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In honour of Prof. Ekhaguere at 70

Non-commutative generalizations of some classical fixed point and selection results

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Abstract. This work concerns the establishment of the non-commutative generalization of the classical Leray Schauder fixed point theorem, Arsela-Ascoli theorem and Micheal selection theorem in a locally convex space. This results will be employed subsequently in establishing the existence of solutions of some classes of impulsive quantum stochastic differential inclusions.

Keywords: fixed point, multivalued functions, locally convex space, upper and lower semicontinuous operators.

1. Introduction

Solutions have been proffered to series of problems arising in quantum calculus, in particular quantum stochastic differential equations in the framework of Hudson and Parthasarathy formulation. Problems such as existence of solutions; both analytic and numerical approaches [Ayoola,GOS, Ogundiran, Bishop etc]. We observed that in an attempt to establish solution to problems arising in quantum stochastic differential inclusion, fixed point and selection theorems are employed.

In quantum calculus, the setting is non-commutative and processes are operator valued.

In this paper we establish the non-commutative generalization of Leray-Schauder fixed point theorem, Arzela-Ascoli theorem and Micheal selection theorem.

The rest of the paper is organised as follows. Section 2 contains fundamental structures and definitions that we use in the sequel. In section 3, we establish the non-commutative analoque of Leray-Schauder fixed point theorem. In section 4, we establish the non-commutative analoque of Arsela-Ascoli fixed point theorem. In section 5 we established some auxiliary results that are use in establishing Michael selection theorem; while section 6 is on non-commutative analogue of Michael selection theorem.

2. Fundamental structures and definitions

In this section we state some fundamental structures and definition that will be use in the sequel.

(i) Let $\mathbb{D}$ be some pre-Hilbert space whose completion is R and γ is a fixed Hilbert space. $L^2_\gamma(\mathbb{R}_+), t \in \mathbb{R}$ is a space of square integrable γ - values maps on $\mathbb{R}_+$. The inner product of the Hilbert space $R \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)$ will be denoted by $\langle ., . \rangle$ and $||.||$ the norm induced by $\langle ., . \rangle$. Let $\mathbb{E}, t > 0$ be linear space generated by the exponential vector in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$,

$$\mathcal{A} \equiv L^+_\omega(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)), t > 0$$

where $\underline{\otimes}$ denotes the algebraic tensor product and $\mathbb{I}_t$ denotes the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma([0, t))), t > 0$ For every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that $\eta = c\underline{\otimes}e(\alpha), \quad \xi = d\underline{\otimes}e(\beta), \quad \alpha, \beta \in$

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$L^2_{\gamma,loc}(\mathbb{R}_+)$ define

$$\|x\|_{\eta,\xi} = |\langle \eta, x\xi \rangle|, \quad x \in \mathcal{A}$$

then the family of seminorms

$$\{\| \cdot \|_{\eta,\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_ω, called the weak topology on $\mathcal{A}$. The completion of the locally convex spaces $(\mathbf{A}, \tau_\omega)$ is denoted by $\tilde{\mathcal{A}}$.

(ii) Given a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, a single valued map $f : \mathbb{R}^m \to \mathbb{R}^n$ is a selection if $f(x) \in F(x) \quad \forall x \in \mathbb{R}$.

(iii) Upper and Lower Semi continuous Multivalued Maps: Let $\mathcal{N} \subseteq \tilde{\mathcal{A}}$ and $I \subseteq \mathbb{R}_+$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, (t, x), (t_0, x_0) \in I \times \mathcal{N}$ and real numbers $\epsilon, \delta_{\eta,\xi} > 0$, we define the map $d_{\eta,\xi} : [I \times \mathcal{N}] \to \mathbb{R}_+$ by

$$d_{\eta,\xi}((t, x), (t_0, x_0)) = \max\{|t - t_0|, \|x - x_0\|_{\eta,\xi}\}.$$

For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

$$\tilde{\mathcal{A}}(\eta, \xi) = \{x_{\eta,\xi} = \langle \eta, x\xi \rangle, x \in \tilde{\mathcal{A}}\}$$

$$B_{\eta,\xi,\epsilon}(0) = \{x_{\eta,\xi} \in \tilde{\mathcal{A}}(\eta, \xi) : |x_{\eta,\xi}| < \epsilon\}$$

$$B_{\delta_{\eta,\xi}}(t_0, x_0) = \{(t, x) \in I \times \mathcal{N} : d_{\eta,\xi}((t, x), (t_0, x_0)) < \delta_{\eta,\xi}\}.$$

(iv) A map $\phi : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be upper semi continuous at a point $(t_0, x_0) \in I \times \mathcal{N}$, if for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exists $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0, x_0), \epsilon) > 0$ such that

$$\phi(t, x)(\eta, \xi) \subset \phi(t_0, x_0)(\eta, \xi) + B_{\eta,\xi,\epsilon}(0)$$

on $B_{\delta_{\eta,\xi}}(t_0, x_0)$. The map ϕ is said to be upper semi continuous if it is upper semi continuous at every point $(t, x) \in I \times \mathcal{N}$. Furthermore, for a sesquilinear formed valued map we define

$$B_{\mathbb{P},\epsilon}(0) = \{\varphi(t, x)(\eta, \xi) \in \mathbb{P}(t, x)(\eta, \xi) : |\varphi(t, x)(\eta, \xi)| < \epsilon\}$$

(v) A sesquilinear form valued multifunction $\mathbb{P} : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be upper semi continuous at a point $(t_0, x_0) \in I \times \mathcal{N}$ if for every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exist $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0, x_0), \epsilon) > 0$ such that

$$\mathbb{P}(t, x)(\eta, \xi) \subset \mathbb{P}(t_0, x_0)(\eta, \xi) + B_{\mathbb{P},\epsilon}(0)$$

on $B_{\delta_{\eta,\xi}}(t_0, x_0)$. The map $\mathbb{P}$ is said to be upper semi continuous if it is upper semi continuous at every point $(t, x) \in I \times \mathcal{N}$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $\Phi : I \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ be a closed multivalued map. For each pair $(t, x), (t^{'}, x^{'}) \in I \times \tilde{\mathcal{A}}$ we define

$$d_{\eta,\xi}((t, x), (t^{'}, x^{'})) = \max\{|t - t^{'}|, ||x - x^{'}||_{\eta\xi}\}$$

$$B_{\eta,\xi,\delta}(t_0, x_o) = \{(t, x) \in I \times \tilde{\mathcal{A}} : d_{\eta,\xi}((t_0, x_0), (t, x))(\eta, \xi) < \delta\} \quad \text{and}$$

$$B_{\eta\xi,\epsilon}(\Phi(t,x) = \{y \in \tilde{\mathcal{A}} : \inf_{k\in\Phi(t,x)} ||y-k||_{\eta,\xi} < \epsilon\}.$$

(vi) A map $\Phi : I\times\mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be lower semi continuous at a point $(t_0, x_0) \in I\times \mathcal{N}$, with respect to the seminorm $||.||_{\eta\xi}$ if for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exists $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0, x_0), \epsilon) > 0$ such that for each $y_o \in \Phi(t_o, x_o)$, $\inf_{y\in\Phi(t,x)} ||y-y_o||_{\eta\xi} < \epsilon$, $\forall \quad y \in \mathcal{N}$, almost all $t \in I$ and $d_{\eta,\xi}((t,x),(t',x')) < \delta_{\eta,\xi}$. If Φ is lower semi continuous at every point $(t_0, x_0) \in I\times\mathcal{N}$ with respect to the seminorm $||.||_{\eta\xi}$, then it will be said to be lower semi continuous on $I\times\mathcal{N}$.

(vii) A sesquilinear form valued multifunction $\mathbb{P} : I\times\mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be lower semi continuous at a point $(t_0, x_0) \in I\times\mathcal{N}$, with respect to the seminorm $||.||_{\eta\xi}$ if for every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exist $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0, x_0), \epsilon) > 0$ such that for each $y_{\eta\xi,0} \in \mathbb{P}(t_0, x_0)(\eta, \xi)$ $\inf_{y_{\eta\xi}\in\mathbb{P}(t,x)} |y_{\eta\xi,0} - y_{\eta\xi}| < \epsilon$, $\forall \quad y \in \mathcal{N}$, almost all $t \in I$ and $d_{\eta,\xi}((t,x),(t_0,x_0)) < \delta_{\eta,\xi}$

(viii) Let $M \subset \tilde{\mathcal{A}}$, for an arbitrary pair $\eta, \xi \in \mathbb{D}\otimes\mathbb{E}$, a map $\varphi : M \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ is said to be locally Lipschitizian if for every $x \in M$ there exist a neighbourhood $V(x)$ such that $\forall \quad x_1, x_2 \in V(x)$, there exists $L_{\eta,\xi} > 0$, such that

$$|\varphi(x_1)(\eta,\xi) - \varphi(x_2)(\eta,\xi)| < L_{\eta,\xi}\|x_1 - x_2\|_{\eta,\xi}$$

(ix) Let $M \subset \tilde{\mathcal{A}}$, a family of open subsets $\{\Omega_i\}_{i\in I}$ of M, such that $M = \cup_{i\in I}\Omega_i$ is called an open covering of M. Let $\{\Omega_i\}_{i\in I}$ and $\{\omega_j\}_{j\in J}$ be two coverings of M. $\{\Omega_i\}_{i\in I}$ is a refinement of $\{\omega_j\}_{j\in J}$ if for every $i \in I$, there exist $j \in J$, $J \subset I$ such that $\Omega_i \subset \omega_j$. Let $\{\Omega_i\}_{i\in I}$ be a covering, if $J \subset I$ and $\{\Omega_j\}_{j\in J}$ is again a covering, then $\{\Omega_j\}_{j\in J}$ is a sub covering.

(x) A covering $\{\Omega_i\}_{i\in I}$ of M is called locally finite if for every $x \in M$, there exists a neighbourhood V of x such that $\Omega_i \cap V \neq \emptyset$ only for a finite number of indexes.

(xi) Let $\varphi : M \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$, the closure of this set is called a support of φ, $(supp(\varphi))$. A family $\{\varphi_i\}$ is called a locally Lipschitzian partition of unity if for all $i \in I$
(a) φ_i is locally Lipschitzian and non negative
(b) the support of φ_i are closed locally finite covering of M;
(c) for each $x \in M$, $\sum_{i=I}\varphi_i(x) = 1$.
We say that a partition of unity $\{\varphi_i\}_{i\in I}$ is subordinated to a covering $\{\Omega_i\}_{i\in I}$ if $\forall \quad i \in I, supp(\varphi_i) \in \Omega_i$.

(xii) Let $v_0, v_1, ..., v_n$ be an affine independent set of $n+1$ points in $\tilde{\mathcal{A}}$. The convex hull $\{x \in \tilde{\mathcal{A}} : x = \sum_{i=0}^n \lambda_i v_i, \quad 0 \le Re\lambda \le 1$ and $0 \le Im\lambda \le 1, \quad \sum_{i=0}^n |\lambda_i| = 1\}$ is called closed n-simplex and is denoted by $v_0, v_1, ..., v_n$. The points $v_0, v_1, ..., v_n$. are called the vertices of the simplex. For $0 \le k \le n$, $0 \le i_0 \le i_1 \le .... \le i_k \le n$, the k-simplex $v_{i_0}, v_{i_1}, ..., v_{i_k}$ is a subset of the n-simplex $v_0, v_1, ..., v_n$; is called the k-dimensional face (or simply k face of $v_0, v_1, ..., v_n$. in addition if $y = \sum_{i=0}^\infty \lambda_i v_i$, we let $\chi(y) = [i : \lambda_i > 0]$.

(xiii) Let K be a non empty set, and $\phi : K \to 2^K$ a multifunction, an element $x \in K$ is said to be a fixed point of ϕ if $x \in \varphi(x)$.

3. Non-commutative analogue of Leray-Schauder fixed point theorem

THEOREM 3.1 [23] *Let $K \neq \emptyset$, $K \subset \tilde{\mathcal{A}}$ be compact and satisfies the following*
(i) $\Phi : K \to 2^K$ be upper semi continuous and compact map
(ii) The set $\{X \in K : X = \lambda\Phi(X)$ for some $\lambda \in \mathbb{C}$ such that $0 \le Re\lambda \le 1$ and $0 \le Im\lambda \le 1\}$ is bounded.
(iii) $\Phi(X(t))$ is a non-empty closed and convex subset of K for each $X(t) \in K$,
then there exist $y(t) \in K$ such that $y(t) \in \Phi(y(t))$.

THEOREM 3.2 *(Leray-Schauder's Principle) Suppose that given any arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ the map $N : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ satisfies the following conditions :*
(i) $K \subset \tilde{\mathcal{A}}$ and U be open K.

(ii) $S : K \subset \tilde{\mathcal{A}} \to K$ *is upper semi continuous and compact.*
(iii) there exists an $r > 0$ *such that if* $x = \lambda N x$ *with* $\lambda \in \mathbb{C}$,

$$\|x\|_{\eta,\xi} \leq r$$

Then the equation $x = N(x)$ *has a fixed point.*

THEOREM 3.3 *(Leray-Schauder's Theorem) Let* U *and* $\overline{U}$ *denote respectively the open and closed subsets of a convex set* K *of* $\tilde{\mathcal{A}}$ *such that* $0 \in U$ *and let* $N : \overline{U} \to K$ *be a compact and semi continuous map. Then either*
(i) The equation $x = Nx$ *has a solution in* $\overline{U}$ *or*
(ii) There exists a point $u \in \delta U$ *such that* $u = \lambda N u$ *for some* $\lambda \in \mathbb{C}$ *such that* $Re\lambda \in (0,1)$ *and* $Im\lambda \in (0,1)$, *where* δU *is a boundary of* U.

Proof. Let assume $N \backslash \delta U$ as a fixed point, then we define a map (from Leray- Schauder's Principle)

$$S(x) = \begin{cases} N(x), \text{if} \quad \|N(x)\|_{\eta,\xi} \quad \leq 2r \\ \dfrac{2rN(x)}{||N(x)||_{\eta\xi}}, \quad \text{if} \quad ||N(x)||_{\eta,\xi} > 2r \end{cases} .$$

We claim that $S : K \to K$ is compact on $K = \{x : \|x\|_{\eta,\xi} \leq 2r\}$. To establish compactness, let $\{x_n\}$ be a sequence in K, we consider (a) a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\|N(y_n)\|_{\eta\xi} \leq 2r$ for all n (b) a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\|N(y_n)\|_{\eta\xi} > 2r$ for all n.

In case (a) we have $\{z_n\}$ of $\{y_n\}$ such that $S(z_n) = N(z_n) \to y$ as $n \to \infty$. In case (b), let

$$\{z_n\} = \frac{1}{\|N(z_n)\|_{\eta,\xi}} \to \infty \quad \text{and} \quad N(z_n) \to y \quad \text{as} \quad n \to \infty.$$

Using Schaefer's theorem (Theorem 3.1) we have $x \in K$ for which $S(x) = x$ if $\|N(y_n)\|_{\eta\xi} \leq 2r$ then, $N(x) = S(x) = x$. The other case $\|N(x)\|_{\eta\xi} > 2r$ is impossible, for otherwise

$$S(x) = \lambda N(x) = x, \quad \lambda = \frac{2r}{||N(x)||_{\eta\xi}}, \quad Re\lambda, Im\lambda \in (0,1).$$

Let assume $N \backslash \delta U$ is fixed point free. From condition (ii) $u \in \delta U$. Let $F : \overline{U} \to K$ be a constant map $u \to 0$, consider the compact homotopy $H_\lambda : \overline{U} \to K$ given by $H(u,\lambda) = \lambda F(u)$ joining N to F, if $H_\lambda : \overline{U} \to K$ is fixed point free on δU then by Leray - Schauder's (Theorem 3.2) F as a fixed point. If the homotopy is not fixed point free on δU then there is an $x \in \delta U$ with $x = \lambda N(x)$, Since $Re\lambda \neq 0$ because $0 \notin \delta U$ and $\lambda \neq 1$ because $N \backslash \delta U$ has been assumed to be fixed point free, then condition (ii) is satisfied. ■

4. Non-commutative analogue of Arsela-Ascoli theorem

THEOREM 4.1 *Let* $X : I \to \tilde{\mathcal{A}}$ *be a stochastic process that satisfy the following conditions : (i) For any arbitrary pair* $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, *let* $K \subset \tilde{\mathcal{A}}$ *such that* $F : K \to K$ *is a compact map. (ii)* $\|f(x)\|_{\eta\xi} \leq m$ *for each* $x \in X, f \in F$ *and* $m < \infty$. *(iii) For every* $\epsilon > 0$ *(depending on* η, ξ*) there exist* $\delta_{\eta\xi}$ *such that for every* $x, y \in X$,

$$d(x,y)(\eta,\xi) < \delta_{\eta\xi}.$$

Then

$$\langle \eta, (f(x) - f(y))\xi \rangle < \epsilon \quad \forall \quad f \in F, \quad x, y \in X.$$

Proof. Given that $\mathcal{F}$ is bounded and equicontinuous on K. We claim that we can define a sequence $\{f_{\eta\xi,n}\}$ in $\mathcal{F}$. Let $K_{\mathbb{Q}}$ denote the subsets of K such that

$$K_{\mathbb{Q}} = \{x_{\eta\xi 1}, x_{\eta\xi,2}, x_{\eta\xi 3}, ..., \}$$

The sequence

$$\{f_{\eta\xi,1}(x_1), f_{\eta\xi,2}(x_2), f_{\eta\xi,3}(x_3), \cdot\}$$

is bounded and therefore contains a convergence subsequence. Let

$$\{f_{\eta\xi,1,1}(x_1), f_{\eta\xi,1,2}(x_1), f_{\eta\xi,1,3}(x_1), \cdots\}$$

be the subsequence that converges, next the sequence

$$\{f_{\eta\xi,1,1}(x_2), f_{\eta\xi,1,2}(x_2), f_{\eta\xi,1,3}(x_2), \cdots\}$$

is also bounded, it has a convergent subsequence

$$\{f_{\eta\xi,2,1}(x_2), f_{\eta\xi,2,2}(x_2), f_{\eta\xi,2,3}(x_2), ...\}$$

the sequence

$$\{f_{\eta\xi,2,1}(x_3), f_{\eta\xi,2,2}(x_3), f_{\eta\xi,2,3}(x_3), \cdots\}$$

is bounded and it contain convergent subsequence, continuing this process gives rise to an array

$$\begin{array}{cccc} f_{1,1} & f_{1,2} & f_{1,3} & .... \\ f_{2,1} & f_{2,2} & f_{2,3} & .... \\ f_{3,1} & f_{3,2} & f_{3,3} & .... \\ . & . & . & .... \\ . & . & . & .... \\ . & . & . & .... \end{array}$$

The first row is a sequence maps that converges at x_1. The second row is a subsequence of the first row and it converges at x_1 and x_2. The third row is a subsequence of the second row and it converges at x_1, x_2, x_3 and so on. Considering the sequence down the diagonal, it is a subsequence of the original sequence $\{f_{\eta\xi,n}\}$ and it converges at $x \in X$. By the equicontinuity property of $\{f_{\eta\xi,n}\}$, given any $x, y \in X$ and any $\epsilon > 0$ consider $\delta_{\eta\xi} > 0$ there exists $|x - y|_{\eta\xi} < \delta_{\eta\xi} \quad \{f_{\eta\xi,n}(x)\}$ converges. Let $n, m > N \Rightarrow |f_{\eta\xi,n}(x) - |f_{\eta\xi,m}(x)| < \epsilon$ hence

$$|f_{\eta\xi}(x) - |f_{\eta\xi,n}(x)| + |f_{\eta\xi,n}(x) - |f_{\eta\xi,m}(x)| + |f_{\eta\xi,m}(x) - |f_{\eta\xi,m}(x)| < 3\epsilon,$$

which implies

$$d(x, y)(\eta, \xi) < \delta_{\eta\xi}$$

Then

$$\langle \eta, (f(x) - f(y))\xi \rangle < \epsilon \quad \forall \quad f \in F, \quad x, y \in X.$$

■

5. Lemmas

The following lemmas would be employed in the the prove of Micheal selection theorem.

Lemma 5.1 *Assume that $\mathbb{P} : (I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ is a non-empty compact valued multivalued map such that*

(a) $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ *is measurable*
(b) $x \to \mathbb{P}(t,x)(\eta,\xi)$ *is lower semi continuous for* $t \in I$ *and for each* $r > 0$*, there exists a map* $h_r \in L'_{loc}(I, \mathbb{R}_+)$ *such that*
$|\mathbb{P}(t,x)(\eta,\xi)| = \inf\{||\nu||_{\eta,\xi} : \nu_{\eta\xi} \in \mathbb{P}(t,x)(\eta,\xi)\} \leq h_r(t)$ *for a.e* $t \in I$ *and* $x \in \tilde{\mathcal{A}}$ *such that* $||x||_{\eta,\xi} \leq r$*. Then* $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ *is lower semi continuous.*

Proof. For arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. For any point $(t_0,x_0) \in I \times \tilde{\mathcal{A}}$ with respect to the seminorm $||.||_{\eta\xi}$, given $r > 0$, there exists $\delta_{\eta\xi} > 0$, such that for each $y_{\eta\xi,0} \in \mathbb{P}(t_0,x_0)(\eta,\xi)$

$$\inf_{y_{\eta\xi,\nu} \in \mathbb{P}(t,x)(\eta,\xi)} |y_{\eta\xi,0} - y_{\eta\xi,\nu}| < h_r(t), \quad \forall \quad x \in \tilde{\mathcal{A}}, \text{almost all} \quad t \in I \quad \text{and}$$

$$d_{\eta\xi}((t,x),(t_0,x_0) < \delta_{\eta\xi}.$$

Let

$$d_{\eta\xi} = \min\{\delta_{\eta\xi}(t_0,x_0)\}.$$

Then for any $r > 0$, there exist $\delta_{\eta\xi,\nu}$ such that for each $y_{\eta\xi,0} \in \mathbb{P}(t_0,x_0)(\eta,\xi)$

$$\inf_{y_{\eta\xi} \in \mathbb{P}(t,x)(\eta,\xi)} |y_{\eta\xi,0} - y_{\eta,\xi}| < \delta_{\eta\xi,\nu} \quad \forall \quad x \in \tilde{\mathcal{A}}, \quad \text{almost all} \quad t \in I \quad \text{and}$$

$$d_{\eta\xi}((t,x)(t_0,x_0) < \delta_{\eta\xi,\nu},$$

where $\delta_{\eta\xi} = \min\{\delta_{\eta\xi,\nu}\}$. This implies that the map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semi continuous at (t_0,x_0). Since $(t_0,x_0) \in I \times \tilde{\mathcal{A}}$ is arbitrary , then $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semicontinuous. ■

Lemma 5.2 *Assume that $F_1 : (I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ is lower semi continuous and $F_2(I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ has an open graph corresponding to each pair $\eta,\xi \in \mathbb{D}\otimes\mathbb{E}$ and*

$$F_1(t,x)(\eta,\xi) \cap F_2(t,x)(\eta,\xi) \neq \emptyset \quad \text{for each} \quad (t,x) \in (I \times \tilde{\mathcal{A}}).$$

Then the sesquilinear form valued map $F_1 \cap F_2 : (I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ is lower semi continuous.

Proof. Fix (t^*,x^*) in $(I \times \tilde{\mathcal{A}})$, let $y(t^*,x^*)(\eta,\xi) \in F_1 \cap F_2(t^*,x^*)(\eta,\xi)$ and $\omega > 0$. For some $\sigma > 0, |y(t^*,x^*)(\eta,\xi) - F_1(t^*,x^*)(\eta,\xi)| = F_1(t^*,x^*) - \sigma$. There exists δ_1 such that to any $(t,x) \in I \times \tilde{\mathcal{A}}$ with $d_{\eta\xi}((t,x)(t^*,x^*)) < \delta_1$, we associate $y(t,x)(\eta,\xi)$ in $F_2(t,x)(\eta,\xi)$ so that

$$|y(t,x)(\eta,\xi) - y(t^*,x^*)(\eta,\xi)| < \min\{\omega, \frac{\sigma}{3}\}$$

and δ_2 such that

$$d_{\eta\xi}((t,x)(t^*,x^*) < \delta_2.$$

This implies that

$$F_1(t,x) > F_1(t^*,x^*) - \frac{\sigma}{3}$$

and δ_2 then we have

$$d_{\eta\xi}((t,x)(t^*,x^*) < \delta_3.$$

which implies

$$|F_1(t^*,x^*)(\eta,\xi) - F_1(t,x)(\eta,\xi)| < \frac{\sigma}{3}.$$

Then we have

$$d_{\eta\xi}((t,x)(t^*,x^*) < \min\{\delta_1,\delta_2,\delta_3\}$$

$$|y(t,x)(\eta,\xi) - F_1(t,x)(\eta\xi)| \leq$$

$$|y(t,x)(\eta,\xi) - y(t^*,x^*)(\eta\xi)| + |F_1(t^*,x^*)(\eta,\xi) - F_1(t,x)(\eta,\xi)| < \frac{\sigma}{3} + F_1(t,x)(\eta\xi) - \delta + \frac{\sigma}{3}$$

which means

$$y(t,x)(\eta,\xi) \in F_1 \cap F_2(t,x)(\eta,\xi)$$

and

$$|y(t^*,x^*)(\eta\xi) - y(t,x)(\eta\xi)| < \omega.$$

■

LEMMA 5.3 *Assume that the map* $\mathbb{P} : (I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ *satisfies the following conditions :*

(i) $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ *is convex valued.*
(ii) $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ *is lower semi continuous on* $(I \times \tilde{\mathcal{A}})$.

Then for each $\epsilon > 0$ *and each pair* $\eta,\xi \in \mathbb{D}\otimes\mathbb{E}$ *there exists a continuous sesquilinear form* $F_\epsilon : (I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ *such that for every*

$$(t,x) \in (I \times \tilde{\mathcal{A}}), \quad f_{\eta\xi,\epsilon}(t,x) \in V(\mathbb{P}(t,x)(\eta,\xi),\epsilon),$$

where $V(\mathbb{P}(t,x)(\eta,\xi),\epsilon)$ *is a neighbourhood of* $\mathbb{P}(t,x)(\eta,\xi)$ *and* $f_{\eta\xi,\epsilon}(t,x) = \langle \eta, f_\epsilon(t,x)\xi\rangle$.

Proof. Since $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semi continuous, we associate to each

$$(t,x) \in (I \times \tilde{\mathcal{A}}), \quad y_{\eta\xi} \in \mathbb{P}(t,x)(\eta,\xi)$$

an open neighbourhood $U(t,x)$ of (t,x) such that $\mathbb{P}(t,x)(\eta,\xi) = \bigcup U(t,x)$. Let $\mathbb{P}(t^{'},x^{'})(\eta,\xi) = \bigcup \omega(t,x)$, where $\omega(t,x) \subset U(t,x)$ otherwise, suppose there is $(t,x) \in \omega \backslash U$. There exists $\alpha \supset \omega$ such that $(t,x) \notin \alpha$, this implies that $(t,x) \notin \alpha$ since ω is locally finite, there exists a neighbourhood $V(\mathbb{P}(t^{'},x^{'})(\eta,\xi),\epsilon)$ of $\mathbb{P}(t',x')(\eta,\xi)$ such that

$$V(\mathbb{P}(t^{'},x^{'})(\eta,\xi),\epsilon) \cap B(y_{\eta\xi},\epsilon) \neq \emptyset, \quad \text{for all} \quad (t',x') \in U(t,x),$$

where $y_{\eta\xi} = \langle \eta, y\xi\rangle, y \in \tilde{\mathcal{A}}$. Hence the space $(I \times \tilde{\mathcal{A}})$ is paracompact. Then, there exists a locally finite refinement $\{U'(t,x), (t,x) \in (I \times \tilde{\mathcal{A}})\}$. Moreover, to each locally finite covering, we associate a locally Lipschitz partition of unity $\{\phi_{t,x}\}, (t,x) \in (I \times \tilde{\mathcal{A}})$. Hence we define

$$f_\epsilon(s,u) = \sum_{(t,x)\in(I\times\tilde{\mathcal{A}})} \phi_{t,x}(s,u)y.$$

Then f_ϵ is continuous as a locally finite sum of continuous maps. Also if $\phi(t,x) > 0 \quad \forall (t,x) \in U' \subset U(t,x)$, then $y_{\eta\xi} \in V(\mathbb{P}(t,x)(\eta,\xi);\epsilon)$. This implies that

$$\langle \eta, f_\epsilon(s,u)\xi\rangle = \sum_{(t,x)\in(I\times\tilde{\mathcal{A}})} \phi_{t,x}(s,u)\langle \eta, y\xi\rangle \in V(\mathbb{P}(t,x)(\eta,\xi);\epsilon).$$

■

6. Non-commutative analogue of Michael's selection theorem

Theorem 6.1 *Assume that the map $\mathbb{P} : (I\times\tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ satisfies the following conditions : (i) for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \mathbb{P}(t,x)(\eta,\xi)$ is closed and convex in $\mathbb{C}$. (ii) The map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is lower semi continuous on $(I\times\tilde{\mathcal{A}})$, then there exists a continuous map $f : (I\times\tilde{\mathcal{A}}) \to sesq(\mathbb{D}\otimes\mathbb{E})^2$ which is a selection of $\mathbb{P}(t,x)(\eta,\xi)$.*

Proof. We employ the principle of mathematical induction as follows. A sequence of continuous sesquilinear valued map is defined as follows

$$f_n : (I \times \tilde{\mathcal{A}}) \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$$

satisfying the following assertions:
(i) For all $(t,x) \in (I \times \tilde{\mathcal{A}})$, arbitrary pair $\eta,\xi \in \mathbb{D}\otimes\mathbb{E}$,

$$d(f_{n,\eta\xi}(t,x), \mathbb{P}(t,x)(\eta,\xi)) < \frac{1}{2^n} \quad \forall \quad n \in \mathbb{N},$$

(ii)

$$|f_{n,\eta\xi}(t,x) - f_{\eta\xi}(t,x)| < \frac{1}{2^{n-2}} \quad \text{for each} \quad n = 2,3,...$$

(a) Case $n = 1$. The conclusion of the theorem follows by putting $\epsilon = \dfrac{1}{2}$ in lemma 3. This implies $f_{\eta\xi,\frac{1}{2}}(t,x) \in V(\mathbb{P}(t,x)(\eta,\xi), \frac{1}{2})$.
(b) Assume that we have defined the maps $f_1, ..., f_n : (I \times \tilde{\mathcal{A}}) \to sesq(\mathbb{D}\otimes\mathbb{E})^2$ such that (i) - (ii) holds.

We shall construct the following map $f_{n+1} : (I \times \tilde{\mathcal{A}}) \to sesq(\mathbb{D}\otimes\mathbb{E})^2$ satisfying (i) and (ii). Consider the multivalued map

$$\mathbb{P}_{n+1}(t,x)(\eta,\xi) = \mathbb{P}(t,x)(\eta,\xi) \cap B(f_{n,\eta\xi}(t,x), \frac{1}{2^n})$$

from condition (ii), for each $(t,x) \in (I \times \tilde{\mathcal{A}})$ we have

$$\mathbb{P}_{n+1}(t,x)(\eta,\xi) \neq \emptyset \quad \forall \quad (t,x) \in (I \times \tilde{\mathcal{A}}).$$

By lemma 2 the map $(t,x) \to \mathbb{P}_{n+1}(t,x)(\eta,\xi)$ is lower semi continuous. From lemma 3 applied to $\mathbb{P}_{n+1}(t,x)(\eta,\xi)$, there exists a continuous map

$$f_{n+1} : (I \times \tilde{\mathcal{A}}) \to sesq(\mathbb{D} \otimes \mathbb{E})^2$$

such that

$$f_{n+1}(t,x)(\eta,\xi) \in V^0(\mathbb{P}(t,x)(\eta,\xi), \frac{1}{2^{n+1}}),$$

where

$$V^0(\mathbb{P}(t,x)(\eta,\xi), \frac{1}{2^{n+1}}) \quad \text{is a neighbourhood} \quad \mathbb{P}(t,x)(\eta,\xi)$$

and

$$d(f_{n,\eta\xi}(t,x), \mathbb{P}(t,x)(\eta,\xi)) < \frac{1}{2^n}.$$

Hence it follows that

$$d(f_{n+1,\eta\xi}(t,x), \mathbb{P}(t,x)(\eta,\xi)) < \frac{1}{2^{n+1}}$$

and

$$f_{n+1} : (t,x)(\eta,\xi) \in V^0(\mathbb{P}(t,x)(\eta,\xi), \frac{1}{2^{n+1}})$$

which completes the induction.

From (ii) we obtain that the sequence $\{f_{n,\eta,\xi}\}_{n\in\mathbb{N}}$ is a uniform Cauchy sequence in $\mathbb{C}$ which converges to a continuous function $f_{\eta\xi} : (I \times \tilde{\mathcal{A}}) \to \mathbb{C}$. From (i) and the fact that $\mathbb{P}(t,x)(\eta,\xi)$ is closed for each $(t,x) \in (I \times \tilde{\mathcal{A}})$, we have $f_{\eta\xi}(t,x) \in \mathbb{P}(t,x)(\eta,\xi) \quad \forall \quad (t,x) \in (I \times \tilde{\mathcal{A}})$ as $\mathbb{P}(t,x)(\eta,\xi) \subseteq 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ then there exist a map $f : (I \times \tilde{\mathcal{A}}) \to sesq(\mathbb{D} \otimes \mathbb{E})^2$ such that

$$f_{\eta\xi}(t,x) = \langle \eta, f(t,x)\xi \rangle.$$

■

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