qquad (4.18)
\end{aligned}
$$

Using (3.3),

$$
\begin{aligned}
B_5 &\leq Ch^{3/2}\|g\|_{H^2(\Gamma)}\|P_h u - u_h\|_{H^1(\Omega)} \\
&\leq Ch^3(\varepsilon+1)\|g\|^2_{H^2(\Gamma)} + \frac{1}{4\varepsilon}\|e(t)\|^2_{H^1(\Omega)} + Ch^2\|u\|_X^2 \qquad (4.19)
\end{aligned}
$$

In view of (4.5) and (4.10) we substitute (4.14) – (4.18) into (4.13) and simplify the resulting expression taking $\varepsilon = 3/\mu_1$ we obtain, for h sufficiently small,

$$
\frac{1}{4}\frac{d}{dt}\|e(t)\|^2_{L^2(\Omega)} + \frac{\mu_1}{2}\|e(t)\|^2_{H^1(\Omega)} \leq \gamma\|e(t)\|^2_{L^2(\Omega)} + Ch^2\left(\|g\|^2_{H^2(\Gamma)} + \|u\|_X^2 + \|u_t\|_X^2\right)
$$

where $\gamma > 0$ depends on μ_1 and μ_3. It follows that

$$
\begin{aligned}
\exp(-4\gamma T)\|e(t)\|^2_{L^2(0,T;H^1(\Omega))} &\leq \exp(-4\gamma T)\|e(T)\|^2_{L^2(\Omega)} + \int_0^T \exp(-4\gamma t)\|e(t)\|^2_{H^1(\Omega)}\, dt \\
&\leq \|e(0)\|^2_{H^1(\Omega)} + Ch^2\int_0^T \exp(-4\gamma t)\left(\|g\|^2_{H^2(\Gamma)} + \|u\|_X^2 + \|u_t\|_X^2\right)\, dt
\end{aligned}
$$

The result follows by taking $u_{0,h} = \pi_h u_0$. □

Proof of Theorem 4.2 We have

$$
\begin{aligned}
\|u - u_h\|^2_{L^2(\Omega)} &\leq 2(\|u - P_h u\|^2_{L^2(\Omega)} + \|P_h u - u_h\|^2_{L^2(\Omega)}) \\
&\leq Ch^4\|u\|_X^2 + 2\|P_h u - u_h\|^2_{L^2(\Omega)} \qquad (4.20)
\end{aligned}
$$

Using (4.3), it is easy to observe that

$$
\begin{aligned}
((u_h - P_h u)_t, v_h) + A_h(u_h : u_h - P_h u, v_h) &= ((u - P_h u)_t, v_h) + (f(x,u_h), v_h)_h - (f(x,u), v_h) \\
&\quad + \langle g_h, v_h\rangle_{\Gamma_h} - \langle g, v_h\rangle_\Gamma + A_h(u : P_h u, v_h) - A_h(u_h : P_h u, v_h)
\end{aligned}
$$

We take $v_h = u_h - P_h u$ and make use of the fact that $\nabla P_h u$ is constant on $K \in \mathcal{T}_h$, and obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_h - P_h u\|^2_{L^2(\Omega)} + \mu_1\|u_h - P_h u\|^2_{H^1(\Omega)} &\leq C(\mu_1,\varepsilon)\|u_h - P_h u\|^2_{L^2(\Omega)} + C(\varepsilon)\|u - P_h u\|^2_{L^2(\Omega)} \\
&\quad + \|(u - P_h u)_t\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|u_h - P_h u\|^2_{H^1(\Omega)} \\
&\quad + B_4 + B_5 \qquad (4.21)
\end{aligned}
$$

From (3.2) and (4.17)

$$
B_4 \ \leq \ C(\mu_3,\varepsilon)h^4\|u\|_X^2 + \frac{1}{4\varepsilon}\|P_h u - u_h\|^2_{H^1(\Omega)} + \frac{5}{4}\|P_h u - u_u\|^2_{L^2(\Omega)} \qquad (4.22)
$$

From (3.2) and (3.3),

$$
B_5 \leq \varepsilon Ch^4\|g\|^2_{H^2(\Gamma)} + \frac{1}{4\varepsilon}\|P_h u - u_h\|^2_{H^1(\Omega)} \qquad [\text{because } D^\alpha(P_h u - u_h) = 0 \text{ for } |\alpha| = 2] \quad (4.23)
$$

Substitute (4.22) and (4.23) into (4.21), using (4.5) and (4.10) with $\varepsilon = \frac{3}{4\mu_1}$

$$\frac{1}{2}\frac{d}{dt}\|P_h u - u_h\|^2_{L^2(\Omega)} \ \leq \ \gamma\|P_h u - u_h\|^2_{L^2(\Omega)} + Ch^4\left(\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)}\right)$$

With $u_{h,0} = \pi_h u_0$, it follows that

$$\|(P_h u - u_h)(t)\|^2_{L^2(\Omega)} \ \leq \ Ch^4\left[\exp(2\gamma t)\|u_0\|^2_X + \int_0^t \exp(2\gamma(t-s))(\|u\|^2_X + \|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)})\, ds\right] \tag{4.24}$$

The result follows by substituting (4.24) into (4.20) and taking the supremum with respect to t over $[0, T]$. □

4.2 *Fully discrete method*

Now we discuss a fully discrete scheme based on 2-step backward difference approximation. Optimal order error estimate in the $L^2(0, T; L^2(\Omega))$-norm is derived.
The interval [0,T] is divided into M equally spaced (for simplicity) subintervals:

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk$, $k = T/M$ being the time step. For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the backward difference quotient defined by

$$\partial_k w^n = \frac{3w^n - 4w^{n-1} + w^{n-2}}{2k}$$

The fully discrete finite element approximation to (2.4) is defined as follows:
Let $U_h^0 = \pi_h u_0$, find $U_h^n \in S_h$, for $n = 2, 3, \ldots, M$, such that

$$U_h^1 = u_0 + k\left[\nabla\cdot(a(x,u_0)\nabla u_0) + f(x,u_0)\right] \tag{4.25}$$

$$(\partial_k U_h^n, v_h) + A_h(2U_h^{n-1} - U_h^{n-2} : U_h^n, v_h) = (f(x, 2U_h^{n-1} - U_h^{n-2}), v_h)_h + \langle g_h^n, v_h\rangle_{\Gamma_h} \tag{4.26}$$

$\forall\, v_h \in S_h$. If u_{tt} is defined for $t \in (0, T]$, it can be shown using Taylor expansion that

$$\|U_h^n - 2U_h^{n-1} + U_h^{n-2}\| \ \leq \ \lambda k^2$$

For $\lambda \geq 0$ and k sufficiently small. We have the following stability result:

Lemma 4.5 Suppose the conditions of Assumption 2.1 are satisfied, there exists a constant C independent of h and k such that for the solution of (4.25) – (4.26)

$$\|U_h^n\|^2_{L^2(\Omega)} \leq C(1+k^2)\|u_0\|^2_{L^2(\Omega)} + Ck\sum_{i=1}^n \|g_h^i\|^2_{H^{1/2}(\Gamma)} + Ck^3\ , \quad n = 1, 2, \ldots, M \tag{4.27}$$

Proof Taking $v_h = U_h^n$ in (4.26), we obtain by simple calculation

$$\begin{aligned}\frac{1}{k}\|U_h^n\|^2_{L^2(\Omega)} + \mu_1\|\nabla U_h^n\|^2_{L^2(\Omega)} &\leq \frac{1}{2k}\|U_h^{n-1}\|_{L^2(\Omega)}\|U_h^n\|_{L^2(\Omega)} + \frac{1}{2k}\|U_h^n - 2U_h^{n-1} + U_h^{n-2}\|_{L^2(\Omega)}\|U_h^n\|_{L^2(\Omega)} \\ &\quad + \mu_3\|2U_h^{n-1} - U_h^{n-2}\|_{L^2(\Omega)}\|U_h^n\|_{L^2(\Omega)} + \|g_h^n\|_{H^{1/2}(\Gamma)}\|U_h^n\|_{L^2(\Omega)} \\ &\leq \frac{1}{2k}\|U_h^{n-1}\|^2_{L^2(\Omega)} + \left(\frac{1}{2k} + \frac{3\mu_3}{2} + \frac{3}{4}\right)\|U_h^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|g_h^n\|^2_{H^{1/2}(\Gamma)} \\ &\quad + \frac{1}{4}(1+2\mu_3)\lambda^2 k^2\end{aligned}$$

It follows that

$$(1 - 3(0.5+\mu_3)k)\,\|U_h^n\|^2_{L^2(\Omega)} + 2\mu_1 k\|\nabla U_h^n\|^2_{L^2(\Omega)} \;\leq\; \|U_h^{n-1}\|^2_{L^2(\Omega)} + k\|g_h^n\|^2_{H^{1/2}(\Gamma)} + \frac{1}{2}(1+2\mu_3)\lambda^2 k^3$$

For $0 < k \leq k_0 < \dfrac{1}{3(0.5+\mu_3)}$, there is a $c_0 = \dfrac{3(0.5+\mu_3)}{(1 - 3(0.5+\mu_3)k_0)}$ such that

$$\|U_h^n\|^2_{L^2(\Omega)} + 2\mu_1 k\|\nabla U_h^n\|^2_{L^2(\Omega)} \;\leq\; (1+c_0k)\left[\|U_h^{n-1}\|^2_{L^2(\Omega)} + k\|g_h^n\|^2_{H^{1/2}(\Gamma)} + \frac{1}{2}(1+2\mu_3)\lambda^2 k^3\right]$$

By iteration on n we have

$$\begin{aligned}\|U_h^n\|^2_{L^2(\Omega)} + 2\mu_1 k\|\nabla U_h^n\|^2_{L^2(\Omega)} &\leq \|U_h^1\|^2_{L^2(\Omega)}\sum_{i=2}^{n}(1+c_0k)^{i-1} \\ &\quad + \sum_{i=2}^{n}(1+c_0k)^{n-i+1}\left[k\|g_h^i\|^2_{H^{1/2}(\Gamma)} + (1+2\mu_3)\lambda^2 k^4\right] \\ &\leq (1+c_0k_0)^{n-1}(n-1)\left[\|U_h^1\|^2_{L^2(\Omega)} + k\sum_{i=2}^{n}\|g_h^i\|^2_{H^{1/2}(\Gamma)} + (1+2\mu_3)\lambda^2 k^3\right]\end{aligned} \tag{4.28}$$

(4.27) follows from the last inequality, (4.25) and Assumption 2.1. The result below establishes the convergence of the fully discrete solution to the exact solution in the $L^2(0,T;L^2(\Omega))$-norm.

Theorem 4.6 Let u^n and U_h^n be the solutions of (2.4) and (4.25) − (4.26) respectively. Suppose that the conditions of Assumption 2.1 are satisfied for every $a : \Omega\times\mathbb{R}\to\mathbb{R}$, $f : \Omega\times\mathbb{R}\to\mathbb{R}$, $g(x,t)$ and u_{ttt} is defined for $\Omega\times[0,T]$. There exists a positive constant C in dependent of h and k such that

$$\|u^n - U_h^n\|_{L^2(\Omega)} \leq \left[h^2 + k^2\right] C(u,g) \tag{4.29}$$

Proof Let $z^n = U_h^n - P_h u^n$ then

$$\begin{aligned}&(\partial_k z^n, v_h) + A_h(2U_h^{n-1} - U_h^{n-2} : z^n, v_h) \\ &\quad = (\partial_k(u^n - P_hu^n), v_h) - (\partial_k u^n - u_t^n, v_h) + A_h(u^n : P_hu^n, v_h) - A_h(2U_h^{n-1} - U_h^{n-2} : P_hu^n, v_h) \\ &\qquad + (f(x, 2U_h^{n-1} - U_h^{n-2}), v_h)_h - (f(x,u^n), v_h) + \langle g_h^n, v_h\rangle_{\Gamma_h} - \langle g^n, v_h\rangle_\Gamma \\ &\quad \leq (\partial_k(u^n - P_hu^n), v_h) - (\partial_k u^n - u_t^n, v_h) + C\|u^n - 2U_h^{n-1} + U_h^{n-2}\|_{L^2(\Omega)}\|v_h\|_{H^1(\Omega)} \\ &\qquad + Ch^2\|u^n\|_X\|v_h\|_{H^1(\Omega)} + \mu_3\|u^n - 2U_h^{n-1} + U_h^{n-2}\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)} + Ch^2\|g^n\|_{H^2(\Gamma)}\|v_n\|_{H^1(\Omega)}\end{aligned}$$

where we have made use of Holder's inequality, Young's inequality (3.2), (3.3) and the fact that $\nabla P_h u^n$ is constant on $K \in \mathcal{T}_h$, in the last inequality.

$(\partial_k z^n, z^n) + A_h(2U_h^{n-1} - U_h^{n-2} : z^n, z^n)$

$$\leq \|\partial_k(u^n - P_h u^n)\|_{L^2(\Omega)}\|z^n\|_{L^2(\Omega)} + \|\partial_k u^n - u_t^n\|_{L^2(\Omega)}\|z^n\|_{L^2(\Omega)} + Ch^2\|u^n\|_X\|z^n\|_{H^1(\Omega)}$$
$$+ (C+\mu_3)\left(\|u^n - P_h u^n\|_{L^2(\Omega)} + \|z^n\|_{L^2(\Omega)} + \|U_h^n - 2U_h^{n-1} + U_h^{n-2}\|_{L^2(\Omega)}\right)\|z^n\|_{H^1(\Omega)}$$
$$+Ch^2\|g^n\|_{H^2(\Gamma)}\|z^n\|_{H^1(\Omega)}$$

By Young's inequality,

$$\frac{1}{2k}\|z^n\|^2_{L^2(\Omega)} \leq \frac{1}{2k}\|z^{n-1}\|^2_{L^2(\Omega)} + \left(\frac{3}{4} + \mu_1\right)\|z^n\|^2_{L^2(\Omega)} + \|\partial_k(u^n - P_h u^n)\|^2_{L^2(\Omega)} + \|\partial_k u^n - u_t^n\|^2_{L^2(\Omega)}$$
$$\frac{5}{4\mu_1}C^2h^4\left(\|u^n\|_X^2 + \|g^n\|^2_{H^2(\Gamma)}\right) + \frac{\lambda^2 k^3}{4}$$
$$+ \frac{5}{4\mu_1}(C+\mu_3)^2\left(\|u^n - P_h u^n\|^2_{L^2(\Omega)} + \|z^n\|^2_{L^2(\Omega)} + \|U_h^n - 2U_h^{n-1} + U_h^{n-2}\|^2_{L^2(\Omega)}\right)$$

It follows that

$$(1 - Ck)\|z^n\|^2_{L^2(\Omega)} \leq \|z^{n-1}\|^2_{L^2(\Omega)} + 2k\|\partial_k(u^n - P_h u^n)\|^2_{L^2(\Omega)} + 2k\|\partial_k u^n - u_t^n\|^2_{L^2(\Omega)}$$
$$Ch^4 k\left(\|u^n\|_X^2 + \|g^n\|^2_{H^2(\Gamma)}\right) + C\lambda^2 k^4$$

Following the argument that led to (4.28),

$$\|z^n\|^2_{L^2(\Omega)} \leq C\|z^1\|^2_{L^2(\Omega)} + Ck\sum_{j=2}^{n}\|\partial_k(u^j - P_h u^j)\|^2_{L^2(\Omega)} + Ck\sum_{j=2}^{n}\|\partial_k u^j - u_t^j\|^2_{L^2(\Omega)}$$
$$+ Ch^4 k\sum_{j=2}^{n}(\|u^j\|_X^2 + \|g^j\|^2_{H^2(\Gamma)}) + C\lambda^2 k^4$$
$$\leq C\|z^1\|^2_{L^2(\Omega)} + C\int_0^{t_n}\|(u - P_h u)_t\|^2_{L^2(\Omega)}\,dt + Ck^4\int_0^{t_n}\|u_{ttt}\|^2_{L^2(\Omega)}\,dt$$
$$+ Ch^4\int_0^{t_n}\left[\|u\|_X^2 + \|g\|^2_{H^2(\Gamma)}\right]\,dt + C\lambda^2 k^4$$
$$\leq C\|z^1\|^2_{L^2(\Omega)} + Ch^4\int_0^{t_n}\left[\|u\|_X^2 + \|u_t\|_X^2 + \|g\|^2_{H^2(\Gamma)}\right]\,dt$$
$$+ Ck^4\int_0^{t_n}\|u_{ttt}\|^2_{L^2(\Omega)}\,dt + C\lambda^2 k^4 \tag{4.30}$$

where use is made of (4.12) to obtain (4.30). We have, from (3.1), (4.5) and (4.30) with $U_h^0 = \pi_h u_0$,

$$\|u^n - U_h^n\|^2_{L^2(\Omega)} \leq C\|z^1\|^2_{L^2(\Omega)} + Ch^4\left[\|u^n\|_X^2 + \int_0^{t_n}\left(\|u\|_X^2 + \|u_t\|_X^2 + \|g\|^2_{H^2(\Gamma)}\right)\,dt\right]$$
$$+ Ck^4\int_0^{t_n}\|u_{ttt}\|^2_{L^2(\Omega)}\,dt + C\lambda^2 k^4$$
$$\leq Ch^4\left[\|u_0\|_X^2 + \int_0^{t_n}\left(\|u\|_X^2 + \|u_t\|_X^2 + \|g\|^2_{H^2(\Gamma)}\right)\,dt\right]$$
$$+ Ck^4\left[\int_0^{t_n}\|u_{ttt}\|^2_{L^2(\Omega)}\,dt + \lambda^2 + \zeta^2\right] \tag{4.31}$$

We use the fact that $\|u^1 - u_0 - ku_t\| \leq \zeta k^2$ for $\zeta \geq 0$ to obtain (4.31). (4.29) follows immediately.

5. Numerical results

For our numerical experiment, globally continuous piecewise linear finite element functions based on a uniform triangulation described in section 3 are used. The numerical experiments of this section are based on fully discrete scheme.

Example 5.1 We present the results of computation of a two-dimensional non-linear parabolic interface problem in the domain $\Omega = (-1,1) \times (-1,1)$ with $\Omega_1 = (-0.5, 0.5) \times (-0.5, 0.5)$, $\Omega_2 = \Omega \setminus \Omega_1$ and $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$. Γ is made of straight lines (see Figure 2).

Consider the problem (2.1) – (2.3) in $\Omega \times (0, 10]$. We choose a problem with a known solution as follows:

$$u = \begin{cases} \frac{3}{8}(28x^2y^2 - 8x^2 - 8y^2 + 3)\frac{t}{1+t} & \text{in} \quad \Omega_1 \times (0, 10] \\ \frac{1}{2}(x^2y^2 - x^2 - y^2 + 1)\frac{t}{1+t} & \text{in} \quad \Omega_2 \times (0, 10] \end{cases}$$

$$a = \begin{cases} \frac{u^2}{1+u^2} & \text{in} \quad \Omega_1 \times (0, 10] \\ \exp u & \text{in} \quad \Omega_2 \times (0, 10] \end{cases}$$

The source function f, the interface function g and the initial data u_0 are determined from the choice of u and a: The L^2-norm errors at $t = 5$ are presented in Table 1

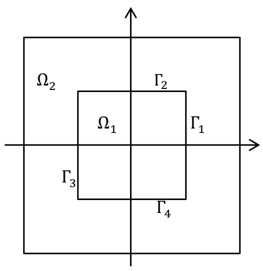

Figure 2. Computational domain

Table 1. Numerical results for Example 5.1

h	Error ($k = 0.1$)
0.064629	2.65905×10^{-4}
0.0315913	8.70106×10^{-5}
0.0168371	4.83378×10^{-5}

k	Error ($h = 0.0315913$)
0.25	2.98596×10^{-4}
0.125	1.06249×10^{-4}
0.0625	8.70106×10^{-5}

The data presented in Table 1 indicate that

$$\|u - u_h\|_{H^1(\Omega)} = O\left(h^{2.056} + k^{2.342}\right)$$

Example 5.2 We consider a parabolic problem of the form (2.1) – (2.3) in the domain $\Omega = (-1,1) \times (-1,1)$ with $\Omega_1 = (-1,0) \times (-1,1)$, $\Omega_2 = (0,1) \times (-1,1)$ and Γ is the line $x = 0$.

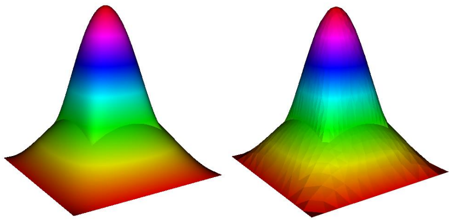

Figure 3. FE solution of Example 5.1 with $t = 5$, $k = 0.125$ and mesh sizes 0.0315913 & 0.235702 respectively

For the exact solution, we choose

$$u = \begin{cases} (1-y^2)x(1+x)\sin t & \text{in} \quad \Omega_1 \times (0,10] \\ (1-y^2)\sin(4\pi x)t\cos t & \text{in} \quad \Omega_2 \times (0,10] \end{cases}$$

We choose

$$a = \begin{cases} \dfrac{1}{1+u^2} & \text{in} \quad \Omega_1 \times (0,10] \\ 3 & \text{in} \quad \Omega_2 \times (0,10] \end{cases}$$

The source function f, the interface function g and the initial data u_0 are determined from the choice of u and a. The L^2-norm errors at $t = 5$ are presented in Table 2

Table 2. Numerical results for Example 5.2

h	Error $(k = 0.1)$
0.127515	7.32726×10^{-2}
0.0653869	1.82268×10^{-2}
0.0329586	4.59465×10^{-3}
0.0170309	1.15467×10^{-3}

k	Error $(h = 0.0329586)$
0.125	4.61233×10^{-3}
0.1	4.59465×10^{-3}
0.0625	4.57646×10^{-3}

The data presented in Table 2 indicate that

$$\|u - u_h\|_{L^2(\Omega)} = O\left(h^{2.088} + k^{2.163}\right)$$

6. Conclusion

Solution of a second order nonlinear parabolic interface problem by FE-BDS is presented. The convergence of the finite element solution to the exact solution on a two-dimensional convex polygonal domain is analyzed. The spatial discretisation was done using quasi-uniform triangular elements with the unknown function approximated using piecewise linear functions. Discretization in time is based on linearized 2-step implicit scheme. It was assumed that the mesh fits interface.

We showed that convergence rate of optimal order in $L^2(0,T;L^2(\Omega))$-norm and $L^2(0,T;H^1(\Omega))$-norm could be obtained for semi-discrete scheme. Convergence rate of optimal order in $L^2(\Omega)$-norm is obtained for the fully discrete scheme. Examples were given to confirm the theoretical result.

In this work, we analyzed the stability and convergence of the fully discrete scheme, however, the maximum principle of the scheme is a good area of interest which the authors might look into for future research. This is possible in view of I. Farago et.al [9].

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