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In honour of Prof. Ekhaguere at 70

On the solvability of a second order boundary value problem at resonance with integral boundary condition

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Abstract. This paper investigates the existence of solutions for the following differential equation

$$x''(t) = f(t, x(t), x'(t)) + e(t)$$

subject to the boundary conditions

$$x'(0) = 0,\ x(1) = \int_0^1 x(s)dA(s)$$

where $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, $A : [0,1] \longrightarrow [0,\infty)$ is a non decreasing function with $A(0) = 0,\ A(1) = 1$ and $e(t) \in L^1[0,1]$. Our method of proof is based on coincidence degree arguments.

Keywords: resonance, integral boundary condition, second order, coincidence degree.

1. Introduction

This paper deals with the following second order boundary value problem

$$x''(t) = f(t, x(t, x(t), x'(t)) + e(t) \tag{1.1}$$

subject to the boundary conditions

$$x'(0) = 0, x(1) = \int_0^1 x(s)dA(s) \tag{1.2}$$

where $f : [0,1] \times R^2 \longrightarrow \mathbb{R}$ is a continuous function $A : [0,1] \longrightarrow [0,\infty)$ is a non decreasing function with $A(0) = 0$, and $A(1) = 1$. The integral is the Riemann Stieltjes integral and $e(t) \in L^1[0,1]$. The boundary value problem (1.1) - (1.2) is said to be a problem at resonance if the linear equation

$$x''(t) = 0,\ t \in (0,1) \tag{1.3}$$

with the boundary conditions (1.2) has non trivial solutions. If (1.3) - (1.2) has only the trivial solution then the problem is said to be at non resonance. In [3] the authors studied the non resonance boundary value problem

$$x''(t) + q(t)f(t, x(t), x'(t)) = 0 \tag{1.4}$$

$$x(0) = 0, x'(1) = \int_0^1 x'(s)dg(s) \tag{1.5}$$

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under the conditions $g(0) = 0$, $0 \leq g(1) < 1$. They obtained existence results by using the Krasnoselskii's fixed point theorem.

Motivated by the above results, the aim of this paper is to establish existence results for (1.1) - (1.2) by using the coincidence degree theory of Mawhin.

2. Preliminaries

Let X and Z be real Banach spaces and let $L : domL \subset X \longrightarrow Z$ be a linear Fredholm operator of index zero. Let $P : X \longrightarrow X$ and $Q : Z \longrightarrow Z$ be continuous projections such that $ImP = \ker L$, $\ker Q = ImL$ and $X = \ker L \oplus \ker P$, $Z = ImL \oplus ImQ$. It follows that $L|_{domL \cap \ker P} : domL \cap \ker P \longrightarrow ImL$ is invertible. We denote this inverse by K_p. If Ω is an open bounded subset of X such that $domL \cap \Omega \neq \phi$ then the mapping $N : X \longrightarrow Z$ is called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \longrightarrow X$ is compact.

In what follows, we shall use the following abstract existence results of Mawhin [6].

Theorem 2.1 (6) *Let L be a Fredholm operator of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(domL \backslash_{\ker L}) \cap \partial\Omega] \times (0, 1)$

(ii) $Nx \notin ImL$ *for every* $x \in \ker L \cap \partial\Omega$

(iii) $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$

Then the equation $Lx = Nx$ has at least one solution in $domL \cap \bar{\Omega}$.

For $x \in C^1[0, 1]$ we use the norms

$$\|x\|_\infty = \max_{t \in [0,1]} |x(t)| \text{ and } \|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$$

and we denote the norm in $L^1[0, 1]$ by $\| \cdot \|_1$. We will use the Sobolev space $W^{2,1}(0, 1)$ defined by

$$W^{2,1}(0, 1) = \{x : [0, 1] \longrightarrow \mathbb{R} | x, x' \text{ are absolutely continuous on } [0, 1] \text{ with } x'' \in L^1[0, 1] \tag{2.1}$$

Let $X = C^1[0, 1]$, $Z = L^1[0, 1]$. L is the linear operator from $domL \subset X \longrightarrow Z$ with

$$domL = \left\{ x \in W^{2,1}(0, 1); x'(0) = 0,\ x(1) = \int_0^1 x(s)dA(s) \right\} \tag{2.2}$$

We define $L : domL \subset X \longrightarrow Z$ by

$$Lx = x''(t),\ x \in domL \tag{2.3}$$

and

$$N : X \longrightarrow Z$$

by

$$Nx = f(t, x(t), x'(t)) + e(t),\ t \in (0, 1) \tag{2.4}$$

Then the bvp (1.1) - (1.2) becomes

$$Lx = Nx$$

LEMMA 2.2 *If* $\int_0^1 s^2 dA(s) \neq 1,\ A(1) = 1, A(0) = 0$ *then*

(i) $\ker L = \{x \in domL : x = c,\ c \in \mathbb{R}\}$

(ii) $ImL = \left\{ y \in z : \int_0^1 \int_0^s y(\tau)d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau)d\tau dt\ dA(s) = 0 \right\}$

(iii) $L : domL \subset X \longrightarrow Z$ *is a Fredholm operator of index zero and furthermore the linear continuous projection* $Q : Z \longrightarrow Z$ *can be written as*

$$Qy = \frac{2}{1 - \int_0^1 s^2 dA(s)} \left[\int_0^1 \int_0^s (\tau)d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau)d\tau dt dA(s) \right]$$

(iv) The linear operator $K_p : ImL \longrightarrow domL \cap \ker P$ *can be defined as*

$$K_p y = \int_0^t \int_0^s y(\tau)d\tau ds$$

(v) $\|K_p y\| \leq \|y\|_1$

Proof. (i) For $x \in \ker L$, we have $x''(t) = 0$ and hence

$$x(t) = a_0 + a_1 t,\ a_i \in \mathbb{R} \tag{2.5}$$

In view of $x'(1) = 0$, $x(t) = \int_0^1 x(s)dA(s)$ we derive that

$$\ker L = \{x \in domL : x = c, \quad c \in \mathbb{R}\}$$

(ii) We next show that

$$ImL = \left\{ y \in Z : \int_0^1 \int_0^s y(\tau)d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau)d\tau dt dA(s) = 0 \right\}$$

We consider the problem

$$x''(t) = y(t) \tag{2.6}$$

We show that problem (2.6) has a solution $x(t)$ satisfying

$$x'(0) = 0,\ x(1) = \int_0^1 x(s)dA(s) \tag{2.7}$$

If and only if,

$$\int_0^1 \int_0^s y(\tau)d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau)d\tau dt dA(s) = 0 \tag{2.8}$$

Suppose (2.6) has a solution $x(t)$ satisfying (2.7) then from (2.6) we obtain

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \int_0^t \int_0^s y(\tau) d\tau ds \\ &= x(0) + \int_0^t \int_0^s y(\tau) d\tau ds \\ x(1) &= x(0) + \int_0^1 \int_0^s y(\tau) d\tau ds = \int_0^1 x(s) dA(s) \\ &= \int_0^1 \left[x(0) + \int_0^s \int_0^t y(\tau) d\tau dt \right] dA(s) \\ &= A(1)x(0) + \int_0^1 \int_0^s \int_0^t y(\tau) d\tau dt dA(s) \end{aligned}$$

Since $A(1) = 1$ we have

$$\int_0^1 \int_0^s y(\tau) d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau) d\tau dt dA(s) = 0$$

If however (2.8) holds; then setting

$$x(t) = c + \int_0^t \int_0^s y(\tau) d\tau ds$$

where c is an arbitrary constant then $x(t)$ is a solution of (2.6) satisfying (2.7).

(iii) For $y \in Z$, we define the projection Q by

$$Qy = \frac{2}{1 - \int_0^1 s^2 dA(s)} \left[\int_0^1 \int_0^s y(\tau) d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau) d\tau dt dA(s) \right]$$

Let $y_1 = y - Qy$ then

$$\begin{aligned} &\int_0^1 \int_0^s y_1(\tau) d\tau ds - \int_0^1 \int_0^s \int_0^t y_1(\tau) d\tau dt dA(s) \\ &= \int_0^1 \int_0^s y(\tau) d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau) d\tau dt dA(s) - Qy \frac{\left(1 - \int_0^1 s^2 dA(s)\right)}{2} \\ &= 0. \end{aligned}$$

Thus $y_1 \in ImL$ and $Z = ImL + \mathbb{R}$ and since $ImL \cap \mathbb{R} = \{0\}$ we conclude that $Z = ImL \oplus \mathbb{R}$. Therefore,

$$\dim \ker L = \dim \mathbb{R} = codimImL = 1$$

Hence L is a Fredholm operator of index zero.

(iv) Let $P : X \longrightarrow X$ be defined as

$$Px = c, \; c \in \mathbb{R} \tag{2.9}$$

We define the generalized inverse
$K_p : ImL \longrightarrow domL \cap \ker P$ as

$$K_p y = \int_0^t \int_0^s y(\tau) d\tau ds \tag{2.10}$$

For $y \in ImL$, we have

$$(LK_p)y(t) = [K_p y]'' = y(t)$$

and for $x \in domL \cap \ker P$ we obtain

$$(K_pL)x(t) = \int_0^t \int_0^s x''(\tau)d\tau ds = x(t) - x(0)$$

since $x \in domL \cap \ker P, Px = x(0) = 0$.
Therefore,

$$(K_pL)x(t) = x(t)$$

We thus conclude that

$$K_p = (L|_{domL\cap\ker P})^{-1}$$

(v) From the definition of K_p, we derive that

$$\|K_p\|_\infty \leq \int_0^1 \int_0^1 [y(\tau)d\tau ds = \|y\|_1 \tag{2.11}$$

$$(K_py)'(t) = \int_0^t y(\tau)d\tau \leq \int_0^1 |y(\tau)|d\tau$$

$$\|(K_py)'\|_\infty \leq \|y\|_1 \tag{2.12}$$

Therefore from (2.11) and (2.12) we conclude that

$$\|K_py\| \leq \|y\|_1 \tag{2.13}$$

■

3. Existence results

THEOREM 3.1 *Let $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function. Assume that*

(A1) There exist functions $a(t), b(t), r(t) \in L^1[0,1]$ such that for all $(x_1, x_2) \in \mathbb{R}^2$ $t \in [0,1]$

$$|f(t, x_1, x_2)| \leq a(t)|x_1| + b(t)|x_2| + r(t) \tag{3.1}$$

(A2) There exist a constant $M_1 > 0$ such that for $x \in domL$, if $x(t) > M_1$ for all $t \in [0,1]$, then

$$\int_0^1 \int_0^s y(\tau)d\tau ds - \int_0^1 \int_0^s \int_0^t y(\tau)d\tau dt dA(s) \neq 0 \tag{3.2}$$

(A3) There exists a constant $M_2 > 0$ such that for $c \in \mathbb{R}, |c| > M_2$ then either

$$c \cdot QN(c) \geq 0 \quad or \quad c \cdot QN(c) \leq 0 \tag{3.3}$$

Then for $e(t) \in L^1[0,1]$ the bvp (1.1) - (1.2) with $A(0) = 0$, $A(1) = 1$, $\int_0^1 s^2 dA(s) \neq 1$ has at least one solution in $C^1[0,1]$ provided

$$\|a\|_1 + \|b\|_1 < \frac{1}{2} \tag{3.4}$$

To prove Theorem 3.1 we require the following Lemmas.

LEMMA 3.2 *Let $\Omega_1 = \{x \in domL\backslash_{\ker L} : Lx = \lambda Nx \ \ \lambda \in (0,1]\}$ Then Ω_1 is bounded in X.*

Proof. Let $x \in \Omega_1$. We assume that $Lx = \lambda Nx$ for $0 < \lambda \leq 1$. Then $Nx \in ImL = \ker Q$ and hence from (A2) there exist $t_0 \in [0,1]$ such that $|x(t_0)| \leq M_1$. Therefore

$$x(0) = x(t_0) - \int_0^{t_0} x'(s)ds$$

$$|x(0)| \leq M_1 + \|x'\|_1 \leq M_1 + \|x'\|_\infty \tag{3.5}$$

We also note that since $x'(0) = 0$,

$$\|x'\|_\infty \leq \|x''\|_1 \tag{3.6}$$

From (3.5) and (3.6) we get

$$\|Px\| = |x(0)| \leq M_1 + \|Nx\|_1 \tag{3.7}$$

For $x \in \Omega_1, x \in domL\backslash_{kerL}$, then $(I-P)x \in domL \cap \ker P$

$$\|(1-P)x\| = \|K_p L(1-P)x\| \leq \|L(1-P)x\|_1 = \|Lx\|_1 \leq \|Nx\|_1 \tag{3.8}$$

where I is the identity operator on X.
Using (3.7) and (3.8) we obtain

$$\|x\| = \|Px + (I-P)x\| \leq \|Px\| + \|(I-P)x\| \leq M_1 + 2\|Nx\|_1 \tag{3.9}$$

By (A1) and the definition of N we derive that

$$\|Nx\|_1 \leq \int_0^1 |f(s,x(s),x'(s)) + e(s)|ds \leq \|a\|_1\|x\|_\infty + \|b\|_1\|x'\|_\infty + \|r\|_1 + \|e\|_1$$
$$\leq \|a\|_1\|x\| + \|b\|_1\|x\| + \|r\|_1 + \|e\|_1 \tag{3.10}$$

Combining (3.9) and (3.10) we obtain

$$\|x\| \leq \frac{2\|r\|_1 + 2\|e\|_1 + M_1}{1 - 2(\|a\|_1 + \|b\|_1)} \tag{3.11}$$

From (A1) and (3.11) we derive

$$\|x''\|_1 = \|Lx\|_1 \leq \|Nx\|_1 \leq \|a\|_1\|x\| + \|b\|_1\|x\| + \|r\|_1 + \|e\|_1$$
$$\leq \frac{(2\|r\|_1 + 2\|e\|_1 + M_1)}{1 - 2(\|a\|_1 + \|b\|_1)}[\|a\|_1 + \|b\|_1] + \|r\|_1 + \|e\|_1$$

Since $\|a\|_1 + \|b\|_1 < \frac{1}{2}$ we conclude that Ω_1 is bounded in X. ■

LEMMA 3.3 *The set $\Omega_2 = \{x \in \ker L : Nx \in ImL\}$ is a bounded set in X.*

Proof. Let $x \in \Omega_2, x = c,\ c \in \mathbb{R}$, and $QNx = 0$. Therefore,

$$\int_0^1 \int_0^s [f(\tau, c, 0) + e(\tau)]d\tau ds - \int_0^1 \int_0^s \int_0^t [f(\tau, c, 0) + e(\tau)]d\tau dt dA(s) = 0.$$

From (A2) there exist $t_0 \in [0, 1]$ such that $\|x(t_0| \leq M_1$. That is $|c| \leq M_1$ hence

$$\|x\| = \max\{\|x\|_\infty,\ \|x'\|_\infty\} = |c| \leq M_1.$$

Therefore Ω_2 is bounded in X. ■

LEMMA 3.4 *Let*

$$\Omega_3^+ = \{x \in kerL : \lambda x + (1 - \lambda)QNx = 0,\ \lambda \in [0, 1]\} \tag{3.12}$$

and

$$\Omega_3^- = \{x \in \ker L : -\lambda x + (1 - \lambda)QNx = 0,\ \lambda \in [0, 1]\} \tag{3.13}$$

Then Ω_3^+ and Ω_3^- are bounded in X provided (3.12) and (3.13) are satisfied respectively.

Proof. Let $x \in \Omega_3^+$. Then, there exist $c \in \mathbb{R}$ such that $x(t) = c$. From the first part of (3.3) we have for $|c| > M_2$, $c \cdot QN(c) \geq 0$. From (3.12) we have

$$(1 - \lambda)QNx = -\lambda x \tag{3.14}$$

If $\lambda = 0$, it follows that $QNx = 0$ and therefore $Nx \in \ker Q = ImL$, that is $Nx \in \Omega_2$ and by Lemma 3.2 we can deduce that $\|x\| \leq M_1$. However, if $\lambda \in (0, 1]$ and $|c| > M_2$ then by the first part of (3.3) we derive

$$0 \leq (1 - \lambda)c \cdot QN(c) = -\lambda|c|^2 < 0.$$

which is a contradition. Thus $\|x\| = |c| \leq M_2$. Therefore Ω_3^+ is bounded. By a similar argument we can prove that Ω_3^- is bounded in X. ■

THEOREM 3.5 *Let the assumption (A1)-(A3) hold. Then problem (1.1)-(1.2) has at least one solution in X.*

Proof. We will show that all the condition of theorem 2.1 are satisfied. Let Ω be a bounded subset of X such that $\cup_{i=1}^3 \bar{\Omega}_i \subset \Omega$. It is easily seen that conditions (i) and (ii) of Theorem 2.1 are satisfied by using Lemma 3.1 and Lemma 3.2. To verify the third condition we apply the invariance under a homotopy property of the degree. That is we set

$$H(x, \lambda) = \pm\lambda x + (1 - \lambda)QNx.$$

Let $I : ImQ \longrightarrow \ker L$ be the identity operator. By Lemma 3.3 we know that $H(x, \lambda) \neq 0$ for $(x, \lambda) \in \ker L \cap \partial\Omega \times [0, 1]$. Therefore,

$$\begin{aligned} deg(QN|_{\ker L},\ \Omega \cap \ker L, 0) &= deg(H(\cdot, 1),\ \Omega \cap \ker L, 0) \\ &= deg(\pm I, \Omega \cap \ker L, 0) = \pm 1 \neq 0 \end{aligned}$$

This proves Theorem 3.1. ■

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