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In honour of Prof. Ekhaguere at 70

Existence of solution of impulsive quantum stochastic differential inclusion

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Abstract. By employing the non-commutative analoque of Leray-Schauder fixed point theorem, Arsela-Ascoli theorem and Michael selection theorem, we establish the existence of solution of impulsive quantum stochastic differential inclusions(IQSDI) in the framework of Hudson and Parthasarathy formulation of quantum stochastic calculus. The result hold in an infinite dimensional locally convex space. Important properties of these solutions are studied.

Keywords: non-commutative analogue, impulsive quantum stochastic differential inclusion, selections, fixed point, infinite dimensional locally convex space, Upper and lower semicontinuous operators.

1. Introduction

For well over a century, differential equations have been used in modeling the dynamics of changing processes. A great deal of the modeling development has been accompanied by a rich theory for differential equations. The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of "impulses". As a consequence, classical impulsive differential equations have found application in modeling impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a classical theory of impulsive differential equations has been given extensive attention. Much attention has also been devoted to modeling natural phenomena with differential equations, both ordinary and functional, for which the part governing the derivative(s) is not known as a single-valued function. Our consideration in this paper concerns the establishment of a solution of impulsive quantum stochastic differential inclusions in the framework of Hudson-Phathasarathy formulation of quantum stochastic calculus.

The plan for the rest of the paper is as follows: section 2 contains fundamental structures and definitions that we use in the sequel. In section 3 we assemble some auxiliary results that are use in establishing the main result. The main result concerning the existence of solution to impulsive quantum stochastic differential inclusion is established in section 4.

2. Fundamental structures and definitions

In this section we state some fundamental structures and definition that will be use in the sequel. Given a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, a single valued map $f : \mathbb{R}^m \to \mathbb{R}^n$ is a selection if $f(x) \in F(x) \quad \forall x \in \mathbb{R}$.

(i) *Upper and Lower Semi continuous Multivalued Maps*: Let $\mathcal{N} \subseteq \tilde{\mathcal{A}}$ and $I \subseteq \mathbb{R}_+$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, (t, x), (t_0, x_0) \in I \times \mathcal{N}$ and real numbers $\epsilon, \delta_{\eta,\xi} > 0$, we define the map $d_{\eta,\xi}$:

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$[I \times \mathcal{N}] \to \mathbb{R}_+$ by

$$d_{\eta,\xi}((t,x),(t_0,x_0)) = \max\{|t-t_0|, \|x-x_0\|_{\eta,\xi}\}.$$

The following shall be employed in what follows. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

$$\tilde{\mathcal{A}}(\eta,\xi) = \{x_{\eta,\xi} = \langle \eta, x\xi \rangle, x \in \tilde{\mathcal{A}}\}$$

$$B_{\eta,\xi,\epsilon}(0) = \{x_{\eta,\xi} \in \tilde{\mathcal{A}}(\eta,\xi) : |x_{\eta,\xi}| < \epsilon\}$$

$$B_{\delta_{\eta,\xi}}(t_0,x_0) = \{(t,x) \in I \times \mathcal{N} : d_{\eta,\xi}((t,x),(t_0,x_0)) < \delta_{\eta,\xi}\}.$$

(ii) A map $\phi : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be upper semi continuous at a point $(t_0, x_0) \in I \times \mathcal{N}$, if for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exists $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that

$$\phi(t,x)(\eta,\xi) \subset \phi(t_0,x_0)(\eta,\xi) + B_{\eta,\xi,\epsilon}(0)$$

on $B_{\delta_{\eta,\xi}}(t_0,x_0)$. The map ϕ is said to be upper semi continuous if it is upper semi continuous at every point $(t,x) \in I \times \mathcal{N}$. Furthermore, for a sesquilinear formed valued map we define

$$B_{\mathbb{P},\epsilon}(0) = \{\varphi(t,x)(\eta,\xi) \in \mathbb{P}(t,x)(\eta,\xi) : |\varphi(t,x)(\eta,\xi)| < \epsilon\}$$

(iii) A sesquilinear form valued multifunction $\mathbb{P} : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be upper semi continuous at a point $(t_0,x_0) \in I \times \mathcal{N}$ if for every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exist $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that

$$\mathbb{P}(t,x)(\eta,\xi) \subset \mathbb{P}(t_0,x_0)(\eta,\xi) + B_{\mathbb{P},\epsilon}(0)$$

on $B_{\delta_{\eta,\xi}}(t_0,x_0)$. The map $\mathbb{P}$ is said to be upper semi continuous if it is upper semi continuous at every point $(t,x) \in I \times \mathcal{N}$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $\Phi : I \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ be a closed multivalued map. For each pair $(t,x),(t^{'},x^{'}) \in I \times \tilde{\mathcal{A}}$ we define

$$d_{\eta,\xi}((t,x),(t^{'},x^{'})) = \max\{|t-t^{'}|, ||x-x^{'}||_{\eta\xi}\}$$

$$B_{\eta,\xi,\delta}(t_0,x_o) = \{(t,x) \in I \times \tilde{\mathcal{A}} : d_{\eta,\xi}((t_0,x_0),(t,x))(\eta,\xi) < \delta\} \quad \text{and}$$

$$B_{\eta\xi,\epsilon}(\Phi(t,x) = \{y \in \tilde{\mathcal{A}} : \inf_{k\in\Phi(t,x)} ||y-k||_{\eta,\xi} < \epsilon\}.$$

(iv) A map $\Phi : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be lower semi continuous at a point $(t_0,x_0) \in I \times \mathcal{N}$, with respect to the seminorm $||.||_{\eta\xi}$ if for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exists $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that for each $y_o \in \Phi(t_o,x_o)$, $\inf_{y\in\Phi(t,x)} ||y-y_o||_{\eta\xi} < \epsilon$, $\forall \; y \in \mathcal{N}$, almost all $t \in I$ and $d_{\eta,\xi}((t,x),(t^{'},x^{'})) < \delta_{\eta,\xi}$. If Φ is lower semi continuous at every point $(t_0,x_0) \in I \times \mathcal{N}$ with respect to the seminorm $||.||_{\eta\xi}$, then it will be said to be lower semi continuous on $I \times \mathcal{N}$.

(v) A sesquilinear form valued multifunction $\mathbb{P} : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be lower semi continuous at a point $(t_0,x_0) \in I \times \mathcal{N}$, with respect to the seminorm $||.||_{\eta\xi}$ if for every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exist $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that for each

$y_{\eta\xi,0} \in \mathbb{P}(t_0,x_0)(\eta,\xi)$ $\inf_{y_{\eta\xi}\in\mathbb{P}(t,x)}|y_{\eta\xi,0} - y_{\eta\xi}| < \epsilon, \forall y \in \mathcal{N}$, almost all $t \in I$ and $d_{\eta,\xi}((t,x),(t_0,x_0)) < \delta_{\eta,\xi}$

(vi) The space $Pad(I,\tilde{\mathcal{A}})_{wac} = \{X : I \to \tilde{\mathcal{A}} : X$ is adapted and weakly absolutely continuous everywhere except for some t_k at which $X(t_k^-)$ and $X(t_k^+)$, $k = 1,2,...,m$ exists and $X(t_k^-) = X(t_k^+)\}$.

(vii) For each pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we define the space of complex valued numbers associated with (i) as $Pad(I,\tilde{\mathcal{A}})_{wac,\eta\xi} = \{\langle\eta,\Phi(.)\xi\rangle : \Phi \in Pad(I,\tilde{\mathcal{A}})_{wac}\}$.

(viii) On $Pad(I,\tilde{\mathcal{A}})_{wac}$, we define a seminorm

$$\|\Phi\|_{p,\eta\xi} = \sup\{\|\Phi(t)\|_{\eta\xi}, t \in I\}, \tag{2.0}$$

and denote by $P_{wac}(\tilde{\mathcal{A}})$ the completion of the locally convex space whose topology is generated by the seminorm in (2.0).

(ix) Let $E,F,G,H \in L^2_{loc}(I\times\tilde{\mathcal{A}})_{mvs}$ and (t_0,x_0) be a fixed point of $I\times\tilde{\mathcal{A}}$. Then a relation of the form

$$dX(t) \in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) + G(t,X(t))dA_g^+(t) + H(t,x(t))dt$$

for almost all $t \in I\backslash\{t_k\}_{k=1}^m$,

$$\Delta X_{t=t_k} = J_k(X(t_k)), \qquad t = t_k, k = 1,2,...,m \tag{2.1}$$

$$X(t_0) = \Phi(t), t \in I$$

or equivalently

$$\frac{d}{dt}[\langle\eta,X(t)\xi\rangle] \in \mathbb{P}(t,X(t))(\eta,\xi) \qquad \text{almost all t } \in I\backslash\{t_k\}_{k=1}^m,$$

$$\langle\eta,\Delta X_{t=t_k}\xi\rangle = \langle\eta,J_kX(t_k)\xi\rangle, \quad t = t_k, k = 1,2,...,m, \tag{2.2}$$

$$\langle\eta,X(t_0)\xi\rangle = \langle\eta,\phi(t)\xi\rangle, t \in I,$$

where $0 = t_0 < t_1 < t_2 < ..... < t_m < t_{m+1} = T, I = [0,T]$, is called impulsive quantum stochastic differential inclusions (IQSDI). *Note*: The map $\mathbb{P} : I\times\tilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ is a multivalued sesquilinear form having non empty, compact values. $X(t_0) \in \tilde{\mathcal{A}}$, $J_k \in C(\tilde{\mathcal{A}},\tilde{\mathcal{A}})$, $k = 1,2,...,m.\Delta X|_{t=t_k} = X(t_k^+) - X(t_k^-)$, $X(t_k^-), X(t_k^+)$ represent the left and the right limit of $X(t)$.

For any process $X : I \to \tilde{\mathcal{A}}$ and any $t \in I$, $X(t)$ represents the history of the state from previous time up to the present time t , the map J_k characterize the jump of the solutions at impulse points $t_k, k = 1,2,...,m$.

(x) By solution of Impulsive quantum stochastic differential inclusion (2.1) or equivalently (2.2) we mean a stochastic process $\Phi : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ lying in the space $P_{wac}(\tilde{\mathcal{A}}) \cap wac((t_k,t_{k+1}),\tilde{\mathcal{A}}), 0 \le k \le m$, satisfying

$$\frac{d}{dt}[\langle\eta,\Phi(t)\xi\rangle] \in \mathbb{P}(t,\Phi(t))(\eta,\xi) \qquad \text{almost all t } \in I\backslash[t_k]_{k=1}^m$$

and the condition

$$\Delta\Phi|_{t=t_k} = J_k(\Phi(t_k^-)) \quad \text{and} \quad \Phi(0) = X_0.$$

The following theorems shall be employ to prove our main result.

3. Theorems

THEOREM 3.1 *Let U and $\overline{U}$ denote respectively the open and closed subsets of a convex set K of $\tilde{\mathcal{A}}$ such that $0 \in U$ and let $N : \overline{U} \to K$ be a compact and semi continuous map. Then either*
(i) The equation $x = Nx$ has a solution in $\overline{U}$ or
(ii) There exists a point $u \in \delta U$ such that $u = \lambda Nu$ for some $\lambda \in \mathbb{C}$ such that $Re\lambda \in (0,1)$ and $Im\lambda \in (0,1)$, where δU is a boundary of U.

THEOREM 3.2 *Let $X : I \to \tilde{\mathcal{A}}$ be a stochastic process that satisfy the following conditions :*
(i) For any arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,
let $K \subset \tilde{\mathcal{A}}$ such that $F : K \to K$ is a compact map.
(ii) $\|f(x)\|_{\eta\xi} \leq m$ for each $x \in X, f \in F$ and $m < \infty$.
(iii) For every $\epsilon > 0$ (depending on η, ξ) there exist $\delta_{\eta\xi}$ such that for every $x, y \in X$,

$$d(x,y)(\eta,\xi) < \delta_{\eta\xi}.$$

Then,

$$\langle \eta, (f(x) - f(y))\xi \rangle < \epsilon \quad \forall \quad f \in F, \quad x, y \in X.$$

Next, we shall establish the a priori estimates on possible solutions of problem (2.1)-(2.2).

THEOREM 3.3 *Suppose that the following hold for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. (i) There exists a continuous non-decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and*

$$p \in L^1(I, \mathbb{R}_+) \quad \text{such that} \quad |\mathbb{P}(t,x)(\eta,\xi)| \leq p(t)\phi(||X||_{\eta,\xi})$$

$$\text{for a.e} \quad t \in I \quad \text{and} \quad x \in \tilde{\mathcal{A}} \tag{3.3.1}$$

with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1,\eta,\xi}}^{\infty} \frac{du}{\phi(u)}, \quad k = 1, ..., m+1, \tag{3.3.2}$$

where

$$N_{0,\eta,\xi} = ||x_0||_{\eta,\xi},$$

$$N_{k-1,\eta,\xi} = \sup_{||x||_{\eta,\xi} \in [-M_{k-2}, M_{k-2}]} ||J_{k-1}(x)||_{\eta,\xi} + M_{k-2},$$

$$M_{k-2} = \Gamma_{k-1}^{-1} \int_{t_{k-2}}^{t_{k-1}} p(s)ds, \quad \text{for} \quad k = 1, ..., m+1, \tag{3.3.3}$$

and

$$\Gamma_l(z) = \int_{N_{l-1,\eta,\xi}}^{z} \frac{du}{\phi(u)}, \quad z \geq N_{l-1} \quad l \in [1, ..., m+1]. \tag{3.3.4}$$

Then, for each $k = 1, ..., m+1$ there exists a constant $M_{k-1,\eta,\xi}$ such that

$$\sup\{||X(t)||_{\eta,\xi} : t \in [t_k, t_{k+1}]\} \leq M_{k-1,\eta,\xi} \tag{3.3.5}$$

for each solution X of the problem (2.1 - 2.2).

Proof. Let X be a possible solution of (2.2). Then $X|_{[0,t_1]}$ is a solution to

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi) \qquad \text{almost all} \quad t \in [0, t_1], X(0) = X_0. \tag{3.3.7}$$

Since

$$\frac{d}{dt}|\langle \eta, X(t)\xi\rangle| \leq |\frac{d}{dt}\langle \eta, X(t)\xi\rangle|, \tag{3.3.8}$$

we have

$$\frac{d}{dt}|\langle \eta, X(t)\xi\rangle| \leq p(t)\phi(\|X(t)\|_{\eta\xi}), \quad \text{for a.e} \quad t \in [0, t_1]. \tag{3.3.9}$$

Let $t^* \in [0, t_1]$ such that

$$\sup\{\|X(t)\|_{\eta\xi} : t \in [0, t_1]\} = \|X(t^*)\|_{\eta\xi}\}, \tag{3.3.10}$$

then

$$\frac{\frac{d}{dt}|\langle \eta, X(t)\xi\rangle|}{\phi(\|X(t)\|_{\eta\xi})} \leq p(t) \text{ for a.e} \quad t \in [0, t_1]). \tag{3.3.11}$$

From inequality (3.3.11), it follows that

$$\int_0^{t^*} \frac{\frac{d}{dt}|\langle \eta, X(s)\xi\rangle|}{\phi(\|X(s)\|_{\eta\xi})} \leq \int_0^{t^*} p(s)ds. \tag{3.3.12}$$

Using change of variable formula, we get

$$\Gamma_1(\|X(t^*)\|_{\eta\xi}) = \int_{||X_0||_{\eta,\xi}}^{||X(t^*||_{\eta,\xi}} \frac{du}{\phi(u)} \leq \int_0^{t^*} p(s)ds \leq \int_0^{t_1} p(s)ds. \tag{3.3.13}$$

Given that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $p \in L^1(I, \mathbb{R}_+)$ such that

$$|\mathbb{P}(t, X)(\eta, \xi)| \leq p(t)\phi(||X||_{\eta,\xi}),$$

we obtain that

$$\|X(t^*)\|_{\eta\xi} \leq \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right)$$

Hence,

$$\|X(t^*)\|_{\eta\xi} = sup\{[\|X(t)\|_{\eta\xi} : t \in [0, t_1]\} \leq \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right) := M_0.$$

Now $X|_{[t_1,t_2]}$ is a solution to

$$\frac{d}{dt}\langle\eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta,\xi) \qquad \text{almost all} \quad t\in[t_1,t_2]$$

$$\Delta X|_{t=t_1} = J_k(X(t_1)). \tag{3.3.14}$$

Then

$$\frac{d}{dt}|\langle\eta, X(t)\xi\rangle| \leq p(t)\phi(||X(t)||_{\eta,\xi}) \qquad \text{almost all} \quad t\in[t_1,t_2]. \tag{3.3.16}$$

Let $t^*\in[t_1,t_2]$ such that

$$\sup\{\|X(t)\|_{\eta\xi} : t\in[t_1,t_2]\} = \|X(t^*)\|_{\eta\xi}. \tag{3.3.17}$$

Then

$$\frac{\frac{d}{dt}|\langle\eta, X(t)\xi\rangle|}{\phi(\|X(t)\|_{\eta\xi}} \leq p(t). \tag{3.3.18}$$

From this inequality, it follows that

$$\int_{t_1}^{t^*}\frac{\frac{d}{dt}|\langle\eta, X(s)\xi\rangle|}{(|X(s)|}ds \leq \int_{t_1}^{t^*}p(s)ds. \tag{3.3.19}$$

Proceeding as above we obtain

$$\Gamma_2|X(t^*)| = \int_{N_1}^{|X(t^*)|}\frac{du}{\phi(u)} \leq \int_{t_1}^{t^*}p(s)ds \leq \int_{t_1}^{t_2}p(s)ds. \tag{3.3.20}$$

This yields

$$|X(t^*)| = \sup\{|X(t)| : t\in[t_1,t_2]\} \leq \Gamma_2^{-1}\left(\int_{t_1}^{t_2}p(s)ds\right) := M_1. \tag{3.3.21}$$

Continuing this process and taken into account that $X|_{[t_m,T]}$ is a solution to the problem,

$$\frac{d}{dt}\langle\eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta,\xi) \qquad \text{almost all} \quad t\in[t_m,T]$$

$$\Delta X|_{t=t_m} = J_k(X(t_m)), \tag{3.3.22}$$

then there exist a constant M_m such that

$$\sup\{\|X(t)\|_{\eta\xi} : t\in[t_m,T]\} \quad \leq \quad \Gamma_{m+1}^{-1}\left(\int_{t_m}^{T}p(s)ds\right) := M_m. \tag{3.3.23}$$

Consequently for each X to (2.2), we have

$$\|X\|_{\eta,\xi} \leq \max\{\|X_0\|_{\eta\xi}, M_{k-1} : k = 1, ..., m+1\}. \quad (3.3.24)$$

■

THEOREM 3.4 *Assume that the map $\mathbb{P} : (I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ satisfies the following conditions :*
(i) for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \mathbb{P}(t,x)(\eta,\xi)$ is closed and convex in $\mathbb{C}$.
(ii) The map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semi continuous on $(I \times \tilde{\mathcal{A}})$,
then there exists a continuous map $f : (I \times \tilde{\mathcal{A}}) \to sesq(\mathbb{D} \otimes \mathbb{E})^2$
which is a selection of $\mathbb{P}(t,x)(\eta,\xi)$.

4. Main result

The following theorem furnish our main result.

THEOREM 4.1 *Suppose that the following hypothesis are satisfied*
(i) The map $\mathbb{P} : I \times \tilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is such that for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, (t,x) \in I \times \tilde{\mathcal{A}}, \mathbb{P}(t,x)(\eta,\xi)$ is closed and convex in $\mathbb{C}$, the space of complex numbers.
(ii) The map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semi continuous and measurable on $(I \times \tilde{\mathcal{A}})$.
(iii) For every $r > 0$, there exists function $h_{\eta\xi,r} : I \to \mathbb{R}$ lying in $L^{'}(I, \mathbb{R}_+)$, such that $|\mathbb{P}(t,x)(\eta,\xi)| = \sup\{|v_{\eta\xi}| : v_{\eta\xi} \in \mathbb{P}(t,x)(\eta,\xi)\} \leq h_{\eta\xi,r}$, for a.e $t \in I$ and $x \in \tilde{\mathcal{A}}$ with $||x||_{\eta\xi} \leq r$.
Then the impulsive problem (2.1) -(2.2) has a solution.

Proof. Let

$$f : P_{wac}(\tilde{\mathcal{A}}) \to L'_{loc}(\tilde{\mathcal{A}})$$

such that

$$f(x) \in \mathcal{F}(x) \quad \forall \quad y \in P_{wac}(\tilde{\mathcal{A}}).$$

Consider the single valued problem

$$\left.\begin{array}{l} \frac{d}{dt}\langle\eta, X(t)\xi\rangle = F(X(t))(\eta,\xi) \ t \in I, t \neq t_k, k = 1,2,...m \\ \Delta X|_{t=t_k} = J_k(X(t_k^-))\xi \qquad t = t_k, k = 1,2,...,m \\ X(0) = X_0. \end{array}\right\} \quad (4.1)$$

Let

$$N(X)(t)(\eta,\xi) = \|N(X)(t)\|_{\eta\xi} = |\langle\eta, N(X)(t)\xi\rangle| = \langle\eta, x_0\xi\rangle + \int_0^t |\langle\eta, (E(t,X(t))d\Lambda_\pi(t) +$$

$$F(t,X(t))dA_f(t) + G(t,X(t))dA_g^+(t) + H(t,X(t))dt)\xi\rangle| + \sum_{0<t_k<t} \langle\eta, J_k(X(t_k^-))\xi\rangle,$$

where

$$(E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) + G(t,X(t))dA_g^+(t) + H(t,X(t))dt \equiv \mathbb{P}(t,X(t))(\eta,\xi).$$

We now transform problem (4.1) into a fixed point problem by considering the operators

$$N_{\eta\xi}(X)(t) = x_0 + \int_0^t f(X(s))(\eta,\xi) + \sum_{0<t_k<t} \langle \eta, J_k(X(t_k^-))\xi \rangle. \tag{4.2}$$

We show that $N_{\eta\xi}$ is compact for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. That is

$$N(X)(t) = x_0 + \sum \int_t^{t_0} E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) +$$

$$G(t, X(t))dA_g^+(t) + H(t, X(t))dt + \sum_{0<t_k<t} J_k(X(t_k^-))$$

$$N : P_{wac}(\tilde{\mathcal{A}}) \to P_{wac}(\tilde{\mathcal{A}}).$$

Step 1 : N is continuous. Let $\{X_n\}$ be a sequence such that $X_n \to X \in P_{wac}(\tilde{\mathcal{A}})$.

$$\begin{aligned}\|N(X_n(t)) - N(X(t))\|_{\eta,\xi} &\leq \int_0^t |\mathbb{P}(s, X_n(s))(\eta,\xi) - \mathbb{P}(s, X(s))(\eta,\xi)|ds \\ &\quad + \sum_{0<t_k<t} ||J_k(X_n(t_k^-)) - J_k(X(t_k^-))||_{\eta,\xi} \\ &\leq \int_0^T |\mathbb{P}(s, X_n(s))(\eta,\xi) - \mathbb{P}(s, X(s))(\eta,\xi)|ds \\ &\quad + \textstyle\sum_{0<t_k<t} ||J_k(X_n(t_k^-)) - J_k(X(t_k^-))||_{\eta,\xi}.\end{aligned} \tag{4.3}$$

Since $\mathbb{P}$ and $J_K, k = 1, 2, ..., m$ are continuous, then

$$\begin{aligned}\|N(X_n) - N(X))||_{\eta,\xi} &\leq \|\mathbb{P}(t, X_n(s))(\eta,\xi) - \mathbb{P}(t, X(s))(\eta,\xi)\|_{\eta,\xi} \\ &\quad + \textstyle\sum_{0<t_k<t} |J_k(X_n(t_k^-)) - J_k(X(t_k^-))| \to 0\end{aligned} \tag{4.5}$$

as $n \to \infty$ which implies that N is continuous.

Step 2 : N maps bounded set into bounded sets in $P_{wac}(\tilde{\mathcal{A}})$. Let $X \in B_q = [x \in P_{wac}(\tilde{\mathcal{A}}) : ||x||_{\eta,\xi} \leq q]$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ we have that

$$\|N(X)\|_{\eta,\xi} \leq q,$$

since $J_k, k = 1, ..., m$ are continuous from assumption (iii), we have

$$\begin{aligned}\|N(X(t))\|_{\eta,\xi} &\leq \|X_0\|_{\eta\xi} + \int_0^t |\mathbb{P}(t, X(s))(\eta,\xi)|ds + \sum_{0<t_k<t} ||J_k(X(t_k^-))||_{\eta,\xi} \\ &\leq \|X_0\|_{\eta\xi} + ||h_q||_{L'} + \textstyle\sum_{k=1}^m ||J_k(x(t_k^-)||_{\eta,\xi} := l.\end{aligned} \tag{4.6}$$

Step 3 : N maps bounded set into equicontinuous sets of $P_{wac}(\tilde{\mathcal{A}})$. Let $r_1, r_2, \in I$ and let $B_q = [X \in P_{wac}(\tilde{\mathcal{A}}) : ||X||_{\eta,\xi} \leq q]$ be a bounded set of $P_{wac}(\tilde{\mathcal{A}})$. Then

$$\|N(X)(r_2) - N(X)(r_1)\|_{\eta,\xi} \leq \int_{r_1}^{r_2} |h_q(s))|ds + \sum_{0<t_k<r_2-r_1} ||J_k(x(t_k^-)||_{\eta,\xi}. \tag{4.7}$$

As $r_2 \to r_1$, the right hand side of the above inequality tends to zero. This established the equicontinuity of the case where $t \neq t_i, i = 1, 2, ..., m$. To examine equicontinuity at $t = t_i$ we have

$$\begin{aligned}
&\|N(X)(r_2) - N(X)(r_1)\|_{\eta,\xi} \leq |\mathbb{P}(r_2, X_n(s))(\eta,\xi) - \mathbb{P}(r_1, X(s))(\eta,\xi)|ds \\
&\sum_{0<t_k<t} \|J_k(X_n(t_k^-)) - J_k(X(t_k^-))\|_{\eta,\xi}\|X_0\|_{\eta,\xi} \\
&+ \int_{r_1}^{r_2} |\mathbb{P}(r_2 - s, X(s))(\eta,\xi) - \mathbb{P}(r_1 - s, X(s))(\eta,\xi)|ds \\
&+ \sum_{0<t_k<t} \|J_k(X_n(t_k^-)) - J_k(X(t_k^-))\|_{\eta,\xi}(B(X(s))|ds \\
&+ \int_{r_1}^{r_2} |\mathbb{P}(r_2 - s, X(s))(\eta,\xi)|(B(x(s))|ds \\
&+ \int_{0}^{r_1} |\mathbb{P}(r_2 - s, X(s))(\eta,\xi) - \mathbb{P}(r_1 - s, X(s))(\eta,\xi)|\phi_q(s)ds \\
&+ \int_{r_1}^{r_2} |\mathbb{P}(r_2 - s, X(s))(\eta,\xi)| \\
&+ \sum_{r_1<0<r_2} J_k|\mathbb{P}(r_2 - t_k, X(s)) - \mathbb{P}(r_1 - t_k, X(s))|. \qquad (4.8)
\end{aligned}$$

The right hand side of (4.8) tends to zero as $r_2 - r_1 \to 0$. To show equicontinuity at the left limit $t = t_k^-$ fix $\delta_1 > 0$ such that $[t_k : k \neq i] \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < h < \delta_1$, we have

$$\begin{aligned}
&|N(X)(t_i) - N(X)(t_i - h)| \leq |\mathbb{P}((t_i, X(s))(\eta,\xi) - \mathbb{P}((t_i - h, X(s))(\eta,\xi)|||X_0||_{\eta,\xi} \\
&+ \int_0^{t_i-h} |\mathbb{P}(t_i - s, X(s))(\eta,\xi) - \mathbb{P}(t_i - h - s, X(s))(\eta,\xi)|(B(X(s))|ds \\
&+ \int_{t_i-h}^{t_i} |\mathbb{P}(t_i - h, X(s))(\eta,\xi)|(B(x(s))|ds \\
&+ \int_0^{t_i-h} |\mathbb{P}(t_i - s, X(s)) - \mathbb{P}(t_i - h - s, X(s))|(B(X(s))|ds \\
&+ \int_{t_i-h}^{t_i} |\mathbb{P}(t - i - s, X(s))(\eta,\xi)|(B(X(s))|ds \\
&+ \int_0^{t_i-h} |[\mathbb{P}(t - i - h - s, X(s)) - \mathbb{P}(t_i - s, X(s))]\phi_q(s)|ds \\
&+ \int_0^{t_i-h} |[\mathbb{P}(t_i - h - s, X(s))]\phi_q(s)|ds \\
&+ \sum_{k=1}^{l-1}[\mathbb{P}(t_i - h - t_k, X(s)) - \mathbb{P}(t_i - t_k s, X(s))]J_k(X(t_k^-)).
\end{aligned}$$

To show equicontinuity at the right limit $t = t_k^+$, fix $\delta_2 > 0$ such that $[t_k : k \neq i] \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. For $0 < h < \delta_2$, we have

$$\begin{aligned}
|N(x)(t_i + h) - N(x)(t_i)| &\leq |[\mathbb{P}(t_i + h, X(s)) - \mathbb{P}(t_i, X(s))||X_0||_{\eta,\xi} + \textstyle\int_0^{t_i} |\mathbb{P}(t_i + h - s, X(s)) - \\
&\mathbb{P}(t_i - s, X(s))|(B(X(s))|ds + \textstyle\int_{t_i}^{t_i+h} |\mathbb{P}(t_i - h, X(s))|(B(X(s))|ds + \int_0^{t_i} |\mathbb{P}(t_i + h - s, X(s)) \\
&-\mathbb{P}(t_i - s, X(s))|\phi_q(s)ds| + \textstyle\int_{t_i}^{t_i+h} |\mathbb{P}(t_i - h, X(s))(\phi_q(s)|ds + \sum_{0<t_k<t_i}[\mathbb{P}(t_i - h - t_k) - \\
&\mathbb{P}(t_i - t_k, X(s))] + \textstyle\sum_{t_i<t_k<t_{i+1}} |\mathbb{P}(t_i - h - t_k, X(s))J_k(X(t_k^+))|. \qquad (4.9)
\end{aligned}$$

The right hand tends to zero as $h \to 0$. Set

$$U = [X \in P_{wac}(\tilde{\mathcal{A}}) : \|x\|_{P_{wac}} \leq \max[X_0, M_{k-1} : k = 1, ..., m+1].$$

As a consequence of steps 1,2 and 3, we can conclude that

$$N : \overline{U} \to P_{wac}(\tilde{\mathcal{A}})$$

is compact. From the choice of U there is no $y \in \delta U$ such that $x = \lambda N x$ for any $\lambda \in \mathbb{C}$ such that $Re\lambda \in (0,1)$ and $Im\lambda \in (0,1)$ As a result , we deduce that N has a fixed point $x \in U$ which is a solution to problem (2.1 - 1.2). ■

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