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Certain subclass of univalent functions using generalised Salagean differential operator involving modified sigmoid function

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Abstract. By making use of a generalised Salagean differential operator involving modified real sigmoid function, the authors defined a certain subclass of univalent holomorphic functions denoted by class $G_\gamma(\beta, \xi, \mu, \pi)$. Various geometric properties of the class were investigated and established.

Keywords: holomorphic function, sigmoid function, generalised Sălăgean operator, convex set, closure property.

1. Introduction and Preliminaries

The study of subclasses of analytic and univalent function f(z) and their geometric properties is significant. Various authors such as [1],[2], [3],[5], [6]and [7] have successfully defined and investigated certain subclasses of univalent functions using differential operators. Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let

$$\gamma(s) = \frac{2}{1+e^{-s}} = 1 + \frac{1}{2}s - \frac{1}{24}s^3 + \frac{1}{240}s^5 - \frac{17}{40320}s^7 + \dots \tag{2}$$

be a modified sigmoid function with $\gamma(0) = 1$. Let A_γ denote the class of function of the form

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \tag{3}$$

(3) is of the form (1) for $\gamma = 1$. Note $A_1 \equiv A$. We denote by T the subclass of A consisting of functions $f(z) \in A$ which are analytic and univalent in U and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \tag{4}$$

Similarly, we have $f_\gamma(z) \in T_\gamma$ defined as

$$f(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \tag{5}$$

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We define an identity function involving the modified sigmoid function as

$$e_\gamma(z) = z \tag{6}$$

DEFINITION 1.1 *Salagean [7] introduced a differential operator called Salagean differential operator D^n defined as*

$$D^n : A \to A, n \in N_0 = N \cup \{0\}$$

$$D^0 f(z) = f(z)$$

$$D' f(z) = Df(z) = z\left(1 - \sum_{k=2}^{\infty} a_k z^{k-1}\right) = z - \sum_{k=2}^{\infty} k a_k z^k$$

$\vdots$

$$D^n f(z) = D\left[D^{n-1} f(z)\right], = z - \sum_{k=2}^{\infty} k^n a_k z^k$$

The basic geometric properties such as classes of convex functions, starlike functions, close-to-convex functions, coefficient estimate, radius of convexity problems, growth and distortion properties and other properties have been investigated by various authors. Darus and Ibrahim [2] introduced and studied

$$A^m(\alpha, \beta) f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)(\mu - \alpha)\beta]^m a_k z^k$$

Let $f_\gamma(z) \in A_\gamma$ and let $D^n_{\alpha,\lambda} f(z)$ be a generalised Salagean differential operator involving modified sigmoid function with $D^n_{\alpha,\lambda} : A_\gamma \to A_\gamma$, $n \in N_0 = N \cup \{0\}$ defined as follow See [1] and [4]

$$D^n_{\alpha,\lambda} f_\gamma(z) = \gamma^n(s) z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)[(k-1)(\lambda - \alpha) + 1]^n a_k z^k \tag{7}$$

Let $[(k-1)(\lambda - \alpha) + k] = C_k[\lambda, \alpha]$. Hence,

$$D^{n+1}_{\alpha,\lambda} f_\gamma(z) = \gamma^{n+1}(s) z + \sum_{k=2}^{\infty} \gamma^{n+2}(s) C_k[\lambda, \alpha]^{n+1} a_k z^k$$

DEFINITION 1.2 *A function $f_\gamma \in A_\gamma$ defined by (3) is said to belong to the class $G_\gamma(\beta, \xi, \mu, \pi)$ if*

$$\left| \frac{\frac{D^{n+1}_{\alpha,\lambda} f(z)}{D^n_{\alpha,\lambda} f(z)} - \xi}{3\mu\left(\frac{D^{n+1}_{\alpha,\lambda} f(z)}{D^n_{\alpha,\lambda} f(z)}\right) - \beta\left(\frac{D^{n+1}_{\alpha,\lambda} f(z)}{D^n_{\alpha,\lambda} f(z)} - \xi\right)} \right| < \pi$$

$\pi > 0, 0 < \xi \leq 1, 0 < \beta \leq 1, 0 < \mu \leq 1.$

2. Main Result

In this section we prove the main result.

THEOREM 2.1 *Coefficient Estimates for class $G_\gamma(\beta,\xi,\mu,\pi)$. If a function $f_\gamma(z)$ defined by (3) belongs to the the class $G_\gamma(\beta,\xi,\mu,\pi)$ then*

$$\sum_{k=2}^{\infty}\gamma\left\{(3\pi\mu-\beta\pi)[(k-1)(\lambda-\alpha)+k]-\beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n a_k \leq [3\gamma\pi\mu-\gamma(1-\xi)(\pi\beta-1)]$$

Proof.

$$D^n_{\alpha,\lambda}f_\gamma(z)=\gamma^n(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k$$

$$D^{n+1}_{\alpha,\lambda}f_\gamma(z)=\gamma^{n+1}(s)z+\sum_{k=2}^{\infty}\gamma^{n+2}(s)C_k[\lambda,\alpha]^{n+1} a_k z^k$$

Thus

$$\frac{D^{n+1}_{\alpha,\lambda}f_\gamma(z)}{D^n_{\alpha,\lambda}f_\gamma(z)}-\xi=\frac{\gamma^n(s)[\gamma(s)-\xi]z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n\left[\gamma(s)C_k[\lambda,\alpha]-\xi\right]a_k z^k}{\gamma^n(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k} \tag{8}$$

$$3\mu\left(\frac{D^{n+1}_{\alpha,\lambda}f_\gamma(z)}{D^n_{\alpha,\lambda}f_\gamma(z)}\right)=\frac{3\mu\gamma^{n+1}(s)z+\sum_{k=2}^{\infty}3\mu\gamma^{n+2}(s)C_k[\lambda,\alpha]^{n+1}a_k z^k}{\gamma^n(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k}$$

$$\beta\left(\frac{D^{n+1}_{\alpha,\lambda}f_\gamma(z)}{D^n_{\alpha,\lambda}f_\gamma(z)}-\xi\right)$$

$$=\frac{\beta\gamma^{n+1}(s)z+\sum_{k=2}^{\infty}\beta\gamma^{n+2}(s)C_k[\lambda,\alpha]^{n+1}a_k z^k-\beta\xi\gamma^n(s)z-\sum_{k=2}^{\infty}\beta\xi\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k}{\gamma^n(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k}.$$

$$\frac{\beta\gamma^n(s)[\gamma(s)-\xi]z+\sum_{k=2}^{\infty}\beta\gamma^{n+1}(s)C_k[\lambda,\alpha]^n\left[\gamma(s)C_k[\lambda,\alpha]-\xi\right]a_k z^k}{\gamma^n(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k}$$

$$3\mu\left(\frac{D^{n+1}_{\alpha,\lambda}f_\gamma(z)}{D^n_{\alpha,\lambda}f_\gamma(z)}\right)-\beta\left(\frac{D^{n+1}_{\alpha,\lambda}f_\gamma(z)}{D^n_{\alpha,\lambda}f_\gamma(z)}-\xi\right)=\frac{3\mu\gamma^{n+1}(s)z+\sum_{k=2}^{\infty}3\mu\gamma^{n+2}(s)C_k[\lambda,\alpha]^{n+1}a_k z^k}{\gamma^n(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k}$$

$$-\frac{\beta\gamma^{n+1}(s)z+\sum_{k=2}^{\infty}\beta\gamma^{n+2}(s)C_k[\lambda,\alpha]^{n+1}a_k z^k-\beta\xi\gamma^n(s)z-\sum_{k=2}^{\infty}\beta\xi\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k}{\gamma^n(s)z+\sum_{k=2}^{\infty}\gamma^{n+1}(s)C_k[\lambda,\alpha]^n a_k z^k}.$$

$$= \frac{\gamma^n(s)[3\mu\gamma(s) - \beta\gamma(s) + \beta\xi]z + \sum_{k=2}^{\infty} \gamma^n(s) C_k[\lambda,\alpha]^n \left\{3\mu\gamma^2 C_k[\lambda,\alpha] - \beta\gamma^2(s) C_k[\lambda,\alpha] + \beta\xi\gamma(s)\right\} a_k z^k}{\gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s) C_k[\lambda,\alpha]^n a_k z^k}. \tag{9}$$

Substituting (8) and (9) in Definition 1.2, we have

$$\left|\gamma^n(s)[\gamma(s) - \xi]z + \sum_{k=2}^{\infty} \gamma^{n+1}(s) C_k[\lambda,\alpha]^n \left[\gamma(s) C_k[\lambda,\alpha] - \xi\right] a_k z^k\right|$$

$$< \left|\pi\gamma^n(s)[3\mu\gamma(s) - \beta\gamma(s) + \beta\xi]z + \sum_{k=2}^{\infty} \pi\gamma^n(s) C_k[\lambda,\alpha]^n \left\{3\mu\gamma^2 C_k[\lambda,\alpha] - \beta\gamma^2(s) C_k[\lambda,\alpha] + \beta\xi\gamma(s)\right\} a_k z^k\right|$$

As $|z| \to 1^+$,

$$\gamma(1-\xi) + \sum_{k=2}^{\infty} \gamma(\gamma-\xi)[(k-1)(\lambda-\alpha)+k]^{n+1} a_k$$

$$\leq \gamma\pi[3\mu - \beta(1-\xi)] + \sum_{k=2}^{\infty} \gamma\pi \left\{(3\mu-\beta)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\} [(k-1)(\lambda-\alpha)+k]^n a_k$$

$$\sum_{k=2}^{\infty} \gamma \left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\} [(k-1)(\lambda-\alpha)+k]^n a_k \leq [3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]$$

which is sharp for

$$f_\gamma(z) = z + \frac{[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]}{\gamma\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n} z^n.$$

■

Corollary 2.2. A function $f(z)$ defined by (3) is said to belong to class $G_1(\beta,\xi,\mu,\pi)$ if

$$\sum_{k=2}^{\infty} \left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\} [(k-1)(\lambda-\alpha)+k]^n a_k \leq [3\pi\mu - (1-\xi)(\pi\beta-1)]$$

which is sharp for

$$f(z) = z + \frac{[3\pi\mu - (1-\xi)(\pi\beta-1)]}{\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n} z^n.$$

Corollary 2.3. A function $f(z)$ defined by (3) is said to belong to class $G_1(1,\xi,\mu,\pi)$ if

$$\sum_{k=2}^{\infty} \left\{(3\pi\mu - \pi)[(k-1)(\lambda-\alpha)+k] - \xi\right\} [(k-1)(\lambda-\alpha)+k]^n a_k \leq [3\pi\mu - (1-\xi)(\pi-1)]$$

which is sharp for

$$f(z) = z + \frac{[3\pi\mu - (1-\xi)(\pi-1)]}{\{(3\pi\mu - \pi)[(k-1)(\lambda-\alpha)+k] - \xi\}\,[(k-1)(\lambda-\alpha)+k]^n} z^n.$$

Corollary 2.4. A function $f(z)$ defined by (3) is said to belong to class $G_1(1,1,\mu,\pi)$ if

$$\sum_{k=2}^{\infty} \{(3\pi\mu - \pi)[(k-1)(\lambda-\alpha)+k] - \xi\}\,[(k-1)(\lambda-\alpha)+k]^n a_k \leq 3\pi\mu$$

which is sharp for

$$f(z) = z + \frac{3\pi\mu}{\{(3\pi\mu - \pi)[(k-1)(\lambda-\alpha)+k] - \xi\}\,[(k-1)(\lambda-\alpha)+k]^n} z^n.$$

Corollary 2.5. A function $f(z)$ defined by (3) is said to belong to class $G_1(1,1,1,\pi)$ if

$$\sum_{k=2}^{\infty} \{(3\pi - \pi)[(k-1)(\lambda-\alpha)+k] - \xi\}\,[(k-1)(\lambda-\alpha)+k]^n a_k \leq 3\pi$$

which is sharp for

$$f(z) = z + \frac{3\pi}{\{(3\pi - \pi)[(k-1)(\lambda-\alpha)+k] - \xi\}\,[(k-1)(\lambda-\alpha)+k]^n} z^n.$$

Next is the radii properties for the class $G_\gamma(\beta,\xi,\mu,\pi)$.

THEOREM 2.2 *(Starlikeness): Let the function $f_\gamma(z)$ defined by (3) be in the class $G_\gamma(\beta,\xi,\mu,\pi)$, then $f_\gamma(z)$ is starlike of order σ $(0 \leq \sigma < 1)$ in $|z| < r_1$, where*

$$r_1 = inf_k \left\{ \frac{(1-\sigma)k\gamma(s)\,\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\}\,[(k-1)(\lambda-\alpha)+k]^n}{(k-\sigma)[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]} \right\}^{\frac{1}{k-1}} \quad k \geq 2.$$

Proof. It suffices to show that $\left|\frac{zf_\gamma'(z)}{f_\gamma(z)} - 1\right| < 1-\sigma,\ |z| < r_1$.
That is,

$$\left|\frac{zf_\gamma'(z)}{f_\gamma(z)} - 1\right| = \left|\frac{z - \sum_{k=2}^{\infty}\gamma(s)ka_kz^k - z + \sum_{k=2}^{\infty}\gamma(s)a_kz^k}{z - \sum_{k=2}^{\infty}\gamma(s)a_kz^k}\right|$$

$$\left|\frac{-\sum_{k=2}^{\infty}\gamma(s)(k-1)a_kz^{k-1}}{1 - \sum_{k=2}^{\infty}\gamma(s)a_kz^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty}\gamma(s)(k-1)a_k|z|^{k-1}}{(1 - \sum_{k=2}^{\infty}\gamma(s)a_k|z|^{k-1})} < 1-\sigma$$

It follows that

$$\sum_{k=2}^{\infty}\gamma(s)\frac{(k-\sigma)|z|^{k-1}}{(1-\sigma)} \leq \frac{1}{a_k}$$

$$|z|^{k-1} \leq \frac{(1-\sigma)k\gamma(s)\,\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\}\,[(k-1)(\lambda-\alpha)+k]^n}{[(k-\sigma)[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]}.$$

Equivalently,

$$|z| \leq \left\{ \frac{(1-\sigma)k\gamma(s)\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n}{(k-\sigma)[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]} \right\}^{\frac{1}{k-1}}; \ |z| < r_1.$$

Thus,

$$r_1 = inf_k \left\{ \frac{(1-\sigma)k\gamma(s)\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n}{(k-\sigma)[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]} \right\}^{\frac{1}{k-1}} \quad k \geq 2$$

which completes the proof. ■

The result is sharp for the function $f_\gamma(z)$ given by

$$f_\gamma(z) = z + \frac{[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]}{\gamma(s)\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n} z^k, \quad k \geq 2.$$

THEOREM 2.3 *(Convexity): Let the function $f_\gamma(z)$ defined by (3) be in the class $G_\gamma(\beta,\xi,\mu,\pi)$ then $f_\gamma(z)$ is convex of order σ $(0 \leq \sigma < 1)$ in $|z| < r_2$, where*

$$r_2 = inf_k \left\{ \frac{(1-\sigma)k\gamma(s)\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n}{k(k-\sigma)[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]} \right\}^{\frac{1}{k-1}} \quad k \geq 2 \tag{10}$$

Proof. It suffices to show that $\left|\frac{zf''_\gamma(z)}{f'_\gamma(z)}\right| < 1-\sigma, \ |z| < r_2$.
The result is sharp for the function $f_\gamma(z)$ given by

$$f_\gamma(z) = z + \frac{[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]}{\gamma(s)\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n} z^k, \quad k \geq 2.$$

■

THEOREM 2.4 *(Close-to-convex): Let the function $f_\gamma(z)$ defined by (3) be in the class $G_\gamma(\beta,\xi,\mu,\pi)$. Then $f_\gamma(z)$ is close-to-convex of order σ $(0 \leq \sigma < 1)$ in $|z| < r_3$, where*

$$r_3 = inf_k \left\{ \frac{(1-\sigma)k\gamma(s)\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n}{[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]} \right\}^{\frac{1}{k-1}} \quad k \geq 2. \tag{11}$$

Proof. It suffices to show that $|f'_\gamma(z) - 1| = 1-\sigma \ \ (0 \leq \sigma < 1$ for $|z| < r_3$. The result is sharp

$$f_\gamma(z) = z + \frac{[3\gamma\pi\mu - \gamma(1-\xi)(\pi\beta-1)]}{\gamma(s)\left\{(3\pi\mu - \beta\pi)[(k-1)(\lambda-\alpha)+k] - \beta\xi\right\}[(k-1)(\lambda-\alpha)+k]^n} z^k, \quad k \geq 2.$$

■

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