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In honour of Prof. Ekhaguere at 70

On specification, stationarity and estimation of parameters of mixed one-dimensional seasonal autoregressive integrated moving average bilinear time series models

J.F. Ojo[a*] and L.O. Adekola[b]

[a]*Department of Statistics, University of Ibadan, Ibadan, Nigeria*; [b]*Department of Physical Sciences, Bells University of Technology, Ota, Nigeria*

Abstract. In practice, many time series processes exhibit both the seasonal and non-seasonal behaviour coupled with their nonlinear nature. Hence, this study considers the Specification, Stationarity and Estimation of the parameters of the full one-dimensional Mixed Seasonal Autoregressive Integrated Moving Average Bilinear Models (MSARIMABL) which are capable of achieving stationarity for all nonlinear seasonal time series. The stationarity and convergence conditions for these models are established. The nonlinear least squares method of minimizing errors and the Newton-Raphson iterative procedure are employed in the estimation of the parameters.

Keywords: mixed, one-dimensional, integrated, seasonal bilinear time series model, least square method, Newton-Raphson iterative procedure.

1. Introduction

The objective of this paper is to consider the estimation of the parameters of the Mixed seasonal one-dimensional Bilinear Time series Model. The model is a nonlinear model with the linear part defined as:

$$\Phi_p(B^s)\varphi_p(B)\nabla_s^D\nabla^d X_t = \Theta_Q(B^s)\theta_q(B)e_t \tag{1}$$

which is called **mixed seasonal ARIMA** and it is denoted by:

$$ARIMA_{(p,d,q)} \times (P, D, Q)_s \tag{2}$$

2. Model Specification

The mixed seasonal autoregressive Integrated Moving Average Bilinear Time Series Model (MSARIMABL) is defined as follows:

$$\psi(B)X_t = \varphi_p(B)X_t\Phi_p(B^s)\nabla_s^D\nabla^d + \theta_q(B)\Theta_Q(B^s)e_t + \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}X_{t-i}e_{t-j} \tag{3}$$

where: $\psi(B) = \varphi_p(B)(1-B)^d = 1 - \psi_1 B - \psi_2 B^2 \ldots - \psi_{p+d}B^{p+d}$, is the autoregressive integrated operator, $\Theta(B^s) = 1 + \Theta_1 B^s - \Theta_2 B^{2s} \ldots - \Theta_Q B^{Qs}$ is the seasonal moving average operator, $\Psi(B) = \Phi(B^s)(1-B^s)^D = 1 - \Psi_1 B - \Psi_2 B^2 .... - \Psi_{P+D}B^{P+D}$ is the seasonal autoregressive integrated operator, $\theta(B) = 1 + \theta_1 B - \theta_2 B^2 \ldots - \theta_q B^q$ is the moving average operator, b_{i1} are the nonlinear

*Corresponding author. Email: jfunminiyiojo@yahoo.co.uk one - dimensional bilinear components. Equation (3) by expansion can be written as:

$$X_t = \psi_1 X_{t-1} + \psi_2 X_{t-2} \ldots + \psi_{p+d} X_{t-p-d} + \Psi_1 X_{t-s} + \Psi_2 X_{t-2s} \ldots + \Psi_{P+D} X_{t-Ps-D} + \theta_1 e_{t-1}$$
$$+\theta_2 e_{t-2} \ldots + \theta_q e_{t-q} + \Theta_1 e_{t-s} + \Theta_2 e_{t-2s} \ldots + \Theta_q e_{t-Qs} + b_{11} X_{t-1} e_{t-1} + \ldots + b_{i1} X_{t-i} e_{t-1} + e_t \tag{4}$$

3. Vector form of the specified mixed SARIMABL $(p, d, q, m, 1)(P, D, Q)_s$ model

It is convenient to study the properties of a process when the model is in the state space form because of the Markovian nature of the model [1]. Given:

$$X_t = \psi_1 X_{t-1} + \psi_2 X_{t-2} .... + \psi_{p+d} X_{t-p-d} + \Psi_1 X_{t-s} + \Psi_2 X_{t-2s} .... + \Psi_{P+D} X_{t-Ps-D} + \theta_1 e_{t-1}$$
$$+ \theta_2 e_{t-2} .... + \theta_q e_{t-q} + \Theta_1 e_{t-s} + \Theta_2 e_{t-2s} .... + \Theta_q e_{t-Qs} + b_{11} X_{t-1} e_{t-1} + ... + b_{i1} X_{t-i} e_{t-1} + e_t$$

$$\Rightarrow X_{t-1} = \psi_1 X_{t-2} + \psi_2 X_{t-3} .... + \psi_{p+d} X_{t-p-d-1} + \Psi_1 X_{t-s-1} + \Psi_2 X_{t-2s-1} .... + \Psi_{P+D} X_{t-Ps-D-1}$$
$$+ \theta_1 e_{t-2} + \theta_2 e_{t-3} .... + \theta_q e_{t-q-1} + \Theta_1 e_{t-s-1} + \Theta_2 e_{t-2s-1} .... + \Theta_Q e_{t-Qs-1}$$
$$+ b_{11} X_{t-2} e_{t-2} + ... + b_{i1} X_{t-i-1} e_{t-2} + e_{t-1}$$

$$X_{t-2} = \psi_1 X_{t-3} + \psi_2 X_{t-4} .... + \psi_{p+d} X_{t-p-d-2} + \Psi_1 X_{t-s-2} + \Psi_2 X_{t-2s-2} .... + \Psi_{P+D} X_{t-Ps-D-2} +$$
$$\theta_1 e_{t-3} + \theta_2 e_{t-4} .... + \theta_q e_{t-q-2} + \Theta_1 e_{t-s-2} + \Theta_2 e_{t-2s-2} .... + \Theta_q e_{t-Qs-2} + b_{11} X_{t-3} e_{t-3} + ... +$$
$$b_{i1} X_{t-i-1} e_{t-3} + e_{t-2}$$

$$\vdots$$
$$\vdots$$

$$X_{t-p-1} = \psi_1 X_{t-p-2} + \psi_2 X_{t--p-3} .... + \psi_{p+d} X_{t-2p-d-1} + \Psi_1 X_{t-s-p-1} + \Psi_2 X_{t-2s-p-1} .... +$$
$$\Psi_{P+D-p-1} X_{t-Ps-D-p-1} + \theta_1 e_{t-p-2} + \theta_2 e_{t-p-3} .... + \theta_q e_{t-q-p-1} + \Theta_1 e_{t-s-p-1} + \Theta_2 e_{t-2s-p-1} +$$
$$.... + \Theta_q e_{t-Qs-p-1} + b_{11} X_{t-p-2} e_{t-p-1} + ... + b_{i1} X_{t-i-p-1} e_{t-p-2} + e_{t-p-1}$$

Let:

$$\underline{\psi} = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & .... & \psi_p & \psi_{p+1} \\ 1 & 0 & 0 & .... & 0 & 0 \\ 0 & 1 & 0 & .... & 0 & 0 \\ \vdots & 0 & 1 & .... & 0 & 0 \\ 0 & 0 & 0 & .... & 1 & 0 \end{pmatrix}, \quad \underline{\Psi} = \begin{pmatrix} \Psi_1 & \Psi_2 & \Psi_3 & .... & \Psi_{P+D} & \Psi_{P+D+1} \\ 1 & 0 & 0 & .... & 0 & 0 \\ 0 & 1 & 0 & .... & 0 & 0 \\ \vdots & 0 & 1 & .... & 0 & 0 \\ 0 & 0 & 0 & .... & 1 & 0 \end{pmatrix},$$

$$\underset{m\times m}{B_j} = \begin{pmatrix} b_{11} & b_{21} & b_{31} & .... & b_{m1} \\ 0 & 0 & 0 & .... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & .... & 0 \end{pmatrix} \quad j = 1, \quad \underline{\theta} = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & .... & \theta_q & \theta_{q+1} \\ 1 & 0 & 0 & .... & 0 & 0 \\ 0 & 1 & 0 & .... & 0 & 0 \\ \vdots & 0 & 1 & .... & 0 & 0 \\ 0 & 0 & 0 & .... & 1 & 0 \end{pmatrix},$$

$$\underline{\Theta} = \begin{pmatrix} \Theta_1 & \Theta_2 & \Theta_3 & .... & \Theta_Q & \Theta_{Qs+1} \\ 1 & 0 & 0 & .... & 0 & 0 \\ 0 & 1 & 0 & .... & 0 & 0 \\ \vdots & 0 & 1 & .... & 0 & 0 \\ 0 & 0 & 0 & .... & 1 & 0 \end{pmatrix},$$

the vectors:

$$\underset{1\times p}{\mathbf{X}'} = (\, X_t \; X_{t-1} \; X_{t-2} \; .... \; X_{t-p-1}\,), \;\; \underset{1\times p}{H'} = (\, 1 \; 0 \; 0 \; .... \; 0\,)$$

and $C' = (\, 1 \; 0 \; 0 \; .... \; 0\,)$, where $H^{'}$ stands for the transpose of a matrix H, $t = \cdots -1, 0, 1, \ldots$. With these notations, we can write the model (4) in the vector form as:

$$\mathbf{X}_t = \underline{\psi}\mathbf{X}_{t-1} + \underline{\Psi}\mathbf{X}_{t-s} + \underline{\theta}\mathrm{e}_{t\text{ - }1} + \underline{\Theta}\mathrm{e}_{t\text{ - }s} + \mathrm{B}\mathbf{X}_{t\text{ - }1}\mathrm{e}_{t\text{ - }1} + \mathrm{C}\mathrm{e}_{t} \tag{5}$$

Hence;

$$X_t = H^{'} X_t \tag{6}$$

4. Stationary and convergence

Following Rao et al. [2] and Sangodoyin et al. [7], a sufficient condition necessary for the existence of strictly stationary process and convergence conforming to the mixed seasonal one-dimensional bilinear model (4) can be achieved through the following theorem.

THEOREM 4.1 *Let $\{e_t, t \in Z\}$ be a sequence of independent and identically distributed random variables defined on the probability space $\{\Omega, F, P\}$ such that; $E(e_t) = 0$ and $E\left(e_t^2\right) = \sigma^2 < \infty$. Let $\underline{\psi}, \underline{\Psi}, \underline{\theta}, \underline{\Theta}$ and B be matrices as defined above such that;*

$$\rho[(\underline{\psi} \otimes \underline{\psi} + 2\underline{\psi} \otimes \underline{\Psi} + \underline{\Psi} \otimes \underline{\Psi} + 2\sigma^2(\underline{\theta} \otimes \underline{\Theta} + \underline{\theta} \otimes \mathrm{B} + \frac{1}{2}\underline{\Theta} \otimes \underline{\Theta} + \underline{\Theta} \otimes \mathrm{B} + \frac{1}{2}\mathrm{B} \otimes \mathrm{B})] = \tau < 1$$

And $\boldsymbol{C}$ be any column vector with components $x_1, c_2, \ldots, c_p$. Then the series of random vectors; $\sum\limits_{r \geqslant 1} \prod\limits_{i=1}^{r} (\underline{\psi} + \underline{\Psi} + \underline{\theta} e_{t-i} + \underline{\Theta} e_{t-is} + \mathrm{B} e_{t-i}) C e_{t-r}$ converges absolutely almost surely as well as in the mean for every fixed t in Z. Moreover, if:

$$X_t = Ce_t + \sum_{r \geqslant 1} \prod_{i=1}^{r} (\underline{\psi} + \underline{\Psi} + \underline{\theta} e_{t-i} + \underline{\Theta} e_{t-is} + \mathrm{B} e_{t-i}) C e_{t-r}, \; t \in Z$$

Then for every $t \in Z, X_t$, is a strictly stationary process conforming to the mixed seasonal bilinear model:

$$\mathbf{X}_t = \underline{\psi}\mathbf{X}_{t-1} + \underline{\Psi}\mathbf{X}_{t-s} + \underline{\theta}\mathrm{e}_{t\text{ - }1} + \underline{\Theta}\mathrm{e}_{t\text{ - }s} + \mathrm{B}\mathbf{X}_{t\text{ - }1}e_{t\text{ - }1} + Ce_t$$

Conversely, if $\{X_t, t \in Z\}$ is a stationary process conforming to the mixed seasonal bilinear model

$$\mathbf{X}_t = \underline{\psi}\mathbf{X}_{t-1} + \underline{\Psi}\mathbf{X}_{t-s} + \underline{\theta}e_{t\text{ - }1} + \underline{\Theta}\mathrm{e}_{t\text{ - }s} + \mathrm{B}\mathbf{X}_{t\text{ - }1}e_{t\text{ - }1} + Ce_t$$

$\forall\, t \in Z$, for some sequences $\{e_t, t \in Z\}$ of independent and identically distributed random variables with $E(e_t) = 0$ and $E\left(e_t^2\right) = \sigma^2 < \infty$ and for some matrices $\underline{\psi}, \underline{\Psi}, \underline{\theta}, \underline{\Theta}$ B and C of orders $p \times p, P \times P, q \times q, Q \times Q, m \times m$ and $p \times 1$ respectively with

$$\rho[(\underline{\psi} \otimes \underline{\psi} + 2\underline{\psi} \otimes \underline{\Psi} + \underline{\Psi} \otimes \underline{\Psi} + 2\sigma^2(\underline{\theta} \otimes \underline{\Theta} + \underline{\theta} \otimes \mathrm{B} + \frac{1}{2}\underline{\Theta} \otimes \underline{\Theta} + \underline{\Theta} \otimes \mathrm{B} + \frac{1}{2}\mathrm{B} \otimes \mathrm{B})] = \tau < 1$$

Then $X_t = Ce_t + \sum\limits_{r \geqslant 1} \prod\limits_{i=1}^{r} (\underline{\psi} + \underline{\Psi} + \underline{\theta} e_{t-i} + \underline{\Theta} e_{t-is} + \mathrm{B} e_{t-i}) C e_{t-r}$, for every $t \in Z$.

Proof. For almost sure convergence, we prove that;

$$\sum_{r\geqslant 1} E\left|\left(\prod_{\mathrm{j}=1}^{\mathrm{r}}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r}\right)_j\right|<\infty\ \forall\ i=1,2,...,p \tag{7}$$

$\Rightarrow \sum\limits_{r\geqslant 1}\prod\limits_{\mathrm{i}=1}^{\mathrm{r}}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r}$ is absolutely convergent almost surely as well as in the mean. $\Rightarrow$ for every $t \in Z, r = 1$ and $n = 1,2,\ldots,p$ $E\left|((\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r})_n\right| \leqslant k_0$ where k_0 is a constant that depends only on $\underline{\psi},\underline{\Psi},\mathrm{B},C$ and σ^2.

Similarly if $r=2$ then there exists some $k_1>0$ such that:

$$E\left|\left(\prod_{i=1}^{r}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r}\right)_1\right|\leqslant k_1\tau^{\left(\frac{r-1}{2}\right)}$$

Now for any $n=1,2,\ldots,p$

$$E\left|\left(\prod_{i=1}^{r}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r}\right)_1\right|\leqslant k_1\tau^{\left(\frac{r-1}{2}\right)}$$
$$=\left(\left(\prod_{i=1}^{r-1}\left(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i}\right)\right)\otimes\left(\prod_{i=1}^{r-1}\left(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i}\right)\right)\right)_{1n;1n}$$
$$=\left(\prod_{i=1}^{r-1}\left(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i}\right)\otimes\left(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i}\right)\right)_{1n;1n}$$
$$=\left(\left(E\left[\left(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_t+\underline{\Theta}e_t+\ \mathrm{B}e_t\right)\otimes\left(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_t+\underline{\Theta}e_t+\ \mathrm{B}e_t\right)\right]\right)^{r-1}\right)_{1n;1n}$$

Hence;

$$E\left(\left(\prod_{i=1}^{r-1}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})\right)_{1n}\right)^2$$
$$=(E([(\underline{\psi}\otimes\underline{\psi}+2\underline{\psi}\otimes\underline{\Psi}+2\underline{\psi}\otimes\underline{\theta}e_t+2\underline{\psi}\otimes\underline{\Theta}e_t+\ 2\underline{\psi}\otimes\mathrm{B}e_t+\underline{\Psi}\otimes\underline{\Psi}+2\underline{\Psi}\otimes\underline{\theta}e_t+2\underline{\Psi}$$
$$\otimes\underline{\Theta}e_t+2\ \underline{\Psi}\otimes\mathrm{B}e_t+2\underline{\theta}\otimes\underline{\Theta}e_t^2+2\underline{\theta}\otimes\mathrm{B}e_t^2+\underline{\Theta}\otimes\underline{\Theta}e_t^2+2\underline{\Theta}\otimes\mathrm{B}e_t^2+\mathrm{B}\otimes\mathrm{B}e_t^2)])^{r-1})_{1n;1n}$$
$$=((\underline{\psi}\otimes\underline{\psi}+2\underline{\psi}\otimes\underline{\Psi}+\underline{\Psi}\otimes\underline{\Psi}+2\sigma^2(\underline{\theta}\otimes\underline{\Theta}+\underline{\theta}\otimes\mathrm{B}+\frac{1}{2}\underline{\Theta}\otimes\underline{\Theta}+\underline{\Theta}\otimes\mathrm{B}+\frac{1}{2}\mathrm{B}\otimes\mathrm{B}))^{r-1})_{1n;1n}\leqslant k\tau^{r-1}$$

Therefore; $E\left|\left(\prod\limits_{i=1}^{r}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r}\right)_1\right| \leqslant k_1\tau^{\left(\frac{r-1}{2}\right)}$ for a suitable choice of k_1. Since $\tau<1$

$$\Rightarrow\sum_{r\geqslant 1}E\left|\left(\prod_{i=1}^{r}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r}\right)_i\right|<\infty\ \forall\ i=1,2,\ldots,p$$

Thus (7) is established. It is then obvious that the vector-valued stochastic process $\{X_t, t\in Z\}$ defined by: $X_t = Ce_t+\sum\limits_{r\geqslant 1}\prod\limits_{i=1}^{r}(\underline{\psi}+\underline{\Psi}+\underline{\theta}e_{t-i}+\underline{\Theta}e_{t-is}+\ \mathrm{B}e_{t-i})Ce_{t-r},\ t\in Z$ is strictly stationary.

Hence;

$$\rho((\underline{\psi}\otimes\underline{\psi}+2\underline{\psi}\otimes\underline{\Psi}+\underline{\Psi}\otimes\underline{\Psi}+2\sigma^2(\underline{\theta}\otimes\underline{\Theta}+\underline{\theta}\otimes \mathrm{B}+\tfrac{1}{2}\underline{\Theta}\otimes\underline{\Theta}+\underline{\Theta}\otimes \mathrm{B}+\tfrac{1}{2}\mathrm{B}\otimes \mathrm{B}))=\tau<1$$

is a sufficient condition for strict stationarity of the mixed seasonal one-dimensional bilinear model (4). Hence the proof is established. ■

5. Estimation of parameters

The estimation procedures of the model is similar to [5]. We shall report the estimation of the parameters of the one dimensional case. Suppose that X_t are generated by equation (4), the sequence of random deviates $\{e_t\}$ could be determined from the relation:

$$e_t = X_t-\psi_1X_{t-1}-\psi_2X_{t-2}....-\psi_{p+1}X_{t-p-1}-\Psi_1X_{t-s}-\Psi_2X_{t-2s}....-\Psi_{P+D}X_{t-Ps-D}-$$
$$\theta_1e_{t-1}-\theta_1e_{t-2}....-\theta_qe_{t-q}-\Theta_1e_{t-s}-\Theta_2X_{t-2s}....-\Theta_Qe_{t-Qs}-b_{11}X_{t-1}e_{t-1}-....-b_{m1}X_{t-i}e_{t-1} \quad (8)$$

To estimate the unknown parameters in (9) we shall minimize the error by obtaining the first and second order partial derivatives of (9) with respect to the parameters; $\psi_1,\psi_2....\psi_{p+1};\ \Psi_1,\ \Psi_2....\Psi_{P+D};\ \theta_1,\ \theta_2....\theta_q;\ \Theta_1,\Theta_2....\Theta_Q$ and b_{i1}

$$\frac{\partial e_t}{\partial\psi_i}=-X_{t-i}-\sum_{i=1}^{q}\theta_i\frac{\partial e_{t-i}}{\partial\psi_i}-\sum_{i=1}^{Q}\Theta_i\frac{\partial e_{t-is}}{\partial\psi_i}-\sum_{i=1}^{m}b_{i1}X_{t-i}\frac{\partial e_{t-1}}{\partial\psi_i} \quad (9)$$

$$\frac{\partial e_t}{\partial\psi_i}=-X_{t-i}-\sum_{i=1}^{q}\theta_i\frac{\partial e_{t-i}}{\partial\psi_i}-\sum_{i=1}^{Q}\Theta_i\frac{\partial e_{t-is}}{\partial\psi_i}-\sum_{i=1}^{m}b_{i1}X_{t-i}\frac{\partial e_{t-1}}{\partial\psi_i} \quad (10)$$

$$\frac{\partial e_t}{\partial\theta_i}=-e_{t-i}-\sum_{i=1}^{Q}\Theta_i\frac{\partial e_{t-is}}{\partial\theta_i}-\sum_{i=1}^{m}b_{i1}X_{t-1}\frac{\partial e_{t-1}}{\partial\theta_i} \quad (11)$$

$$\frac{\partial e_t}{\partial\Theta_i}=-e_{t-is}-\sum_{i=1}^{q}\theta_i\frac{\partial e_{t-i}}{\partial\Theta_i}+\sum_{i=1}^{m}b_{i1}X_{t-i}\frac{\partial e_{t-1}}{\partial\Theta_i} \quad (12)$$

$$\frac{\partial e_t}{\partial b_{i1}}=-X_{t-i}e_{t-1}-\sum_{i=1}^{q}\theta_i\frac{\partial e_{t-i}}{\partial b_{i1}}-\sum_{i=1}^{Q}\Theta_i\frac{\partial e_{t-s}}{\partial b_{i1}}-b_{i1}X_{t-i}\frac{\partial e_{t-1}}{\partial b_{i1}} \quad (13)$$

And the second derivatives;

$$\frac{\partial^2 e_t}{\partial\psi_i^2}=-\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\psi_i^2}-\sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-s}}{\partial\psi_i^2}-\sum_{i=1}^{m}b_{i1}X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\psi_i^2} \quad (14)$$

$$\frac{\partial^2 e_t}{\partial\Psi_i^2}=-\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\Psi_i^2}-\sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-s}}{\partial\Psi_i^2}-\sum_{i=1}^{m}b_{i1}X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\Psi_i^2} \quad (15)$$

$$\frac{\partial^2 e_t}{\partial\theta_i^2}=\frac{\partial e_{t-i}}{\partial\theta_i}-\sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-is}}{\partial\theta_i^2}-\sum_{i=1}^{m}b_{i1}X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\theta_i^2} \quad (16)$$

$$\frac{\partial^2 e_t}{\partial\Theta_i^2}=\frac{\partial e_{t-is}}{\partial\Theta_i}-\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\Theta_i^2}+\sum_{i=1}^{m}b_{i1}X_{t-i}\frac{\partial^2 e_{t-1}}{\partial\Theta_i^2} \quad (17)$$

$$\frac{\partial^2 e_t}{\partial b_{i1}^2}=-X_{t-i}\frac{\partial e_{t-1}}{\partial b_{i1}}-\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial b_{i1}^2}-\sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-s}}{\partial b_{i1}^2}-b_{i1}X_{t-i}\frac{\partial^2 e_{t-1}}{\partial b_{i1}^2} \quad (18)$$

$$\frac{\partial^2 e_t}{\partial\psi_i\Psi_i} = -\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\psi_i\Psi_i} - \sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-is}}{\partial\psi_i\Psi_i} - \sum_{i=1}^{m} b_{i1}X_{t-i}\frac{\partial^2 e_{t-1}}{\partial\psi_i\Psi_i} \tag{19}$$

$$\frac{\partial^2 e_t}{\partial\psi_i\theta_i} = -\sum_{i=1}^{q}\frac{\partial^2 e_{t-i}}{\partial\psi_i\theta_i} - \sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-is}}{\partial\psi_i\theta_i} - \sum_{i=1}^{m} b_{i1}X_{t-i}\frac{\partial^2 e_{t-1}}{\partial\psi_i\theta_i} \tag{20}$$

$$\frac{\partial^2 e_t}{\partial\psi_i\Theta_i} = -\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\psi_i\Theta_i} - \sum_{i=1}^{Q}\frac{\partial^2 e_{t-is}}{\partial\psi_i\Theta_i} - \sum_{i=1}^{m} b_{i1}X_{t-i}\frac{\partial^2 e_{t-1}}{\partial\psi_i\Theta_i} \tag{21}$$

$$\frac{\partial^2 e_t}{\partial\psi_i b_{i1}} = -\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\psi_i b_{i1}} - \sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-is}}{\partial\psi_i b_{i1}} - \sum_{i=1}^{m} X_{t-i}\frac{\partial^2 e_{t-1}}{\partial\psi_i b_{i1}} \tag{22}$$

$$\frac{\partial^2 e_t}{\partial\Psi_i\partial\theta_i} = -\sum_{i=1}^{q}\frac{\partial^2 e_{t-i}}{\partial\Psi_i\partial\theta_i} - \sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-s}}{\partial\Psi_i\partial\theta_i} - \sum_{i=1}^{m} b_{i1}X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\Psi_i\partial\theta_i} \tag{23}$$

$$\frac{\partial^2 e_t}{\partial\Psi_i\partial\Theta_i} = -\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\Psi_i\partial\Theta_i} - \sum_{i=1}^{Q}\frac{\partial^2 e_{t-s}}{\partial\Psi_i\partial\Theta_i} - \sum_{i=1}^{m} b_{i1}X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\Psi\partial\Theta_{ii}} \tag{24}$$

$$\frac{\partial^2 e_t}{\partial\Psi_i\partial b_{i1}} = -\sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\Psi_i} - \sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-s}}{\partial\Psi_i} - \sum_{i=1}^{m} X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\Psi_i b_{i1}} \tag{25}$$

$$\frac{\partial^2 e_t}{\partial\theta_i\partial\Theta_i} = -\frac{\partial e_{t-i}}{\partial\Theta_i} - \sum_{i=1}^{Q}\frac{\partial^2 e_{t-is}}{\partial\theta_i\partial\Theta_i} - \sum_{i=1}^{m} b_{i1}X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\theta_i\partial\Theta_i} \tag{26}$$

$$\frac{\partial^2 e_t}{\partial\theta_i b_{i1}} = -\frac{\partial e_{t-i}}{\partial b_{i1}} - \sum_{i=1}^{Q}\Theta_i\frac{\partial^2 e_{t-is}}{\partial\theta_i b_{i1}} - \sum_{i=1}^{m} X_{t-1}\frac{\partial^2 e_{t-1}}{\partial\theta_i b_{i1}} \tag{27}$$

$$\frac{\partial e_t}{\partial\Theta_i b_{i1}} = -\frac{\partial e_{t-is}}{\partial b_{i1}} - \sum_{i=1}^{q}\theta_i\frac{\partial^2 e_{t-i}}{\partial\Theta_i b_{i1}} + \sum_{i=1}^{m} X_{t-i}\frac{\partial^2 e_{t-1}}{\partial\Theta_i b_{i1}} \tag{28}$$

Proceeding as in [6], we can note that maximizing the likelihood function of $(X_{n_0}, X_{n_0+1}\ldots, X_n)$ is the same as minimizing the function; $V = V(\lambda) = \sum_{t=1}^{n} e_t^2$

$$\underline{\lambda} = \begin{pmatrix} \psi \\ \Psi \\ \theta \\ \Theta \\ b_{m1} \end{pmatrix} \qquad G(\underline{\lambda}) = \begin{pmatrix} \frac{\partial V}{\partial\psi} \\ \frac{\partial V}{\partial\Psi} \\ \frac{\partial V}{\partial\theta} \\ \frac{\partial V}{\partial\psi} \\ \frac{\partial V}{\partial b_{m1}} \end{pmatrix}, \tag{29}$$

and

$$H(\lambda) = \begin{pmatrix} \frac{\partial^2 V}{\partial \psi^2} & \frac{\partial^2 V}{\partial \psi \partial \Psi} & \frac{\partial^2 V}{\partial \psi \partial \theta} & \frac{\partial^2 V}{\partial \psi \partial \Theta} & \frac{\partial^2 V}{\partial \psi \partial b_{m1}} \\ & \frac{\partial^2 V}{\partial \Psi^2} & \frac{\partial^2 V}{\partial \Psi \partial \theta} & \frac{\partial^2 V}{\partial \Psi \partial \theta} & \frac{\partial^2 V}{\partial \Psi \partial b_{m1}} \\ & & \frac{\partial^2 V}{\partial \theta^2} & \frac{\partial^2 V}{\partial \theta \partial \Theta} & \frac{\partial^2 V}{\partial \theta \partial b_{m1}} \\ & & & \frac{\partial^2 V}{\partial \Theta^2} & \frac{\partial^2 V}{\partial \Theta \partial b_{m1}} \\ & & & & \frac{\partial^2 V}{\partial b_{m1}^2} \end{pmatrix} \quad (30)$$

minimizing V(λ) with respect to $\underline{\lambda}$, the normal equations are non-linear in λ.

The solutions of these equations require the application of Newton Raphson algorithm which iterative equation is given as follows:

$$\lambda_{k+1} = \lambda_k - H'(\lambda_k)G(\lambda_k) \quad (31)$$

And can be adopted to obtain the $(k+1)^{th}$ iteration $(\hat{\lambda}_{k+1})$ of the estimates from the k^{th} estimate $(\hat{\lambda}_k)$. The estimates obtained by the iterative equations (31) usually converge having good sets of initial values of the parameters.

6. Conclusion

In this paper we have specified the mixed seasonal Autoregressive Moving Average Bilinear Model and estimated its parameters using the procedure of minimizing error (least squares) and Newton-Raphson iterative procedure.

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