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On fixed point theorems for sums of certain mappings in locally convex spaces

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Abstract. In this paper, we prove some fixed point results for sums of two mappings on a locally convex space.The results extend the fixed point theorems of Vijayaraju for the sums of two mappings on a convex subset of a locally convex space to the sums of two mappings defined on a star-shaped subset, as well as an almost convex subset of a locally convex space.

Keywords: mapping, convex space, fixed point theorem.

1. Introduction

In fixed point theory, conditions under which certain mappings, defined on some spaces, leave some points in the space invariant are investigated. Such invariant points are called fixed points. Some of the earliest fixed point theorems are the Banach contraction mapping principle which states that a strict contraction mapping on a complete metric space into itself has a unique fixed point and the Tychonoff fixed point theorem, which states that a continuous mapping on a compact convex subset of a Hausdorff locally convex space has a fixed point.

Many problems of analysis involve operators which may be split in the form $H = T + S$, such that T is a contraction, S is a continuous operator and H has neither of these properties. Therefore, neither the Tychonoff nor the Banach fixed point theorem directly applies in this case and it therefore becomes desirable to develop fixed point theorems for such cases.

Let K be a nonempty closed convex and bounded subset of a Banach space X, $T : K \to X$ a contraction mapping and $S : K \to X$ a compact mapping. Krasnoselskii [5], in his 1955 paper, proved the existence of a fixed point in K for the sum $T + S$ of the two mappings T and S when they satisfy the condition $Tx + Sy \in K$ for all $x, y \in K$. Since the publication of his result, several improvements and generalizations of Krasnoselskii's result have been obtained by different authors. For example, Nashed and Wong [6] proved the existence of a fixed point for the sum $T + S$ of a nonlinear contraction mapping $T : K \to X$ and a compact mapping $S : K \to X$. Cain and Nashed [1] proved Krasnoselskii's result in the setting of locally convex topological vector spaces. They extended Krasnoselskii's result to the sum $T + S$ of a contraction mapping $T : K \to X$ and a continuous mapping $S : K \to X$, where K is a nonempty complete convex subset of a locally convex space X. The fixed point result of Nashed and Wong was proved in a locally convex space setting when Sehgal and Singh [8] extended the result of Cain and Nashed to a sum $T + S$ of a nonlinear contraction mapping $T : K \to X$ and a continuous mapping $S : K \to X$.

Vijayaraju [9] proved some extensions of the result of Cain and Nashed to the sum of a nonexpansive mapping and a continuous mapping as well as to the sum of an asymptotically nonexpansive mapping and a continuous mapping.

In this paper, we prove some fixed point results which extend the results of Vijayaraju to a star-shaped subset as well as to an almost convex subset of a locally convex space. This will be done using the same method as in the references above.

Throughout this paper, X denotes a Hausdorff locally convex topological vector space and $\mathcal{P} = \{p_\alpha : \alpha \in J\}$, a family of seminorms which defines the topology on X.

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DEFINITION 1 *Let K be a nonempty subset of X and $T : K \to K$. Then (i) T is called a* p_α-contraction *if there is λ_α, $0 < \lambda_\alpha < 1$, such that*

$$p_\alpha(Tx - Ty) \leq \lambda_\alpha p_\alpha(x - y)$$

for each $x, y \in K$ and $\alpha \in J$. If $p_\alpha(Tx - Ty) \leq \lambda_\alpha p_\alpha(x - y)$ for all $p_\alpha \in \mathcal{P}$, then T is called $\mathcal{P}$-contraction (or simply a contraction). If $\lambda_\alpha = 1$, then T is called $\mathcal{P}$-nonexpansive (or simply nonexpansive). (ii) T is asymptotically nonexpansive *if there is a sequence $\{\lambda_n\}$ of real numbers satisfying $\lambda_n \geq 1$ and $\lambda_n \geq \lambda_{n+1}$ for $n = 1, 2, 3, ...$ and $\lambda_n \to 1$ as $n \to \infty$ such that*

$$p_\alpha(T^n x - T^n y) \leq \lambda_n p_\alpha(x - y)$$

for arbitrary $x, y \in K$, each $n \in \mathbb{N}$ and each $\alpha \in J$. (iii) T is uniformly asymptotically regular *with respect to a map $S : K \to K$ if for each $\alpha \in J$ and $\epsilon > 0$, there exists $N = N(\alpha, \epsilon)$ such that*

$$p_\alpha(T^n x - T^{n+1} x + Sx) < \epsilon$$

for all $n \geq N$ and for all $x, y \in K$. If T is uniformly asymptotically regular with respect to the zero operator, we simply say that T is uniformly asymptotically regular. *(iv) T is* asymptotically regular *on K if for each $x \in K$ and $\alpha \in J$*

$$\lim p_\alpha(T^n x - T^{n+1} x) = 0$$

The following result is a generalization of the Banach contraction principle to locally convex spaces due to Cain and Nashed [1]

THEOREM 1 *Let K be a sequentially complete subset of X and $T : K \to K$ a contraction mapping. Then T has a unique fixed point $\bar{x} \in K$ and $T^n x \to \bar{x}$ for every $x \in K$.*

Tychonoff (1935) obtained the following fixed point result.

THEOREM 2 *Let K be a nonempty compact convex subset of X. If $T : K \to K$ is any continuous mapping, then T has a fixed point in K.*

The following notion of almost convex set was introduced by Himmelberg [3].

DEFINITION 2 *A nonempty subset K of a topological vector space X is called* almost convex *if for any neighbourhood V of the origin 0 in X and for any finite set $\{x_1, x_2, \ldots, x_n\} \subseteq K$, there exists a finite set $\{z_1, z_2, \ldots, z_n\} \subseteq K$ such that for each $i \in \{1, 2, 3 \ldots, n\}$, $z_i - x_i \in V$ and $co\{z_1, z_2, \ldots, z_n\} \subseteq K$.*

In the above definition, "co" stands for the convex hull of a set. Clearly, every convex set is almost convex but the converse is not true in general. Park and Tan [7] proved the following generalization of the Tychonoff fixed point theorem.

THEOREM 3 *Let K be a nonempty compact almost convex subset of X, and $T : K \to K$ a continuous mapping. Then T has a fixed point in K.*

DEFINITION 3 *Let K be a subset of a vector space X. Then K is called* star-shaped *if there exists $p \in K$ such that $tp + (1 - t)x \in K$ for all $x \in K$, $0 \leq t \leq 1$. The point p is called a* star-point *and the set of all the star-points of K is called the* star-core *of K.*

Clearly, the star-core is a convex subset of K.

DEFINITION 4 *A mapping T on a convex set K is called* affine *if it satisfies the identity*

$$T(tx + (1 - t)y) = tTx + (1 - t)Ty$$

where $0 < t < 1,\ x, y \in K$.

It is evident that every affine mapping is convex. The following result is well known (see [2]).

THEOREM 4 *If K is a compact star-shaped subset of X and C is the corresponding star-core of K, then C is a compact convex subset of K.*

Hu and Heng [2] proved the following results.

THEOREM 5 *Let K be a nonempty compact star-shaped subset of a topological vector space X. Then every decreasing chain of nonempty compact and star-shaped subsets of K has a nonempty intersection that is compact and star-shaped.*

THEOREM 6 *Suppose K is a star-shaped subset of a topological vector space X and $T : K \to K$ a surjective mapping that is affine on K. Then the star-core of K is invariant under T.*

Applying the above results, we have the following generalization of Tychonoff's fixed point theorem:

THEOREM 7 *Let K be a nonempty compact and star-shaped subset of X. If $T : K \to K$ is an affine continuous mapping, then T has a fixed point in K.*

Proof. Since affine maps preserve star-shapedness and continuous maps preserve compactness, we define a decreasing chain of nonempty, compact and star-shaped subsets of K by $K_1 = K$ and $K_{n+1} = TK_n$, $n = 1, 2, 3, \ldots$ Clearly, $TK_1 \subseteq K_1$. Suppose $TK_n \subseteq K_n$. Then

$$TK_{n+1} = T(TK_n) \subseteq TK_n = K_{n+1}$$

Hence by induction $TK_n \subseteq K_n\ \forall\ n$, showing that K_n is invariant under T and that $K_n \supseteq TK_n = K_{n+1}$, $n = 1, 2, \ldots$.

Applying Theorem 5 and Zorn's lemma, we get a minimal nonempty, compact and star-shaped subset M of K which is invariant under T. We claim that $TM = M$. Suppose that $TM = S \subset M$. Since T is affine and continuous, S is nonempty compact and star-shaped and $TS \subseteq TM = S$. That is, S is a nonempty compact and star-shaped subset of K which is invariant under T. This contradicts the minimality of M. Hence, $TM = M$, that is, $T : M \to M$ is surjective.

Now, let C be the star-core of M. By Theorems 4 and 6, C is a compact convex subset of M and $T : C \to C$. Hence, by the Tychonoff fixed point theorem, T has a fixed point in $C \subset K$. ■

2. Main Result

We prove some fixed point results for the sums of two mappings defined on star-shaped as well as almost convex subsets of a locally convex space. Our results generalize the fixed point results of Vijayaraju [9]. First we establish the following lemma.

LEMMA 1 *Let $T : K \to X$ be an asymptotically nonexpansive mapping of a subset K of X into X. Then $(I - a_nT^n)$ is a homeomorphism on K onto $(I - a_nT^n)K$, where $a_n = \frac{1-\frac{1}{n}}{\lambda_n}$ and λ_n are as in the definition 1(ii).*

Proof. The mapping a_nT^n is a contraction mapping for each n. Since every contraction mapping is continuous, a_nT^n is continuous and so is $I - a_nT^n$. Let $x, y \in K$ such that $(I - a_nT^n)x = (I - a_nT^n)y$. Then $a_n(T^nx - T^ny) = x - y$. a_nT^n is a contraction, implying that

$$p_\alpha(x - y) = a_np_\alpha(T^nx - T^ny) \leq (1 - \frac{1}{n})p_\alpha(x - y)$$

$\Rightarrow \frac{1}{n}p_\alpha(x - y) \leq 0$ and so $p_\alpha(x - y) = 0$. Since X is Hausdorff, we have $x = y$. Showing that $I - a_nT^n$

is one-one. Also,

$$p_\alpha((I - a_nT^n)x - (I - a_nT^n)y) \geq p_\alpha(x - y) - a_np_\alpha(T^nx - T^ny) \geq \frac{1}{n}p_\alpha(x - y)$$

for each α and $n = 1, 2, ...$ This implies that $(I - a_nT^n)^{-1}$ is continuous and so $I - a_nT^n$ is a homeomorphism. ■

Theorem 8 *Let K be a nonempty compact star-shaped subset of X. Let $T, S : K \to X$ be mappings such that*

(i) T is an affine asymptotically nonexpansive self mapping.
(ii) S is an affine continuous mapping.
(iii) T is uniformly asymptotically regular with respect to S and $T^nx + Sy \in K \;\; \forall\, x, y \in K$ and n=1,2,...

Then there is a point $\bar{x} \in K$ such that $T\bar{x} + S\bar{x} = \bar{x}$.

Proof. For each $y \in K$, we define a mapping $F_n : K \to K$ by

$$F_n(x) = a_n(T^nx + Sy) \quad x \in K$$

where $a_n = (1 - \frac{1}{n})/\lambda_n$ and $\{\lambda_n\}$ is as in definition 1(ii). It follows from the fact that T is asymptotically nonexpansive that for $x_1, x_2 \in K$ and $\alpha \in J$,

$$p_\alpha(F_n(x_1) - F_n(x_2)) = a_np_\alpha(T^nx_1 - T^nx_2) \leq (1 - \frac{1}{n})p_\alpha(x_1 - x_2)$$

This shows that F_n is a contraction on K and so by Theorem 1, F_n has a unique fixed point, say, $H_ny \in K$. Hence,

$$H_ny = F_n(H_ny) = a_n(T^n(H_ny) + Sy). \tag{1}$$

As a result of Lemma 1, (1) can be written as

$$H_ny = (I - a_nT^n)^{-1}(a_nS)y$$

from which it follows that H_n is continuous, being a composition of continuous mappings. Also for each n, T^n is affine and so $(I - a_nT^n)^{-1}$ is affine. Therefore, $H_n = (I - a_nT^n)^{-1}(a_nS)$ is an affine continuous mapping. By Theorem 7, H_n has a fixed point, say, $x_n \in K$. Therefore,

$$x_n = H_n(x_n) = a_n(T^nx_n + Sx_n). \tag{2}$$

Hence

$$x_n - T^nx_n - Sx_n = (a_n - 1)(T^nx_n + Sx_n) \to 0 \text{ as } n \to \infty \tag{3}$$

since $a_n \to 1$ as $n \to \infty$ and K is bounded and $T^nx + Sy \in K$, $\forall\, x, y \in K$. As T is uniformly asymptotically regular with respect to S, it follows that

$$T^nx_n - T^{n-1}x_n + Sx_n \to 0 \text{ as } n \to \infty. \tag{4}$$

From (3) and (4), we obtain

$$x_n - T^{n-1}x_n \to 0 \text{ as } n \to \infty. \tag{5}$$

Also,

$$p_\alpha(x_n - (T+S)x_n) \le p_\alpha(x_n - (T^n+S)x_n) + p_\alpha((T^n+S)x_n - (T+S)x_n)$$

$$\le p_\alpha(x_n - (T^n+S)x_n) + \lambda_1 p_\alpha(T^{n-1}x_n - x_n). \tag{6}$$

Using (3) and (5) in (6), we obtain

$$x_n - (T+S)x_n \to 0 \text{ as } n \to \infty. \tag{7}$$

K is compact implies there exists a subsequence (x_β) of the sequence (x_n) such that

$$x_\beta \to \bar{x} \quad \text{for some } \bar{x} \in K.$$

As S and T are continuous, it follows that

$$(I - (T+S))(x_\beta) \to (I - (T+S))(\bar{x})$$

and by (7) we have

$$x_\beta - (T+S)(x_\beta) \to 0.$$

Since X is Hausdorff, it follows that $(I - (T+S))(\bar{x}) = 0$. Hence, $T\bar{x} + S\bar{x} = \bar{x}$ for some $\bar{x} \in K$. This completes the proof. ■

Remark 1 If T is nonexpansive, the condition that T is uniformly asymptotically regular with respect to S can be dropped and so we have the following:

THEOREM 9 *Let K be a nonempty compact star-shaped subset of X. Let $T, S : K \to X$ be mappings such that*

(i) T is an affine nonexpansive mapping.
(ii) S is an affine continuous mapping such that $Tx + Sy \in K \ \forall\, x, y \in K$

Then there is a point $\bar{x} \in K$ such that $T\bar{x} + S\bar{x} = \bar{x}$.

Proof. For each $y \in K$ and $n \in \mathbb{N}$, we define a mapping $F_n : K \to K$ by

$$F_n(x) = \lambda_n(Tx + Sy) \quad x \in K$$

where $\{\lambda_n\}$ is a sequence of real numbers with $0 < \lambda_n < 1$ and $\lambda_n \to 1$ as $n \to \infty$. Mimicking the proof of the above theorem and applying Theorems 1 and 7, we obtain a sequence $\{x_n\}$ in K such that

$$x_n = \lambda_n(Tx_n + Sx_n).$$

As K is compact, there exists a subsequence (x_β) of $\{x_n\}$ such that

$$x_\beta \to \bar{x} \quad \text{for some } \bar{x} \in K$$

Therefore $x_\beta = \lambda_\beta(Tx_\beta + Sx_\beta)$. By the continuity of S and T, it follows that $\bar{x} = (T+S)\bar{x}$. Hence the result. ■

If we proceed as in the proofs of the above theorems and we apply Theorems 1 and 3, we have the following results:

Theorem 10 *Let K be a nonempty compact almost convex subset of X. Let $T, S : K \to X$ be mappings such that*

(i) T is an asymptotically nonexpansive self mapping.
(ii) S is a continuous mapping.
(iii) T is uniformly asymptotically regular with respect to S and $T^n x + Sy \in K \;\; \forall\, x, y \in K$ and n=1,2,...

Then there is a point $\bar{x} \in K$ such that $T\bar{x} + S\bar{x} = \bar{x}$.

Theorem 11 *Let K be a nonempty compact almost convex subset of X. Let $T, S : K \to X$ be mappings such that*

(i) T is a nonexpansive mapping.
(ii) S is a continuous mapping such that $Tx + Sy \in K \;\; \forall\, x, y \in K$

Then there is a point $\bar{x} \in K$ such that $T\bar{x} + S\bar{x} = \bar{x}$.

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