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In honour of Prof. Ekhaguere at 70

On some properties of locally convex partial *-algebraic modules

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Abstract. In this paper, we begin the systematic study of locally convex partial *-algebraic modules. These are generalizations of inner product modules over C*-algebras. We develop some of their properties by extending a number of results in the theory of Hilbert C*-modules to the present partial *-algebraic setting.

Keywords: Hilbert C*-modules, partial *-algebras, locally convex partial *-algebraic modules, module maps, adjointable maps.

1. Introduction

The notion of a *locally convex partial *-algebraic module* was introduced by Ekhaguere [2] in his study of the representation of completely positive maps between partial *-algebras. Locally convex partial *-algebraic modules are generalizations of inner product modules over B*-algebras [9]. These inner product modules [11], now generally known as pre-Hilbert C*-modules, provide a natural generalization of the Hilbert space in which the complex field of scalars is replaced by a C*-algebra. Although the theory of Hilbert C*-modules, in the case of commutative unital C*-algebras, can be traced back to the work of Kaplansky [4], where he proved that derivations of type I AW*-algebras are inner, it was Paschke [9] who gave the general framework. Apart from being interesting on its own, the theory of Hilbert C*-modules has had several areas of applications. For example, the work of Kasparov on KK-theory [5,6], the work of Rieffel on induced representations and Morita equivalence [11,12], and the work of Woronowicz on C*-algebraic quantum group theory [14], etc. For a more detailed bibliography of the theory of Hilbert C*-modules, see [3]. In this paper, we develop some of the properties of locally convex partial *-algebraic modules by extending a number of results from the theory of Hilbert C*-modules [9,10,8,7] to a partial *-algebraic setting.

The paper is organized as follows. In section 2, we outline some of the fundamental notions used in the sequel. See [1,2,13], for more details of these notions. Section 3 outlines the basic notions of a locally convex partial *-algebraic module as introduced in [2]. We develop some basic properties of locally convex partial *-algebraic modules in section 4. In section 5, we study some properties of certain classes of linear maps acting on locally convex partial *-algebraic modules. Most importantly, we develop some properties of a class of adjointable maps which, in the case of Hilbert C*-modules, are operators analogous to the finite-rank operators on a Hilbert space. Finally, in section 6, we introduce the notion of a locally convex partial *-algebraic bimodule. Since we are working with right locally convex partial *-algebraic modules, this notion is natural.

2. Fundamental notions

A partial *-algebra is simply a complex involutive linear space $\mathcal{A}$ with a multiplication that is defined only for certain pairs of compatible elements determined by a relation on $\mathcal{A}$. More precisely, there is the following definition.

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DEFINITION 1 *A partial *-algebra is a quadruple $(\mathcal{A}, \Gamma, \diamond, *)$ comprising:*

(a) *a linear space $\mathcal{A}$ over $\mathbb{C}$;*
(b) *a relation $\Gamma \subseteq \mathcal{A} \times \mathcal{A}$;*
(c) *a partial multiplication, $\diamond$, such that*
  (c1) $(x, y) \in \Gamma$ *if and only if* $x \diamond y \in \mathcal{A}$;
  (c2) $(x, y), (x, z) \in \Gamma$ *implies* $(x, \lambda y + \mu z) \in \Gamma$ *and then* $x \diamond (\lambda y + \mu z) = \lambda(x \diamond y) + \mu(x \diamond z), \forall\, \lambda, \mu \in \mathbb{C}$; *and*
(d) *an involution $(x \mapsto x^*)$ such that*
  (d1) $(x + \lambda y)^* = x^* + \overline{\lambda} y^*, \forall x, y \in \mathcal{A}, \lambda \in \mathbb{C}$ *and* $x^{**} = x, \forall\, x \in \mathcal{A}$;
  (d2) $(x, y) \in \Gamma$ *if and only if* $(y^*, x^*) \in \Gamma$ *and then* $(x \diamond y)^* = y^* \diamond x^*$.

DEFINITION 2 *An element e of a partial *-algebra $\mathcal{B}$ is called a unit, and $\mathcal{B}$ is said to be unital, if $(e, x), (x, e) \in \Gamma$, and then $e^* = e$, and $e \diamond x = x \diamond e = x$, for every $x \in \mathcal{B}$. $\mathcal{B}$ is said to be abelian if, for all $x, y \in \mathcal{B}$, $(x, y), (y, x) \in \Gamma$, and then $x \diamond y = y \diamond x$.*

Remark 1 Partial *-algebras are studied by means of their spaces of multipliers.

DEFINITION 3 *Let $(\mathcal{A}, \Gamma, \diamond, *)$ be a partial *-algebra, $\mathcal{M} \subset \mathcal{A}$ and $x \in \mathcal{A}$. Put $L(x) = \{y \in \mathcal{A} : (y, x) \in \Gamma\}$ (resp., $R(x) = \{y \in \mathcal{A} : (x, y) \in \Gamma\}$, $L(\mathcal{M}) = \bigcap_{x \in \mathcal{M}} L(x) \equiv \{y \in \mathcal{A} : y \in L(x), \forall x \in \mathcal{M}\}$, $R(\mathcal{M}) = \bigcap_{x \in \mathcal{M}} R(x) \equiv \{y \in \mathcal{A} : y \in R(x), \forall x \in \mathcal{M}\}$). Then $L(x)$ (resp., $R(x)$, $L(\mathcal{M})$, $R(\mathcal{M})$) is called the space of left multipliers of x (resp., right multipliers of x, left multipliers of $\mathcal{M}$, right multipliers of $\mathcal{M}$). In particular, elements of $L(\mathcal{A})$ (resp., $R(\mathcal{A})$) are called universal left (resp., universal right) multipliers. $M(\mathcal{A}) \equiv L(\mathcal{A}) \cap R(\mathcal{A})$ is the so-called universal multipliers of $\mathcal{A}$.*

DEFINITION 4 *A partial *-algebra $\mathcal{B}$ is said to be semi-associative if $y \in R(x)$ implies $y \diamond z \in R(x)$ for every $z \in R(\mathcal{B})$ and $(x \diamond y) \diamond z = x \diamond (y \diamond z)$.*

Remark 2 If a partial *-algebra $\mathcal{B}$ is semi-associative, then $L(\mathcal{B})$ and $R(\mathcal{B})$ are algebras, while $M(\mathcal{B})$ is a *-algebra.

DEFINITION 5 *The positive cone of a partial *-algebra $\mathcal{A}$ is the set $\mathcal{A}_+$ given by $\mathcal{A}_+ := \{\sum_{j=1}^n x_j^* \diamond x_j : x_j \in R(\mathcal{A}), n \in \mathbb{N}\}$. We say that $x \in \mathcal{A}$ is positive if $x \in \mathcal{A}_+$ and write $x \geq 0$.*

DEFINITION 6 *Given a Hausdorff locally convex topology τ on $\mathcal{A}$, we call the pair $(\mathcal{A}, \tau)$ a locally convex partial *-algebra if and only if:*

(i) $(\mathcal{A}_0, \tau)$ *is a Hausdorff locally convex space, where* $\mathcal{A}_0$ *is the underlying linear space of* $\mathcal{A}$,
(ii) *the map* $x \in \mathcal{A} \mapsto x^* \in \mathcal{A}$ *is* τ*-continuous,*
(iii) *the map* $x \in \mathcal{A} \mapsto a \diamond x \in \mathcal{A}$ *is* τ*-continuous, for all* $a \in L(\mathcal{A})$ *and*
(iv) *the map* $x \in \mathcal{A} \mapsto x \diamond b \in \mathcal{A}$ *is* τ*-continuous, for all* $b \in R(\mathcal{A})$.

DEFINITION 7 *Let $\mathcal{B}$ be a complex linear space and $\mathcal{B}_0$ a *-algebra contained in $\mathcal{B}$. $\mathcal{B}$ is said to be a quasi *-algebra with distinguised *-algebra $\mathcal{B}_0$ if*

(i) $\mathcal{B}$ *is a bimodule over* $\mathcal{B}_0$ *for which the module action extends the multiplication of* $\mathcal{B}_0$ *such that* $x.(y.b) = (x.y).b$ *and* $x.(b.y) = (x.b).y$, *for all* $b \in \mathcal{B}$ *and* $x, y \in \mathcal{B}_0$;
(ii) *the involution* * *on* $\mathcal{B}$ *extends the involution of* $\mathcal{B}_0$ *such that* $(x.b)^* = b^*.x^*$ *and* $(b.x)^* = x^*.b^*$, *for all* $b \in \mathcal{B}$ *and* $x \in \mathcal{B}_0$.

If $\mathcal{B}$ is a locally convex space with a locally convex topology τ such that

(i) $\mathcal{B}_0$ *is* τ*-dense in* $\mathcal{B}$;
(ii) *the involution* * *is* τ*-continuous;*
(iii) *the left and right module actions are separately* τ*-continuous,*

then $(\mathcal{B}, \mathcal{B}_0)$ is said to be a locally convex quasi *-algebra.

Remark 3 Every quasi *-algebra is a semi-associative partial *-algbera

3. Locally convex partial *-algebraic modules

As in [2], let $(\mathcal{B}, \tau_{\mathcal{B}})$ be a locally convex partial *-algebra, with involution * and partial multiplication written as juxtaposition. Let $\tau_{\mathcal{B}}$ be generated by a family $\{|\cdot|_\alpha : \alpha \in \Delta\}$ of seminorms. In what follows, we assume, without loss of generality, that the family $\{|\cdot|_\alpha : \alpha \in \Delta\}$ of seminorms is directed. Let $\mathcal{D}$ be a linear space which is also a right $R(\mathcal{B})$-module in the sense that $x.a + y.b \in \mathcal{D}$, whenever $x, y \in \mathcal{D}$ and $a, b \in R(\mathcal{B})$, where the action of $R(\mathcal{B})$ on $\mathcal{D}$ is written as $z.c$ for $z \in \mathcal{D}$, $c \in R(\mathcal{B})$. Locally convex partial *-algebraic modules were introduced in [2] as follows.

Definition 8 *A $\mathcal{B}$-valued inner product on $\mathcal{D}$ is a conjugate-bilinear map $\langle\cdot,\cdot\rangle_{\mathcal{B}} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{B}$ satisfying the following:*

(i) $\langle x, x\rangle_{\mathcal{B}} \in \mathcal{B}_+, \forall x \in \mathcal{D}$ *and* $\langle x, x\rangle_{\mathcal{B}} = 0$ *only if* $x = 0$,
(ii) $\langle x, y\rangle_{\mathcal{B}} = \langle y, x\rangle^*_{\mathcal{B}}, \forall x, y \in \mathcal{D}$,
(iii) $\langle x, y.b\rangle_{\mathcal{B}} = \langle x, y\rangle_{\mathcal{B}} b, \forall x, y \in \mathcal{D}, b \in R(\mathcal{B})$

Lemma 1 *Let $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ be a $\mathcal{B}$-valued inner product on $\mathcal{D}$. Define $\|\cdot\|_\alpha : \mathcal{D} \longrightarrow [0, \infty)$ by*

$$\|x\|_\alpha = |\langle x, x\rangle_{\mathcal{B}}|_\alpha^{1/2}, x \in \mathcal{D}, \alpha \in \Delta. \tag{1}$$

Then, the following inequality holds:

$$\frac{1}{2}(|\langle x, y\rangle_{\mathcal{B}}|_\alpha + |\langle y, x\rangle_{\mathcal{B}}|_\alpha) \le \|x\|_\alpha \|y\|_\alpha, \forall x, y \in \mathcal{D}, \alpha \in \Delta. \tag{2}$$

*Moreover, if $|\cdot|_\alpha$ is *-ivariant, i.e., if $|a^*|_\alpha = |a|_\alpha, \forall a \in \mathcal{B}, \alpha \in \Delta$, then the inequality (2) reduces to*

$$|\langle x, y\rangle_{\mathcal{B}}|_\alpha \le \|x\|_\alpha \|y\|_\alpha, \forall x, y \in \mathcal{D}, \alpha \in \Delta. \tag{3}$$

Corollary 1 *If $\|\cdot\|_\alpha : \mathcal{D} \longrightarrow [0, \infty)$ is defined as in Equation (1), then $\|\cdot\|_\alpha$ is a seminorm on $\mathcal{D}$ for each $\alpha \in \Delta$.*

Remark 4 We observe that the family $\{\|\cdot\|_\alpha : \alpha \in \Delta\}$ of seminorms is directed.

Definition 9 *A locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module is a triple $(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{\mathcal{D},\mathcal{B}})$ comprising:*

(a) *a linear space $\mathcal{D}$ which is also a right $R(\mathcal{B})$-module;*
(b) *a $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{B}$; and*
(c) *a locally convex topology $\tau_{\mathcal{D},\mathcal{B}}$ on $\mathcal{D}$ generated by the family $\{\|\cdot\|_\alpha : \alpha \in \Delta\}$ of seminorms given by (1) and, with respect to this topology, the map $l_R(b) : \mathcal{D} \longrightarrow \mathcal{D}$ given by $l_R(b)x = x.b, \forall x \in \mathcal{D}$, is continuous for each $b \in R(\mathcal{B})$; i.e., for each $\alpha \in \Delta, \exists$ a $\beta(\alpha) \in \Delta$ and $K_{\alpha,b} > 0$ such that $\|l_R(b)x\|_\alpha \le K_{\alpha,b}\|x\|_{\beta(\alpha)}$*

Proposition 1 *Let $(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{\mathcal{D},\mathcal{B}})$ be a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module. Then the $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ on $\mathcal{D}$ is $\tau_{\mathcal{B}}$-continuous.*

Proof. Let (x_λ) and (y_μ) be nets in $\mathcal{D}$ such that $x_\lambda \longrightarrow x$ and $y_\mu \longrightarrow y$. Then

$$\begin{aligned}
|\langle x_\lambda, y_\mu\rangle_{\mathcal{B}} - \langle x, y\rangle_{\mathcal{B}}|_\alpha &= |\langle x_\lambda - x, y\rangle_{\mathcal{B}} - \langle x_\lambda, y\rangle_{\mathcal{B}} + \langle x_\lambda, y_\mu\rangle_{\mathcal{B}}|_\alpha \\
&= |\langle x_\lambda - x, y\rangle_{\mathcal{B}} - \langle x_\lambda, y - y_\mu\rangle_{\mathcal{B}}|_\alpha \\
&= |\langle x_\lambda - x, y\rangle_{\mathcal{B}} + \langle x_\lambda, y_\mu - y\rangle_{\mathcal{B}}|_\alpha \\
&\leq |\langle x_\lambda - x, y\rangle_{\mathcal{B}}|_\alpha + |\langle x_\lambda, y_\mu - y\rangle_{\mathcal{B}}|_\alpha \\
&\leq (|\langle x_\lambda - x, y\rangle_{\mathcal{B}}|_\alpha + |\langle y, x_\lambda - x\rangle_{\mathcal{B}}|_\alpha) + (|\langle x_\lambda, y_\mu - y\rangle_{\mathcal{B}}|_\alpha + |\langle y_\mu - y, x_\lambda\rangle_{\mathcal{B}}|_\alpha) \\
&\leq 2(\|x_\lambda - x\|_\alpha \|y\|_\alpha + \|x_\lambda\|_\alpha \|y_\mu - y\|_\alpha) \\
&\leq 2[\|x_\lambda - x\|_\alpha \|y\|_\alpha + (\|x_\lambda - x\|_\alpha + \|x\|_\alpha)\|y_\mu - y\|_\alpha] \\
&= 2(\|x_\lambda - x\|_\alpha \|y\|_\alpha + \|x_\lambda - x\|_\alpha \|y_\mu - y\|_\alpha + \|x\|_\alpha \|y_\mu - y\|_\alpha) \longrightarrow 0.
\end{aligned}$$

Hence $\langle x_\lambda, y_\mu\rangle_{\mathcal{B}} \longrightarrow \langle x, y\rangle_{\mathcal{B}}$. This completes the proof. ■

Remark 5 The lemma above implies that the $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ on $\mathcal{D}$ can be extended to a $\mathcal{B}$-valued inner product on the $\tau_{\mathcal{D},\mathcal{B}}$-completion of $\mathcal{D}$. We shall always denote also by $\langle\cdot,\cdot\rangle_{\mathcal{B}}$, the $\mathcal{B}$-valued inner product on the $\tau_{\mathcal{D},\mathcal{B}}$-completion of $\mathcal{D}$.

EXAMPLE 1 Let $(\mathcal{B}, \tau_{\mathcal{B}})$ be a semi-associative locally convex partial *-algebra whose topology is generated by the family $\{|\cdot|_\alpha : \alpha \in \Delta\}$ of seminorms. If we take $\mathcal{D}$ as $R(\mathcal{B})$ and define $\langle\cdot,\cdot\rangle_{\mathcal{B}} : \mathcal{D}\times\mathcal{D} \longrightarrow \mathcal{B}$ by $\langle x, y\rangle_{\mathcal{B}} = x^*y, \forall x, y \in \mathcal{D}$, then $(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{\mathcal{D},\mathcal{B}})$ is a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module, where we take $\tau_{\mathcal{D},\mathcal{B}}$ to be the locally convex topology on $\mathcal{D}$ generated by $\|x\|_\alpha = |\langle x, x\rangle_{\mathcal{B}}|_\alpha^{1/2}$. Indeed:

(b) (i) $\langle x, x\rangle_{\mathcal{B}} = x^*x \in \mathcal{B}_+, \forall x \in R(\mathcal{B})$ and if $x = 0$, then$\langle x, x\rangle_{\mathcal{B}} = x^*x = 0$;
  (ii) $\langle y, x\rangle_{\mathcal{B}}^* = (y^*x)^* = x^*y = \langle x, y\rangle_{\mathcal{B}}, \forall x, y \in R(\mathcal{B})$,
  (iii) $\langle x, y.b\rangle_{\mathcal{B}} = x^*(y.b) = (x^*y)b = \langle x, y\rangle_{\mathcal{B}}b, \forall x, y \in R(\mathcal{B}), b \in R(\mathcal{B})$; and

(c) The continuity of the map $x \in R(\mathcal{B}) \longmapsto l_R(b)x = x.b \in R(\mathcal{B})$ follows from the assumption that $(\mathcal{B}, \tau_{\mathcal{B}})$ is a locally convex partial *-algebra.

4. Some basic properties of locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-modules

DEFINITION 10 *Let $\mathcal{A}$ be a partial *-algebra and $\mathcal{B}$ a linear subspace of $\mathcal{A}$. Then $\mathcal{B}$ is said to be a left (resp., right) ideal in $\mathcal{A}$, if $a \in L(\mathcal{A})$ and $b \in \mathcal{B}$ (resp., $a \in R(\mathcal{A})$ and $b \in \mathcal{B}$) implies $ab \in \mathcal{B}$ (resp., $ba \in \mathcal{B}$). If $\mathcal{B}$ is both a left and a right ideal in $\mathcal{A}$, then $\mathcal{B}$ is called a two-sided ideal, or simply, an ideal in $\mathcal{A}$.*

PROPOSITION 2 *Let $(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{\mathcal{D},\mathcal{B}})$ be a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module. Define the linear subspace, $\mathcal{M}_{\mathcal{D}}$ of $\mathcal{B}$ by*

$$\mathcal{M}_{\mathcal{D}} = \text{span}\{\langle x, y\rangle_{\mathcal{B}} : x, y \in \mathcal{D}\} \bigcap R(\mathcal{B}).$$

Then $\mathcal{M}_{\mathcal{D}}$ is an ideal in $\mathcal{B}$.

Proof. Take an element m in $\mathcal{M}_{\mathcal{D}}$. Then m may be expressed as $m = \sum_{j=1}^n \lambda_j\langle x_j, y_j\rangle_{\mathcal{B}}$, $\forall\lambda_j \in \mathbb{C}, \forall x_j, y_j \in \mathcal{D}, n \in \mathbb{N}$. So if $b \in R(\mathcal{B})$, then $mb = (\sum_{j=1}^n \lambda_j\langle x_j, y_j\rangle_{\mathcal{B}})b = \sum_{j=1}^n \lambda_j\langle x_j, y_j\rangle_{\mathcal{B}}b = \sum_{j=1}^n \lambda_j\langle x_j, y_j.b\rangle_{\mathcal{B}}$, by Condition (iii) of Definition 8. It follows that $mb \in \mathcal{M}_{\mathcal{D}}$, i.e., $\mathcal{M}_{\mathcal{D}}$ is a right ideal in $\mathcal{B}$, by Definition 10.

To show that $\mathcal{M}_{\mathcal{D}}$ is also a left ideal in $\mathcal{B}$, we first note that conditions (ii) and (iii) of Definition **??** imply $\langle x.b, y\rangle_{\mathcal{B}} = b^*\langle x, y\rangle_{\mathcal{B}}$, $\forall b \in R(\mathcal{B})$. It now follows from this that, if $m \in \mathcal{M}_{\mathcal{D}}$ and $a \in L(\mathcal{B})$, then $am \in \mathcal{M}_{\mathcal{D}}$, i.e., $\mathcal{M}_{\mathcal{D}}$ is a left ideal in $\mathcal{B}$, by Definition 10. Hence $\mathcal{M}_{\mathcal{D}}$ is, indeed, an ideal in $\mathcal{B}$. ■

DEFINITION 11 *Let $(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{\mathcal{D},\mathcal{B}})$ be a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module. Then $\mathcal{D}$ will be called full* *if $\mathcal{M}_\mathcal{D}$ is dense in $R(\mathcal{B})$.*

EXAMPLE 2 Let $\{\mathcal{D}_j\} := \{(\mathcal{D}_j, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{\mathcal{D}_j,\mathcal{B}})\}$ be a finite collection of locally convex $(\mathcal{B}, \tau_\mathcal{B})$-modules. Let $\mathcal{D}$ be the set of n-tuples $x = (x_1, \dots, x_n)$ where $x_j \in \mathcal{D}_j$ $(j = 1, \dots, n)$, and define the following (componentwise) operations: $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$ and $(x_1, \dots, x_n).b = (x_1.b, \dots, x_n.b)$, $b \in R(\mathcal{B})$. Then $\mathcal{D}$ is the direct sum of the $R(\mathcal{B})$-modules $\{\mathcal{D}_j\}$, i.e., $\mathcal{D} := \bigoplus_{j=1}^n \mathcal{D}_j$. $\mathcal{D}$ is also a right $R(\mathcal{B})$-module under these operations. If we define the $\mathcal{B}$-valued inner product on $\mathcal{D}$ by $\langle x, y\rangle_\mathcal{B} := \sum_{j=1}^n \langle x_j, y_j\rangle_\mathcal{B}$, where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathcal{D}$, then $(\mathcal{D}, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{\mathcal{D},\mathcal{B}})$ is a locally convex $(\mathcal{B}, \tau_\mathcal{B})$-module. We shall denote by $\mathcal{D}^n$ the direct sum of n copies of a locally convex $(\mathcal{B}, \tau_\mathcal{B})$-module $\mathcal{D}$. Now let $\mathcal{D} = R(\mathcal{B})^n$, the direct sum of n copies of the locally convex $(\mathcal{B}, \tau_\mathcal{B})$-module $R(\mathcal{B})$. Then $\mathcal{D}$ is full. Here, $\mathcal{M}_\mathcal{D} = \text{span}\{\sum_{j=1}^n a_j^* b_j : a_j, b_j \in R(\mathcal{B}), n \in \mathbb{N}\} \bigcap R(\mathcal{B}) \subset R(\mathcal{B})$. But $\mathcal{B}_+ = \{\sum_{j=1}^n b_j^* b_j : b_j \in R(\mathcal{B}), n \in \mathbb{N}\} \subset \{\sum_{j=1}^n a_j^* b_j : a_j, b_j \in R(\mathcal{B}), n \in \mathbb{N}\}$. This implies that $\text{span}\,\mathcal{B}_+ \bigcap R(\mathcal{B}) \subset \mathcal{M}_\mathcal{D}$. So, if $\text{span}\,\mathcal{B}_+$ coincides with the partial *-algebra $\mathcal{B}$, then it follows that $R(\mathcal{B}) \subset \mathcal{M}_\mathcal{D}$. Hence $\mathcal{M}_\mathcal{D}$ is dense in $R(\mathcal{B})$.

Remark 6 From the preceding example, we have the following result.

PROPOSITION 3 *If the linear span of $\mathcal{B}_+$ coincides with the partial *-algebra $\mathcal{B}$, then the locally convex $(\mathcal{B}, \tau_\mathcal{B})$-module $R(\mathcal{B})$ is full.*

5. Adjointable maps on locally convex $(\mathcal{B}, \tau_\mathcal{B})$-modules

We now turn to a study of some classes of maps acting on locally convex partial *-algebraic modules. Let $(\mathcal{X}, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{\mathcal{X},\mathcal{B}})$ and $(\mathcal{Y}, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{\mathcal{Y},\mathcal{B}})$ be complete locally convex $(\mathcal{B}, \tau_\mathcal{B})$-modules and let $(\mathcal{D}, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{\mathcal{D},\mathcal{B}})$ be a dense locally convex $(\mathcal{B}, \tau_\mathcal{B})$-submodule of $\mathcal{X}$.

DEFINITION 12 *A map $t : \mathcal{X} \longrightarrow \mathcal{Y}$ is called a $(\mathcal{B}, \tau_\mathcal{B})$-module map (or simply, a module map) if and only if $t(x.b) = (tx).b$, $\forall x \in \mathcal{D}, b \in R(\mathcal{B})$. One also says that t is a $(\mathcal{B}, \tau_\mathcal{B})$-linear map. We denote by $\mathcal{L}_\mathcal{B}(\mathcal{X}, \mathcal{Y})$, the set of all linear $(\mathcal{B}, \tau_\mathcal{B})$-module maps from $\mathcal{X}$ to $\mathcal{Y}$.*

DEFINITION 13 *We call a map $t : \mathcal{X} \longrightarrow \mathcal{Y}$ adjointable if there exists a map $t^* : \mathcal{Y} \longrightarrow \mathcal{X}$ such that*

$$\langle tx, y\rangle_{\mathcal{Y},\mathcal{B}} = \langle x, t^* y\rangle_{\mathcal{X},\mathcal{B}}, \ \forall x \in \mathcal{D}, \ y \in \mathcal{Y} \tag{4}$$

The map t^* will be called the adjoint of t.

PROPOSITION 4 *If the map $t : \mathcal{X} \longrightarrow \mathcal{Y}$ is adjointable, then $t \in \mathcal{L}_\mathcal{B}(\mathcal{X}, \mathcal{Y})$.*

Proof. Let $t : \mathcal{X} \longrightarrow \mathcal{Y}$ be adjointable. Then $\forall x, y \in \mathcal{D}$, $z \in \mathcal{Y}$, $\alpha \in \mathbb{C}$ and $b \in R(\mathcal{B})$

$$\begin{aligned}\langle t[(x + \alpha y).b], z\rangle_{\mathcal{Y},\mathcal{B}} &= \langle (x + \alpha y).b, t^* z\rangle_{\mathcal{X},\mathcal{B}} = b^* \langle x + \alpha y, t^* z\rangle_{\mathcal{X}\mathcal{B}} \\ &= b^*(\langle x, t^* z\rangle_{\mathcal{X},\mathcal{B}} + \alpha\langle y, t^* z\rangle_{\mathcal{X},\mathcal{B}}) = b^*\langle x, t^* z\rangle_{\mathcal{X},\mathcal{B}} + \alpha b^*\langle y, t^* z\rangle_{\mathcal{X},\mathcal{B}} \\ &= b^*\langle tx, z\rangle_{\mathcal{Y},\mathcal{B}} + \alpha b^*\langle ty, z\rangle_{\mathcal{Y},\mathcal{B}} = \langle tx.b, z\rangle_{\mathcal{Y},\mathcal{B}} + \langle \alpha(ty).b, z\rangle_{\mathcal{Y},\mathcal{B}} \\ &= \langle tx.b + \alpha(ty).b, z\rangle_{\mathcal{Y},\mathcal{B}}.\end{aligned}$$

This implies that $t[(x + \alpha y).b] = tx.b + \alpha(ty).b$. It follows that t (as well as t^*) is a linear $(\mathcal{B}, \tau_\mathcal{B})$-module map. Hence $t \in \mathcal{L}_\mathcal{B}(\mathcal{X}, \mathcal{Y})$. ■

NOTATION 5.1 *Let $(\mathcal{D}, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{\mathcal{D},\mathcal{B}})$ be a dense locally convex $(\mathcal{B}, \tau_\mathcal{B})$-submodule of $(\mathcal{X}, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{\mathcal{X},\mathcal{B}})$. $\mathcal{L}_\mathcal{B}(\mathcal{D}, \mathcal{X})$ becomes a linear space when furnished with the usual (pointwise) operations of vector addition, $t + s$ and scalar multiplication, λt, $t, s \in \mathcal{L}_\mathcal{B}(\mathcal{D}, \mathcal{X})$, $\lambda \in \mathbb{C}$. Now set $\mathcal{L}^*_\mathcal{B}(\mathcal{D}, \mathcal{X}) := \{t \in \mathcal{L}_\mathcal{B}(\mathcal{D}, \mathcal{X}) : t$ is continuous and adjointable$\}$. Since $\mathcal{D}$ is dense in $\mathcal{X}$, t^* is uniquely determined, and hence, well-defined. It follows that $\mathcal{L}^*_\mathcal{B}(\mathcal{D}, \mathcal{X})$ is a *-invariant linear subspace of $\mathcal{L}_\mathcal{B}(\mathcal{D}, \mathcal{X})$. It is not* *a *-algebra, except $\mathcal{L}^*_{\mathcal{B}}(\mathcal{D},\mathcal{X}) \equiv \mathcal{L}^*_{\mathcal{B}}(\mathcal{D},\mathcal{D}) = \mathcal{L}^*_{\mathcal{B}}(\mathcal{D}) := \{t \in \mathcal{L}^*_{\mathcal{B}}(\mathcal{D},\mathcal{X}) : t\mathcal{D} \subseteq \mathcal{D} \text{ and } t^*\mathcal{D} \subseteq \mathcal{D}\}$. However, if one sets $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X}) := \{t \in \mathcal{L}^*_{\mathcal{B}}(\mathcal{D},\mathcal{X}) : dom(t^*) \supseteq \mathcal{D}\}$, then:*

Proposition 5 *The linear space $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is a partial *-algebra with:*

(i) involution: $t \longmapsto t^+ := t^* \upharpoonright \mathcal{D}$, for all $t \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ and
(ii) partial multiplication, specified by

$$\Gamma = \{(t,s) \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})^2 : s\mathcal{D} \subseteq \mathrm{dom}(t^{+*}) \text{ and } t^+\mathcal{D} \subseteq \mathrm{dom}(s^*)\} t \circ s = t^{+*}s.$$

Definition 14 *A $^+$-invariant linear subspace $\mathcal{M}$ of $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is called a partial *-subalgebra of $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ if $t,s \in \mathcal{M}$, with $t \in L(s)$ implies $t \circ s \in \mathcal{M}$.*

Remark 7 The next result introduces a class of adjointable maps on locally convex partial *-algebraic modules. In the case of Hilbert C*-modules, they are operators analogous to the finite-rank operators on a Hilbert space. Let $\mathcal{X}$ be a complete locally convex $(\mathcal{B},\tau_{\mathcal{B}})$-module. Set $\mathcal{D} = \{z \in \mathcal{X} : \langle x,z\rangle_{\mathcal{B}} \in R(\mathcal{B}), \forall x, \in \mathcal{X}\}$. In what follows, we shall assume that $\mathcal{D}$ is dense in $\mathcal{X}$.

Proposition 6 *For $x,y \in \mathcal{X}$, define the map $\pi^{\mathcal{B}}_{x,y} : \mathcal{D} \longrightarrow \mathcal{X}$ as*

$$\pi^{\mathcal{B}}_{x,y}(z) = x.\langle y,z\rangle_{\mathcal{B}}. \tag{5}$$

Then the map $\pi^{\mathcal{B}}_{x,y}$ is continuous and adjointable with adjoint

$$\left(\pi^{\mathcal{B}}_{x,y}\right)^+ := \left(\pi^{\mathcal{B}}_{x,y}\right)^* \upharpoonright \mathcal{D} = \pi^{\mathcal{B}}_{y,x} \tag{6}$$

Proof. Let $z \in \mathcal{D}$. Then $\|\pi^{\mathcal{B}}_{x,y}(z)\|_\alpha = \|x.\langle y,z\rangle_{\mathcal{B}}\|_\alpha$. Since the right $R(\mathcal{B})$-module action is continuous, it follows that there exist $\beta(\alpha)$ and a constant $K_{\alpha,\langle y,z\rangle_{\mathcal{B}}} > 0$ such that $\|\pi^{\mathcal{B}}_{x,y}(z)\|_\alpha \leq K_{\alpha,\langle y,z\rangle_{\mathcal{B}}}\|x\|_{\beta(\alpha)}$. Let $C_\alpha = \sup\left\{\frac{K_{\alpha,\langle y,z\rangle_{\mathcal{B}}}}{\|y\|_\alpha\|z\|_\alpha} : y \in \mathcal{X}, z \in \mathcal{D} \text{ with } \|y\|_\alpha \neq 0, \|z\|_\alpha \neq 0\right\}$. Then $K_{\alpha,\langle y,z\rangle_{\mathcal{B}}} \leq C_\alpha\|y\|_\alpha\|z\|_\alpha$. So we have $\|\pi^{\mathcal{B}}_{x,y}(z)\|_\alpha \leq C_\alpha\|x\|_{\beta(\alpha)}\|y\|_\alpha\|z\|_\alpha$ i.e., $\|\pi^{\mathcal{B}}_{x,y}(z)\|_\alpha \leq M_{(\alpha,z,y)}\|z\|_{\gamma(\alpha)}$, where $M_{(\alpha,z,y)} = C_\alpha\|x\|_{\beta(\alpha)}\|y\|_\alpha$. Hence $\pi^{\mathcal{B}}_{x,y}$ is continuous. Now let $u,z \in \mathcal{D}$. Then

$$\begin{aligned} *\pi^{\mathcal{B}}_{x,y}(z)u_{\mathcal{B}} &= \langle x.\langle y,z\rangle_{\mathcal{B}}, u\rangle_{\mathcal{B}} = \langle z,y\rangle_{\mathcal{B}}\langle x,u\rangle_{\mathcal{B}} \\ &= \langle z, y.\langle x,u\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} = *z\pi^{\mathcal{B}}_{y,x}(u)_{\mathcal{B}}. \end{aligned}$$

i.e.,

$$*\pi^{\mathcal{B}}_{x,y}(z)u_{\mathcal{B}} = *z\pi^{\mathcal{B}}_{y,x}(u)_{\mathcal{B}}. \tag{7}$$

It follows that $\pi^{\mathcal{B}}_{x,y}$ is adjointable with adjoint $\left(\pi^{\mathcal{B}}_{x,y}\right)^+ := \left(\pi^{\mathcal{B}}_{x,y}\right)^* \upharpoonright \mathcal{D} = \pi^{\mathcal{B}}_{y,x}$. Since $\pi^{\mathcal{B}}_{x,y}$ is continuous, $\pi^{\mathcal{B}}_{x,y} \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. ∎

Remark 8 From the preceding, we note that, since $\mathcal{D} \subseteq \mathrm{dom}((\pi^{\mathcal{B}}_{x,y})^*)$ we have, for $n \in \mathbb{N}$, $\mathrm{dom}((\sum_{j=1}^n \pi^{\mathcal{B}}_{x_j,y_j})^*) \supseteq \mathrm{dom}((\pi^{\mathcal{B}}_{x_1,y_1})^*)\bigcap \mathrm{dom}((\pi^{\mathcal{B}}_{x_2,y_2})^*)\bigcap \cdots \bigcap \mathrm{dom}((\pi^{\mathcal{B}}_{x_n,y_n})^*) \supseteq \mathcal{D}$. It follows that $\sum_{j=1}^n \pi^{\mathcal{B}}_{x_j,y_j} \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. Also, for $\alpha \in \mathbb{C}$ and $\pi^{\mathcal{B}}_{x,y} \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$, it is clear that $\alpha\pi^{\mathcal{B}}_{x,y} \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. So we introduce the linear subspace $\mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X}) = \mathrm{span}\{\pi^{\mathcal{B}}_{x,y} \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X}) : x,y \in \mathcal{X}\}$ of $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$.

Proposition 7 *$\mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is a partial *-subalgebra and an ideal of $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$.*

Proof. We first show that $\mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is $^+$-invariant, i.e., for $T \in \mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$, $T^+ \in \mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$, where $T^+ := T^* \upharpoonright \mathcal{D}$. Indeed, if $T \in \mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$, then T may be expressed as $T = \sum_{j=1}^n \pi^{\mathcal{B}}_{x_j,y_j}, \forall x_j, y_j \in \mathcal{X}$ and $n \in \mathbb{N}$. Applying (6), we have $T^+ = (\sum_{j=1}^n \pi^{\mathcal{B}}_{x_j,y_j})^+ = \sum_{j=1}^n (\pi^{\mathcal{B}}_{x_j,y_j})^+ = \sum_{j=1}^n (\pi^{\mathcal{B}}_{x_j,y_j})^* \upharpoonright \mathcal{D} =$

$\sum_{j=1}^{n}\pi^{\mathcal{B}}_{y_j,x_j} \in \mathcal{K}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. Next, we show that $\mathcal{K}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is a partial *-subalgebra of $\mathcal{L}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. To this end, let $T,S \in \mathcal{K}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$, with $T=\sum_{j=1}^{m}\pi^{\mathcal{B}}_{x_j,y_j}$ and $S=\sum_{k=1}^{n}\pi^{\mathcal{B}}_{u_k,v_k}, \forall u_k,v_k,x_j,y_j \in \mathcal{X}$ and $m,n\in\mathbb{N}$. Then we claim that $T\circ S \in \mathcal{K}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ whenever $u_k{y_j}_{\mathcal{B}} \in M(\mathcal{B})$, $j=1,\cdots m, k=1,\cdots n$. This is seen as follows. For $w,z\in\mathcal{D}$,

$$
\begin{aligned}
*(T\circ S)(z)w_{\mathcal{B}} &= *\left[\left(\sum_{j=1}^{m}\pi^{\mathcal{B}}_{x_j,y_j}\right)\circ\left(\sum_{k=1}^{n}\pi^{\mathcal{B}}_{u_k,v_k}\right)\right](z)w_{\mathcal{B}}\\
&= *\left(\sum_{j=1}^{m}\pi^{\mathcal{B}}_{x_j,y_j}\right)^{+*}\left(\sum_{k=1}^{n}\pi^{\mathcal{B}}_{u_k,v_k}\right)(z)w_{\mathcal{B}}\\
&= *\left(\sum_{k=1}^{n}\pi^{\mathcal{B}}_{u_k,v_k}\right)(z)\left(\sum_{j=1}^{m}\pi^{\mathcal{B}}_{x_j,y_j}\right)^{+}(w)_{\mathcal{B}}\\
&= *\left(\sum_{k=1}^{n}\pi^{\mathcal{B}}_{u_k,v_k}\right)(z)\left(\sum_{j=1}^{m}\pi^{\mathcal{B}}_{y_j,x_j}\right)(w)_{\mathcal{B}}\\
&= *\sum_{k=1}^{n}\pi^{\mathcal{B}}_{u_k,v_k}(z)\sum_{j=1}^{m}\pi^{\mathcal{B}}_{y_j,x_j}(w)_{\mathcal{B}}\\
&= *\sum_{k=1}^{n}(u_k.v_k z_{\mathcal{B}})\sum_{j=1}^{m}(y_j.x_j w_{\mathcal{B}})_{\mathcal{B}}\\
&= \sum_{\substack{1\le j\le m\\1\le k\le n}} *u_k.v_k z_{\mathcal{B}} y_j.x_j w_{\mathcal{B}_{\mathcal{B}}}\\
&= \sum_{\substack{1\le j\le m\\1\le k\le n}} z{v_k}_{\mathcal{B}}\left(*u_k y_j.x_j w_{\mathcal{B}_{\mathcal{B}}}\right)\\
&= \sum_{\substack{1\le j\le m\\1\le k\le n}} z{v_k}_{\mathcal{B}}\left(*u_k {y_j}_{\mathcal{B}} x_j w_{\mathcal{B}}\right)
\end{aligned}
$$

Hence, whenever $*u_k{y_j}_{\mathcal{B}} \in M(\mathcal{B})$, $j=1,\cdots m, k=1,\cdots n$, we have that

$$
\begin{aligned}
*(T\circ S)(z)w_{\mathcal{B}} &= \sum_{\substack{1\le j\le m\\1\le k\le n}} z{v_k}_{\mathcal{B}}\left(*u_k{y_j}_{\mathcal{B}}x_j w_{\mathcal{B}}\right) = \sum_{\substack{1\le j\le m\\1\le k\le n}} z{v_k}_{\mathcal{B}}x_j.*y_j{u_k}_{\mathcal{B}}w_{\mathcal{B}}\\
&= \sum_{\substack{1\le j\le m\\1\le k\le n}} x_j.*y_j{u_k}_{\mathcal{B}}v_k z_{\mathcal{B}}w_{\mathcal{B}} = \sum_{\substack{1\le j\le m\\1\le k\le n}} *\pi^{\mathcal{B}}_{x_j.y_j{u_k}_{\mathcal{B}},v_k}(z)w_{\mathcal{B}}
\end{aligned}
$$

It follows that $T\circ S$ lies in $\mathcal{K}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ with $T\circ S = \sum_{\substack{1\le j\le m\\1\le k\le n}}\pi^{\mathcal{B}}_{x_j.y_j{u_k}_{\mathcal{B}},v_k}$, whenever $*u_k{y_j}_{\mathcal{B}} \in M(\mathcal{B})$, $j=1,\cdots m, k=1,\cdots n$. Finally, we show that $\mathcal{K}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is an ideal of $\mathcal{L}^{+}_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. Let $\sum_{j=1}^{n}\pi^{\mathcal{B}}_{x_j,y_j} \in$

$\mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ and $t \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. For any $u, z \in \mathcal{D}$, we have

$$\begin{aligned} *\left(t \circ \sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j}\right)(u)z_{\mathcal{B}} &= *t^{+*}\left(\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j}\right)(u)z_{\mathcal{B}} = *\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j}(u)t^{+}z_{\mathcal{B}} \\ &= \sum_{j=1}^{n} *\pi^{\mathcal{B}}_{x_j,y_j}(u)t^{+}(z)_{\mathcal{B}} = \sum_{j=1}^{n} *x_j.y_j u_{\mathcal{B}} t^{+}(z)_{\mathcal{B}} \\ &= \sum_{j=1}^{n} *t^{+*}(x_j).y_j u_{\mathcal{B}} z_{\mathcal{B}} = \sum_{j=1}^{n} *\pi^{\mathcal{B}}_{t^{+*}(x_j),y_j}(u)z_{\mathcal{B}} \\ &= *\sum_{j=1}^{n} \pi^{\mathcal{B}}_{t^{+*}(x_j),y_j}(u)z_{\mathcal{B}} = *\left(\sum_{j=1}^{n} \pi^{\mathcal{B}}_{t^{+*}(x_j),y_j}\right)(u)z_{\mathcal{B}} \end{aligned}$$

It follows that

$$t \circ \sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j} = \sum_{j=1}^{n} \pi^{\mathcal{B}}_{t^{+*}(x_j),y_j} \in \mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X}) \tag{8}$$

Thus $\mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is a left ideal of $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. On the other hand, let $\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j} \in \mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ and $s \in \mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. For any $u, z \in \mathcal{D}$, we have

$$\begin{aligned} *\left[\left(\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j}\right) \circ s\right](u)z_{\mathcal{B}} &= *\left(\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j}\right)^{+*} s(u)z_{\mathcal{B}} = *s(u)\left(\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j}\right)^{+}(z)_{\mathcal{B}} \\ &= *s(u)\sum_{j=1}^{n} \pi^{\mathcal{B}}_{y_j,x_j}(z)_{\mathcal{B}} = \sum_{j=1}^{n} *s(u)\pi^{\mathcal{B}}_{y_j,x_j}(z)_{\mathcal{B}} \\ &= \sum_{j=1}^{n} *us^{*}\pi^{\mathcal{B}}_{y_j,x_j}(z)_{\mathcal{B}} = \sum_{j=1}^{n} *u\pi^{\mathcal{B}}_{s^{+}(y_j),x_j}(z)_{\mathcal{B}} \\ &= \sum_{j=1}^{n} *u\left(\pi^{\mathcal{B}}_{x_j,s^{+}(y_j)}\right)^{+}(z)_{\mathcal{B}} = *\left(\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,s^{+}(y_j)}\right)(u)z_{\mathcal{B}} \end{aligned}$$

It follows that

$$\left(\sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,y_j}\right) \circ s = \sum_{j=1}^{n} \pi^{\mathcal{B}}_{x_j,s^{+}(y_j)} \in \mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X}). \tag{9}$$

Thus $\mathcal{K}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$ is a right ideal of $\mathcal{L}^+_{\mathcal{B}}(\mathcal{D},\mathcal{X})$. This completes the proof. ■

6. Locally convex partial *-algebraic bimodules

DEFINITION 15 *Let $(\mathcal{A}, \tau_{\mathcal{A}})$ and $(\mathcal{B}, \tau_{\mathcal{B}})$ be two locally convex, semi-associative partial *-algebras, and let ${}_{\mathcal{A}}\mathcal{D} \equiv (\mathcal{D}, {}_{\mathcal{A}}\langle\cdot,\cdot\rangle, \tau_{\mathcal{D},\mathcal{A}})$ be a (left) locally convex $(\mathcal{A}, \tau_{\mathcal{A}})$-module and $\mathcal{D}_{\mathcal{B}} \equiv (\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{\mathcal{D},\mathcal{B}})$ a (right) locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module. Then $\mathcal{D}$ is said to be a locally convex $(\mathcal{A}, \tau_{\mathcal{A}})$-$(\mathcal{B}, \tau_{\mathcal{B}})$-bimodule if the following properties are satisfied:*

(i) ${}_{\mathcal{A}}\langle x,y\rangle.z = x.\langle y,z\rangle_{\mathcal{B}}$, $\forall x,y,z \in \mathcal{D}$ *such that* ${}_{\mathcal{A}}\langle x,y\rangle \in L(\mathcal{A})$, $\langle y,z\rangle_{\mathcal{B}} \in R(\mathcal{B})$*;*

(ii) *for each* $b \in R(\mathcal{B})$, *the map* $x \in \mathcal{D}_{\mathcal{B}} \longmapsto x.b \in \mathcal{D}_{\mathcal{B}}$ *is* $\tau_{\mathcal{D},\mathcal{A}}$*-continuous; i.e., for each* $\alpha \in \Delta$, $\exists\ \beta(\alpha) \in \Delta$ *and a constant* $K_{\alpha,b} > 0$ *such that* $\|x.b\|_\alpha \le K_{\alpha,b}\|x\|_{\beta(\alpha)}$*;*

(iii) *for each* $a \in L(\mathcal{A})$, *the map* $x \in {}_{\mathcal{A}}\mathcal{D} \longmapsto a.x \in {}_{\mathcal{A}}\mathcal{D}$ *is* $\tau_{\mathcal{D},\mathcal{B}}$*-continuous; i.e., for each* $\gamma \in \Lambda$, $\exists\ \zeta(\gamma) \in \Lambda$ *and a constant* $K_{\gamma,a} > 0$ *such that* $\|a.x\|_\gamma \le K_{\gamma,a}\|x\|_{\zeta(\gamma)}$.

Remark 9 Suppose $a = {}_{\mathcal{A}}\langle x,x\rangle \in L(\mathcal{A})$ and $b = \langle x,x\rangle_{\mathcal{B}} \in R(\mathcal{B})$. Then, by Property (i) of Definition 15, $a.x = x.b$, $\forall x \in \mathcal{D}$. It now follows from Properties (ii) and (iii) of Definition 15 that the topologies $\tau_{\mathcal{D},\mathcal{A}}$ and $\tau_{\mathcal{D},\mathcal{B}}$ are generated by the same family of seminorms $\|x\|_\alpha = |{}_{\mathcal{A}}\langle x,x\rangle|_\alpha^{1/2} = |\langle x,x\rangle_{\mathcal{B}}|_\beta^{1/2} = \|x\|_\beta$.

Definition 16 *A* $\tau_{\mathcal{D},\mathcal{B}}$*-complete locally convex* $(\mathcal{A},\tau_{\mathcal{A}})$-$(\mathcal{B},\tau_{\mathcal{B}})$*-bimodule* $\mathcal{D}$ *will be called an* $(\mathcal{A},\tau_{\mathcal{A}})$-$(\mathcal{B},\tau_{\mathcal{B}})$*-imprimitivity bimodule if it is full both as a left and as a right locally convex partial* **-algebraic module.*

Lemma 2 *Let* $(\mathcal{A},\tau_{\mathcal{A}})$ *and* $(\mathcal{B},\tau_{\mathcal{B}})$ *be two locally convex, semi-associative partial* **-algebras. If* $\mathcal{D}$ *is an* $(\mathcal{A},\tau_{\mathcal{A}})$-$(\mathcal{B},\tau_{\mathcal{B}})$*-imprimitivity bimodule, then the following hold:*

$$\langle a.x,y\rangle_{\mathcal{B}} = \langle x,a^*.y\rangle_{\mathcal{B}} \text{ and } {}_{\mathcal{A}}\langle x.b,y\rangle = {}_{\mathcal{A}}\langle x,y.b^*\rangle, \forall x,y \in \mathcal{D}, a \in L(\mathcal{A}), b \in R(\mathcal{B}). \tag{10}$$

Proof. Since ${}_{\mathcal{A}}\mathcal{D}$ is full, we may set an $a \in L(\mathcal{A})$ as $a = \sum_{j=1}^n {}_{\mathcal{A}}\langle u_j,v_j\rangle$, for some $u_j, v_j \in {}_{\mathcal{A}}\mathcal{D}$, $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\langle a.x,y\rangle_{\mathcal{B}} &= [\Big]\sum_{j=1}^n {}_{\mathcal{A}}\langle u_j v_j\rangle.x, y_{\mathcal{B}} = [\Big] u_j.\sum_{j=1}^n \langle v_j,x\rangle_{\mathcal{B}} y_{\mathcal{B}} = \sum_{j=1}^n \langle x,v_j\rangle_{\mathcal{B}}\langle u_j,y\rangle_{\mathcal{B}}\\ &= \sum_{j=1}^n \langle x, v_j.\langle u_j,y\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} = \sum_{j=1}^n \langle x, {}_{\mathcal{A}}\langle v_j,u_j\rangle.y\rangle_{\mathcal{B}} = [\Big] x\sum_{j=1}^n {}_{\mathcal{A}}\langle v_j,u_j\rangle.y_{\ \ \mathcal{B}} = \langle x,a^*.y\rangle_{\mathcal{B}}\\ &\text{i.e., } \langle a.x,y\rangle_{\mathcal{B}} = \langle x,a^*.y\rangle_{\mathcal{B}}.\end{aligned}$$

Similarly, $\mathcal{D}_{\mathcal{B}}$ is full, so we may set an element $b \in R(\mathcal{B})$ as $b = \sum_{j=1}^n \langle v_j,w_j\rangle_{\mathcal{B}}$, for some $v_j, w_j \in \mathcal{D}_{\mathcal{B}}$, $n \in \mathbb{N}$. Then we have that

$$\begin{aligned}{}_{\mathcal{A}}\langle x.b,y\rangle &= {}_{\mathcal{A}}[\Big] x.\sum_{j=1}^n \langle v_j,w_j\rangle_{\mathcal{B}} y = {}_{\mathcal{A}}[\Big]\sum_{j=1}^n {}_{\mathcal{A}}\langle x,v_j\rangle.w_j y = \sum_{j=1}^n {}_{\mathcal{A}}\langle x,v_j\rangle_{\mathcal{A}}\langle w_j,y\rangle\\ &= \sum_{j=1}^n {}_{\mathcal{A}}\langle x, {}_{\mathcal{A}}\langle y,w_j\rangle.v_j\rangle = \sum_{j=1}^n {}_{\mathcal{A}}\langle x, y.\langle w_j,v_j\rangle_{\mathcal{B}}\rangle = {}_{\mathcal{A}}[\Big] xy.\sum_{j=1}^n \langle w_j,v_j\rangle_{\mathcal{B}} = {}_{\mathcal{A}}\langle x,y.b^*\rangle.\\ &\text{i.e., } {}_{\mathcal{A}}\langle x.b,y\rangle = {}_{\mathcal{A}}\langle x,y.b^*\rangle.\end{aligned}$$

■

Remark 10 Equation (4) and Lemma 2 imply that the elements of $L(\mathcal{A})$ act as adjointable maps on $\mathcal{D}_{\mathcal{B}}$ and the elements of $R(\mathcal{B})$ act as adjointable maps on ${}_{\mathcal{A}}\mathcal{D}$.

Lemma 3 *Let* $(\mathcal{A},\tau_{\mathcal{A}})$ *and* $(\mathcal{B},\tau_{\mathcal{B}})$ *be two locally convex, semi-associative partial* **-algebras and let* $\mathcal{D}$ *be both* ${}_{\mathcal{A}}\mathcal{D}$ *and* $\mathcal{D}_{\mathcal{B}}$*. If* $\mathcal{D}$ *satisfies (10) of Lemma 2, then* $\mathcal{D}$ *satisfies Properties* (ii) *and* (iii) *of Definition 15.*

Proof. Suppose (10) holds for all $x,y \in \mathcal{D}$, $a \in L(\mathcal{A})$, $b \in R(\mathcal{B})$. Then the elements of $L(\mathcal{A})$ act as adjointable maps on $(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{\mathcal{D},\mathcal{B}})$, by Remark 10. Similarly, the elements of $R(\mathcal{B})$ act as adjointable maps on $(\mathcal{D}, {}_{\mathcal{A}}\langle\cdot,\cdot\rangle, \tau_{\mathcal{D},\mathcal{A}})$. The required result now follows, since the left and right module actions are continuous. ■

Corollary 2 *Let $\mathcal{D}$ be both a full (left) locally convex $(\mathcal{A}, \tau_{\mathcal{A}})$-module and a full (right) locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module satisfying Property (i) of Definition 15. Then $\mathcal{D}$ is an $(\mathcal{A}, \tau_{\mathcal{A}})$-$(\mathcal{B}, \tau_{\mathcal{B}})$-imprimitivity bimodule if and only if $\mathcal{D}$ satisfies (10) of Lemma 2.*

Proof. This follows from Lemmas 2 and 3. ■

Example 3 Let $(\mathcal{B}, \tau_{\mathcal{B}})$ be a complete locally convex semi-associative partial *-algebra and let $\mathcal{D} = M(\mathcal{B})$ such that $M(\mathcal{B})$ is an ideal of $\mathcal{B}$. Then $\mathcal{D}$ is both a left $L(\mathcal{B})$- and a right $R(\mathcal{B})$-module. $\mathcal{D}$ is also a $(\mathcal{B}, \tau_{\mathcal{B}}) - (\mathcal{B}, \tau_{\mathcal{B}})$-imprimitivity bimodule with the $\mathcal{B}$-valued inner products ${}_{\mathcal{B}}\langle a, b\rangle = ab^*$, $\forall a, b \in {}_{\mathcal{B}}\mathcal{D}$ and $\langle a, b\rangle_{\mathcal{B}} = a^*b$, $\forall a, b \in \mathcal{D}_{\mathcal{B}}$. To see this, it suffices by Corollary 2, to show that

(i) $\mathcal{D}$ satisfies Property (i) of Definition 15;
(ii) $\mathcal{D}$ satisfies (10) of Lemma 2;
(iii) $\mathcal{D}$ is full both as a (left) locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module and as a (right) locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module. Indeed:
  (i) For all $a, b, c \in \mathcal{D}$, ${}_{\mathcal{B}}\langle a, b\rangle c = (ab^*)c = a(b^*c) = a\langle b, c\rangle_{\mathcal{B}}$
  (ii) For all $a' \in L(\mathcal{B})$, $a, b \in \mathcal{D}$, $\langle a'a, b\rangle_{\mathcal{B}} = (a'a)^*b = (a^*a'^*)b = a^*(a'^*b) = \langle a, a'^*b\rangle_{\mathcal{B}}$. Also, for all $b' \in R(\mathcal{B})$, $a, b \in \mathcal{D}$, ${}_{\mathcal{B}}\langle ab', b\rangle = (ab')b^* = a(b'b^*) = a(bb'^*)^* = {}_{\mathcal{B}}\langle a, bb'^*\rangle$
  (iii) The locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-modules $L(\mathcal{B})$ and $R(\mathcal{B})$ are both full, by Proposition 3.

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