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In honour of Prof. Ekhaguere at 70

On seminorms on quasi *-algebras associated with multiplier algebras

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Abstract. In this paper we consider quasi *-algebra associated with multiplier algebra. We introduce some new notions which enable us to define some useful seminorms on the multiplier algebras.

Keywords: quasi *-algebras, multiplier algebras, seminorms.

1. Introduction and basic definitions

Towards developing an algebraic model for quantum systems, several authors, have come up with various approaches.In particular,some have considered extending the algebraic framework of the theory to accommodate the unbounded linear operators on Hilbert spaces.The extensions have included the considerations of partial *-algebras.

DEFINITION 1.1 *A partial *-algebra is a vector space $\mathcal{A}$ equipped with an involution $*\colon \mathcal{A} \to \mathcal{A}\colon x \to x^*$ and partial multiplication ' $\cdot$ ' defined by a relation $\Gamma \subset \mathcal{A} \times \mathcal{A}$, $\Gamma = \{(x, y) \in \mathcal{A} \times \mathcal{A}\colon x \cdot y \in \mathcal{A}\}$ such that*

(1) $(x, y) \in \Gamma$ *implies* $(y^*, x^*) \subset \Gamma$;
(2) $(x, y_1), (x, y_2) \in \Gamma$ *and* $\lambda, \mu \in \mathbb{C}$ *imply* $(x, \lambda y_1 + \mu y_2) \in \Gamma$
(3) for every $(x, y) \in \Gamma$ *, a product* $xy \in \mathcal{A}$ *is defined, such that* $x \cdot y$ *depends linearly on* y *and satisfies the equality* $(x \cdot y)^* = y^* \cdot x^*$.

Whenever $(x, y) \in \Gamma$, we say that x is a left multiplier of y and y a right multiplier of x and we write $x \in L(y)$, respectively $y \in R(x)$. The product is distributive with respect to addition, that is, $(x, v), (x, z), (y, z) \in \Gamma$ implies $(x, \alpha v + \beta z)$ and $(\alpha x + \beta z) \in \Gamma$, then $(\alpha x + \beta y) \cdot z = \alpha(x.z) + \beta(y \cdot z)$ and $x.(\alpha v + \beta z) = \alpha(x.v) + \beta(x.z)$ for all $\alpha, \beta \in \mathbb{C}$, the complex numbers. The partial multiplication is in general non-associative.

In Antoine et al.[2] a theory for such algebras has been provided.Quasi *algebras are particular types of partial *-algebras. The notion of quasi *algebra was originally due to Lassner [7,8] . Quasi *-agebras are linear spaces,endowed with notions of multiplication under which they may not be closed. [10]. We recall the definition of a quasi *-algebra as given in Schmüdgen [3]. The definition of quasi *-algebra is linked with the notion of a bimodule which we give in the following;

DEFINITION 1.2 *Suppose that A is an algebra.A linear space X is said to be a left A-module if a bilinear mapping $(a, x) \to a.x$ of $A \times X$ into X is specified and satisfies $a_1 \cdot (a_2 \cdot x) = (a_1 a_2) \cdot x$ for $a_1, a_2 \in A$, $x \in X$. X is called a right A-module if the bilinear mapping $(a, x) \to x.a$ of $A \times X$ into X is specified such that $(x \cdot a_1) \cdot a_2 = x \cdot (a_1 a_2)$ for $a_1, a_2 \in A$, $x \in X$. X is called a A-bimodule if it is both a left A-module and right A-module and the module operations satisfy the following axiom $a_1 \cdot (a_2 \cdot x) = (x \cdot a_1) \cdot a_2$*

We now furnish the definition of a quasi *-algebra given in Schmüdgen [3]

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DEFINITION 1.3 *A topological quasi *-algebra is a couple (X, A) comprising a locally convex space X and a *-algebra A which is a linear subspace of X such that*

(1) X is an A-bimodule.The operations $(a, x) \to a.x$ and $(x, a) \to x.a$ extend the multiplication of A, and they are separately continuous bilinear mappings of $A \times X$ resp. $X \times A$ into X ,where A carries the induced topology of X ,
(2) there is a continuous involution $x \to x^*$ on X which extends the involution of A and satisfies $(a \cdot x)^* = x^* \cdot a^*$ and $(x \cdot a)^* = a^* \cdot x^*$, $a \in A$ and $x \in X$,
(3) A is dense in X .

Remark[1] In Trapani [1] different types of seminorms on a quasi *-algebra (U, U_0) are constructed from a suitable family $\mathfrak{F}$ of sesquilinear forms. A sesquilinear form is a map $\Omega : U \times U \to \mathbb{C}$, such that $\mathbb{C}$ is the complex field.$\Omega$is said to be positive if $\Omega(a, a) > 0$, for $a \in U$. On the other hand seminorms plays a relevant role in the theory of respresentations,in the sense that the existence of a bounded seminorm is very closely linked to the existence of a nontrivial bounded *-representation of a *-algebras in a Hilbert space [9]. In [1] the approach of Yood [6] to the partial algebraic situation, namely the quasi *-algebra was studied.

The main aim of this paper is to consider the approach in [1] for the case of a quasi *-algebra determined by a multiplier algebra as we outline in Section 2. We define on such an algebra a family of invariant positive sesquilinear forms. In this framework, the case of left or right invariant sesquilinear forms defined in Trapani [1] is imbedded in our approach. In Section 3, we introduce the notions of *quasi-invariant positive sesquilinear forms*, *normal multiplier* , *quasi-* invaraint* and *quasi-normal* elements and establish some propositions. We observe that when the multiplier-algebra satisfies these conditions it is possible to define a C*-seminorm. In order to have a bounded *- representation, a sesquilinear form must satisfy the *admissibility* condition . This condition of admissibility is given in [6] for the case of positive linear functionals on * -algebras and also in [1] when a positive linear functional is defined with respect to a sesquilinear form. In this work we define the notion of *admissibility* for the case of a family of sesquilinear forms associated to the multiplier algebra. We recall the following definitions from [1]

DEFINITION 1.4 *Let (U, U_0) be a quasi *-algebra with unit $\mathbf{1}$ and p a seminorm on U .We say that p is a Q*- seminorm on (U, U_0) if*

(1) $p(\mathbf{1}) = 1$
(2) $p(a) = p(a^*)$
(3) for each $x \in U_0$ there exists $K_x > 0$ such that $p(ax) < K_x p(a)$, $\forall a \in U$

If p is a Q-seminorm, we can define $p_0(x) = \max\{\sup_{p(a)=1} p(ax), \sup_{p(a)=1} p(xa)\}$.Then $p(x) \leq p_0(x)$ for every $x \in U_0$ and $p(ax) \leq p(a)p_0(x)$ $\forall a \in U, x \in U_0$*

DEFINITION 1.5 *The seminorm p_0 is an m*- seminorm on U_0 if*

(1) $p(x) = p(x^*)$ for every $x \in U_0$
(2) $p(ax) \leq p(a)p_0(x)$ $\forall a \in U, x \in U_0$

A Q*-seminorm p is called an extended C*-seminorm if $p(x^*x) = p(x)^2$ $\forall x \in U$. Note that if p is an extended C*- seminorm on (U, U_0) and p is submultiplicative, that is $p(ax) \leq p(a)p(x)$, $\forall a \in U, x \in U_0$ then $p_0(x) = p(x)$ for every $x \in U_0$.

DEFINITION 1.6 *Let (U, U_0) be a quasi *-algebra with unit $\mathbf{1}$. A positive sesequilinear form Ω on $U \times U$ is called left-invariant if $\Omega(xa, b) = \Omega(a, x^*b)$ and is called right- invariant if $\Omega(ax, b) = \Omega(a, bx^*)$ $\forall a, b \in U, x \in U_0$*

The set of all positive, left-invariant (resp. right-invariant) sesquilinear forms is denoted by $P_l(resp.P_r)$. Due to positivity, any Ω in P_l is hermitian, that is $\Omega(b, a) = \overline{\Omega(a, b)}$ for any $a, b \in U$ and satisfies the Cauchy's inequality $|\Omega(a, b)|^2 \leq \Omega(a, a)\Omega(b, b)$ for $a, b \in U$. The set $\mathfrak{N}(\Omega) = \{a \in U \colon \Omega(a, a) = 0\}$ is a quasi-ideal of U , that is, if $a \in \mathfrak{N}(\Omega)$ and $x \in U_0$ then $xa \in \mathfrak{N}(\Omega)$

2. Quasi*-algebras of multiplier algebras

In this Section we will give a construction of quasi *-algebras from multiplier algebra given in [4]. Let U_o be a *-algebra without unit such that $x \in U_o$ and $x = 0$ whenever $xy = 0\ \forall y \in U_o$.

DEFINITION 2.1 *A multiplier on U_o is a pair (l, r) of linear operators on U_o such that $l(xy) = l(x)y, r(xy) = xr(y)$ and $xl(y) = r(x)y$ for each $x, y \in U_o$.*

Let $M(U_o)$ be the collection of all the multipliers on U_o is a *-algebra with pointwise linear operations of multiplication and involution define as follows;

(1) $(l_1, r_1)(l_2, r_2) = (l_1l_2, r_2r_1)\ \forall (l_1, r_1), (l_2, r_2) \in \Gamma(U_o)$
(2) $(l, r)^* = (r^*, l^*)$ where $l^*(x) \equiv l(x^*)^*$ and $r^*(x) \equiv r(x^*)^*\ x \in U_0$.

The unit is defined by (i, i) where $i(x) = x$. For any element $x \in U_0$, we put $l_x(y) = xy$ and $r_x(y) = yx$, $y \in U_0$ .The map $x \in U_0 \mapsto M(U_0)$ embeds U_o into a *-ideal of $M(U_0)$. Now let p be a Q* seminorm which satisfies the condition of an extended C*-seminorm be define on $M(U_0)$, and put $\mathfrak{N}_\mathfrak{p} = \{(l_x, r_x) \in M(U_o) \colon p((l_x, r_x)) = 0, x \in U_0\}$ is a *-ideal, thus $M(U_0)\backslash\mathfrak{N}_\mathfrak{p}$, is a normed *-algebra with the norm define by $\|(l_x, r_x) + \mathfrak{N}_\mathfrak{p}\|$. Put $\mathfrak{N} \equiv \overline{M(U_0)\backslash\mathfrak{N}_\mathfrak{p}}$ with norm define by $\|\pi_p((l_x, r_x))\| = p((l_x, r_x))$,where π_p is the map $\pi_p \colon M(U_0) \to \overline{M(U_0)\backslash\mathfrak{N}_\mathfrak{p}}$, then $\mathfrak{N}$ is a C*-algebra ,since $\|\pi_p((l_x, r_x)^*(l_x, r_x))\| = p((l_x, r_x)^*(l_x, r_x)) = p((l_x, r_x))^2 = \|\pi_p((l_x, r_x))\|^2$. We denote the norm on $\mathfrak{N}$ by $|(l_x, r_x)|$. Now since $\|r(a)\| = \sup\{\|r(a)x\| : x \in U_0, \|x\| \le 1\} \le \|a\| \sup\{\|l(x)\| : x \in U_0, \|x\| \le 1\}$ and $\|l(a)\| = \sup\{\|xl(a)\| : x \in U_0, \|x\| \le 1\} \le \|a\| \sup\{\|r(x)\| : x \in U_0, \|x\| \le 1\}$. It follows that l is bounded if and only if r is bounded, and we have that $\|l\| = \sup\{\|(l(x)\|, \|x\| \le 1, x \in U_0\}= \sup\{\|r(x)\|, \|x\| \le 1, x \in U_0\} = \|r\|$, where we have $\|l(x)\| \le \|a\| \sup\{\|(r(x)\|, \|x\| \le 1, x \in U_0\}$ and $\|r(x)\| \le \|a\| \sup\{\|l(x)\|, \|x\| \le 1, x \in U_0\}$. Then $M(U_o)$ is a normed *-algebra with norm $\|(l, r)\| = \sup\{\|l(x)\| : \|x\| \le 1, x \in U_0\}$. Let the norm closure of $M(U_0)$ be denoted by $\overline{M(U_0)}$ whence $\mathfrak{N}$ is a dense linear subspace of $\overline{M(U_0)}$. Now we denote the pair $((\overline{M(U_0)}, \|(l, r)\|), (\mathfrak{N}, \|\pi_p((l_x, r_x))\|))$ simply by $(\overline{M(U_0)}, \mathfrak{N})$ with norms given by $\|(l, r)\|$ and $|(l_x, r_x)|$ respectively.

We introduce the notion of a Cauchy Multiplier for a sequence of multipliers $(l_\alpha, r_\alpha)_{\alpha \in \Lambda}$ in the next definition.

DEFINITION 2.2 *Let $\epsilon > 0$ be given, we say that a sequence of multiplier $(l_\alpha, r_\alpha)_{\alpha \in \Lambda} \in M(U_0)$ is said to be Cauchy if there is an index N for which if $\alpha, \beta \ge N$, we have $\|(l_\alpha, r_\alpha) - (l_\beta, r_\beta)\| = \|(l_\alpha - l_\beta, r_\alpha - r_\beta)\| = \sup\{\|l_\alpha(x) - l_\beta(x)\| : \|x\| \le 1, x \in U_0\} < \epsilon$*

Assume that a multiplier sequence $(l_\alpha, r_\alpha)_{\alpha \in \Lambda} \in M(U_0)$ is Cauchy then it is convergent in the norm of $\overline{M(U_0)}$, we denote by $(l, r) \in \overline{M(U_0)}$ the limit for such a sequence and $\overline{M(U_0)}$ the norm completion of $M(U_0)$. The notion of a normal elements of a Multiplier algebra is introduce as follows; we note that the condition of normal elements is given by $A^*A = AA^*$. We have $(l, r)^*(l, r) = (r^*, l^*)(l, r) = (r^*l, rl^*) = ((l^*r)^*, (lr^*)^*) = (lr^*, l^*r)^* = [(l, r)(r^*l^*)]^* = (r^*l^*)^*(l, r)^* = (l, r)(r^*, l^*) = (l, r)(l, r)^*$

DEFINITION 2.3 *We say that a Multiplier algebra is right- normal if $r_{xx^*} = r_{x^*x}$, we have that the equality holds $(l_{x^*x}, r_{xx^*}) = (l_{x^*x}, r_{x^*x})$. We denote the set of all right normal elements by $\mathfrak{N}^n$ and similarly ${}^n\mathfrak{N}$ for left-normal elements.*

Since for $y \in U_0$ we have $l_{xa}(y) = xay = l_x(ay) = l_x(l_a(y))$,thus we write $l_{xa} = l_xl_a$ and also $l_x^*(y) = l_x(y^*)^* = (xy^*)^* = yx^* = r_{x^*}(y)$,and we have $l_x^* = r_{x^*}$ hence we can write $l_{ax}^* = (l_al_x)^* = l_x^*l_a^* = (l_{ax})^* = r_{x^*a^*}$ hence we have $l_{ax}^* = r_{x^*a^*}$ and thus we can extend the multiplication of $\mathfrak{N}$ on $\overline{M(U_0)}$ as follows,

(1) $\mathfrak{N} \times \overline{M(U_0)} \to \overline{M(U_0)}$:
$((l_x, r_x), (l_a, r_a)) \longmapsto (l_{xa}, r_{ax})$

(2) $\overline{M(U_0)} \times \mathfrak{N} \to \overline{M(U_0)}$:
$((l_a, r_a), (l_x, r_x)) \longmapsto (l_{ax}, r_{xa})$

(3) The involution * of $\mathfrak{N}$ is extended on $\overline{M(U_0)}$ denoted also by *
$(l_{ax}, r_{xa})^* = (r_{ax}^*, l_{xa}^*) = (l_{a^*x^*}, r_{x^*a^*})$
$(l_{xa}, r_{ax})^* = (l_{x^*a^*}, r_{a^*x^*})$
$\forall (l_x, r_x) \in \mathfrak{N}$ and $(l_a, r_a) \in \overline{M(U_0)}$

Remark[2] The multiplication and involution above are always defined and continuous, since we have $\|\pi_p((l_{xa}, r_{ax}))\| = \|\pi_p((l_x, r_x)(l_a, r_a))\| \leq |(l_x, r_x)|\|(l_a, r_a)\| = p(l_x, r_x)\|(l_a, r_a)\|$ and $\|\pi_p((l_{ax}, r_{xa}))\| = \|\pi_p((l_a, r_a)(l_x, r_x))\| \leq |(l_x, r_x)|\|(l_a, r_a)\| = p(l_x, r_x)\|(l_a, r_a)\|$

Notation[1] In the sequel we denote (l_{xa}, r_{ax}) by (xa, ax) and similarly also for (ax, xa).

Now let τ be a locally convex topology on $\mathfrak{N}$ such that τ is compatible with the the norm on $\mathfrak{N}$, in the sense that every Cauchy sequence that converge in the norm of $\mathfrak{N}$ also converges in the topology τ where τ is given by a family of seminorms $\{p_\lambda\}_{\lambda \in \Lambda}$ [5] and coincide to the norm topology define on $\mathfrak{N}$. The norm on $\mathfrak{N}$ is extended to $\overline{M(U_0)}$ and the extension is also denoted by $\|\cdot\|$. We have the following,

DEFINITION 2.4 *Let the topology on $\mathfrak{N}$ and its extension to $\overline{M(U_0)}$ be denoted by τ. Then for any $(l_a, r_a) \in \overline{M(U_0)}$ if there exists a sequence $\{(l_x, r_x)_\alpha\}_{\alpha \in \mathit{\Lambda}}$ in $\mathfrak{N}$ such that $\|(l_a, r_a) - (l_x, r_x)_\alpha\| \to 0$ then the norm *-ideal $\mathfrak{N}$ is said to be dense in $\overline{M(U_0)}$.*

We summarize the above discussion in the following

PROPOSITION 2.5 *Let $\mathfrak{N}$ be dense in $\overline{M(U_0)}$, then the pair $(\overline{M(U_0)}, \mathfrak{N})$ is a quasi *-algebra determined by multiplier Algebra.*

3. Seminorm defined by sesquilinear forms on quasi*-algebras of multiplier algebras $(\overline{M(U_0)}, \mathfrak{N})$

Following Trapani [1], we associate the family of seminorms to a family of sesequilinear forms $\mathcal{F}$. Using the notation of the previous section we have the following

DEFINITION 3.1 *A positive sesequilinear form Ω on $\overline{M(U_0)} \times \overline{M(U_0)}$ is called quasi-invariant if Ω satisfies $\Omega((xa, ax), (b, b)) = \Omega((a, a), (x^*b, bx^*))$ $\forall a, b \in U, x \in U_0$.*

Since U_0 is a *-algebra without a unit, we introduce the notion of *quasi-normal* and *quasi *-invariant* in the following; we note that when an element commute with its adjoint then it is said to be normal.We use this analogy to introduce the notion of quasi-normal multipliers.We say an element $a \in U_0$ is quasi-normal if there is an element $x \in U_0$ such that $ax, x^*a \in U_0$ with $ax = x^*a$.

DEFINITION 3.2 *A multiplier $(l_a, r_a) \in \overline{M(U_0)}$ is said to be quasi-normal if there exists an element $(l_x, r_x) \in \overline{M(U_0)}$ such that $(l_a, r_a)(l_x, r_x)^* {=} (l_x, r_x)^*(l_a, r_a)$ implies $(l_{x^*a}, r_{ax^*}) = (l_{ax^*}, r_{x^*a})$ $\forall \in U_0$ and quasi *-invariant if $(l_{xa}, r_{ax}) = (l_{x^*a}, r_{ax^*})$ $\forall a \in U, x \in U_0$*

PROPOSITION 3.3 *If Ω is an invariant positive sesequilinear form on $\overline{M(U_0)} \times \overline{M(U_0)}$ then it satisfies the relations*

(1) $\Omega((l_{xa}, r_{ax}, (l_b, r_b)) = \Omega((l_a, r_a), (l_{x^*b}, r_{bx^*})$
(2) $\Omega((l_{ax}, r_{xa}), (l_b, r_b)) = \Omega((l_a, r_a), (l_{bx^*}, r_{x^*b}))$

Proof. Let Ω be an invariant positive sesequilinear form on $\overline{M(U_0)} \times \overline{M(U_0)}$ then

$$\begin{aligned}\Omega((l_{xa}, r_{ax}), (l_b, r_b)) =& \Omega((xa, ax), (b, b)) = \Omega((xa, ax), (b, b)) = \Omega((x, x)(a, a), (b, b)) \\ =& \Omega((a, a), (x, x)^*(b, b)) = \Omega((a, a), (x^*, x^*)(b, b)) = \Omega((a, a), (x^*b, bx^*)) \\ =& \Omega((l_a, r_a), (l_{x^*b}, r_{bx^*})\end{aligned}$$

Next

$$\begin{aligned}\Omega((l_{ax}, r_{xa}), (l_b, r_b)) =&\Omega(ax, ax), (b, b)) = \Omega((ax, xa), (b, b)) = \Omega((x, x)(a, a), (b, b))\\ =&\Omega((a, a), (b, b)(x, x)^*) = \Omega((a, a), (b, b)(x^*, x^*)) = \Omega((a, a), (bx^*, x^*b))\\ =&\Omega((l_a, r_a), (l_{bx^*}, r_{x^*b})\end{aligned}$$

■

PROPOSITION 3.4 *For a quasi-normal multiplier, the invariant positive sesequilinear form* Ω *on* $\overline{M(U_0)} \times \overline{M(U_0)}$ *satisfies the relation* $\Omega((l_{xa}, r_{ax}), (l_b, r_b)) = \Omega((l_{ax}, r_{xa}), (l_b, r_b)))$

Proof.

$$\begin{aligned}\Omega((l_{xa}, r_{ax}), (l_b, r_b)) =&\Omega((l_a, r_a), (l_{x^*b}, r_{bx^*})\\ =&\Omega((l_a, r_a), (l_{bx^*}, r_{x^*b})\\ =&\Omega((a, a), (bx^*, x^*b))\\ =&\Omega((a, a), (b, b)(x, x)^*)\\ =&\Omega((l_{ax}), r_{xa}), (l_b, r_b))\end{aligned}$$

■

Remark[3] We denote the set of invariant positive sesequilinear forms on $\overline{M(U_0)} \times \overline{M(U_0)}$ by $\mathcal{P}$. We now state a definition for an invariant sesquilinear form to be admissible.The set of positive cone of $\mathcal{P}$ admits a natural partial order $\leq$ defined as follows: Let Ω_1 and $\Omega_2 \in \mathcal{P}$, write $\Omega_1 \leq \Omega_1$ iff $\Omega_1 - \Omega_1 \in \mathcal{P}$

DEFINITION 3.5 *An element* $\Omega \in \mathcal{P}$ *is called admissible if for each* $(l_a, r_a) \in \overline{M(U_0)}$ *there exists* $K_{(l_a,r_a)} > 0$ *such that* $\Omega((l_{xa}, r_{ax}), (l_{xa}, r_{ax})) \leq K_{(l_a,r_a)}\Omega((l_x, r_x), (l_x, r_x))$

The set of all admissible sesequilinear form in $\mathcal{P}$ is denoted by $\mathcal{P}^a$ .If $\Omega \in \mathcal{P}$ and $(l_a, r_a) \in \overline{M(U_0)}$, we can define a *linear* functional $\omega_\Omega^{(l_a,r_a)}$ on $\mathfrak{N}$ by

$$\omega_\Omega^{(l_a,r_a)}((l_x, r_x)) = \Omega((l_{xa}, r_{ax}), (l_a, r_a))$$

The equality $\Omega((xa, ax), (xa, ax)) = \Omega((x^*, x^*)(xa, ax), (a, a)) = \Omega((x^*xa, axx^*), (a, a))$ implies that each $\omega_\Omega^{(l_a,r_a)}$ is positive on $\mathfrak{N}$, and is a state if, and only if, $\Omega((l_a, r_a), (l_a, r_a)) = 1$. Following [1] and [6] we put $\mathcal{F}^0 = \{\omega_\Omega^{(l_a,r_a)} \colon \Omega \in \mathcal{F}, (l_a, r_a) \in \overline{M(U_0)}\}$. We have the set $\mathfrak{N}(\mathcal{F}^0) = \{(l_x, r_x) \in \mathfrak{N}\colon \sup\omega_\Omega^{(a,a)}((l_{x^*x}, r_{xx^*}))\colon \Omega \in \mathcal{P}, (l_a, r_a) \in \overline{M(U_0)}, \Omega(l_a, r_a), (l_a, r_a)) = 1\} < \infty\}$ is a *-subalgebra of $\mathfrak{N}$ and $|(l_x, r_x)|^2 = (\sup\omega_\Omega^{(l_a,r_a)}((l_{x^*x}, r_{xx^*}))\colon \Omega \in \mathcal{P}, (l_a, r_a) \in \overline{M(U_0)}, \Omega((l_a, r_a), (l_a, r_a)) = 1\})$ defines a seminorm on $\mathfrak{N}(\mathcal{F}^0)$. If we assume that the multiplier is normal in the sense that $r_{xx^*} = r_{x^*x}$, $(l_{x^*x}, r_{xx^*}) = (l_{x^*x}, r_{x^*x})$ and the multipler is quasi *-invariant then we have, the equality $\Omega((xa, ax^*), (x^*a, ax)) =\Omega((x^*, x^{**})(xa, ax^*), (a, a)) =\Omega((x^*, x)(xa, ax^*), (a, a)) = \Omega((x^*xa, ax^*x), (a, a))$ and $\omega_\Omega^{(a,a)}((l_{x^*x}, r_{x^*x})) = \Omega((x^*xa, ax^*x), (a, a))$. We put $|(l_x, r_x)|^2 = sup\{\omega_\Omega^{(a,a)}((l_{x^*x}, r_{x^*x}))\}$ this gives us a C*-seminorm on $\mathfrak{N}(\mathcal{F}^0)$. Since $|(l_x, r_x)^*(l_x, r_x)| = p((l_x, r_x)^*(l_x, r_x)) = p(l_x, r_x)^2 = |(l_x, r_x)|^2$. We need to extend it to $\overline{M(U_0)}$ and hence, we have the following: Let $\mathcal{F} \subset \mathcal{P}$, for $(l_a, r_a) \in \overline{M(U_0)}$, we put $p_\mathcal{F}((l_a, r_a)) = \sup_{\mathcal{F}_s} \Omega((l_a, r_a), (l_a, r_a))^{\frac{1}{2}}$, and $\mathcal{F}_s = \{\Omega \in \mathcal{F}\colon \Omega((l_a, r_a), (l_a, r_a)) = 1)\}$ where $\mathcal{F}_s$ denotes the states in $\mathcal{F}$. We set $\Gamma^\mathcal{F}(U) = \{(l_a, r_a) \in \overline{M(U_0)}\colon p_\mathcal{F}((l_a, r_a)) < \infty\}$. Then we have $p_\mathcal{F}((l_{xa}, r_{ax})) = p_\mathcal{F}((xa, ax)) = p_\mathcal{F}((l_x, r_x)(l_a, r_a)) \leq |(l_x, r_x)|p_\mathcal{F}((l_a, r_a))$ for $(l_a, r_a) \in \Gamma^\mathcal{F}(U)$. Thus we have $p_\mathcal{F}((l_{xa}, r_{ax})) \leq |(l_x, r_x)|p_\mathcal{F}((l_a, r_a))$. We summarize the above discussion in the following.

Proposition 3.6 *Let $\mathcal{F} \subseteq \mathcal{P}$, if the multiplier algebra is normal and quasi *-invariant then $p_{\mathcal{F}}$ defines a seminorm on $\overline{M(U_0)}$, which is given by $p_{\mathcal{F}}((l_{xa}, r_{ax})) \leq |(l_x, r_x)| p_{\mathcal{F}}((l_a, r_a))$. for $(a, a) \in \Gamma^{\mathcal{F}}(U)$*

4. Conclusion

We have the following concluding remarks. The absence of examples in this work is deliberate, we refer the reader to Trapani [1] page (105-106) for examples related to what was presented in this work. The study of unbounded *-representations and that of unbounded seminorms are currently subjects of research interest ,we hope to study them in subsequent work.

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