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In honour of Prof. Ekhaguere at 70

Some isomorphisms in K-theory

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Abstract. Let $F(n+1)$ be the complete flag manifold of length n. It is shown that the topological K - group $KO(F(n+1))$ is isomorphic to an algebraic K - group, with a condition given for this identification. Furthermore, an isomorphism is established connecting the Grothendieck group of a commutative ring R and the first algebraic K - group of an R - algebra S.

Keywords: Grothendieck group, algebraic $K-$ theory, topological $K-$ theory, flag manifold, first algebraic $K-$ theory group.

1. Introduction

The computations of K - groups, as well as the determination of relationships among them constitute an important problem in K - theory([1],[3],[4],[6],[10]). This paper investigates topological and algebraic K - groups with the objective of establishing isomorphism relationships among them. Let C be an additive category, then the set of isomorphism classes $\phi(C)$ of C is an abelian monoid. The group completion ([10],[14]) of $\phi(C)$ is called the Grothendieck group of C and is denoted by K_0C.

In algebraic K - theory, the group completion $K(P(R))$ of the category $P(R)$ of finitely generated projective modules over an arbitrary ring R as objects and R-linear maps as morphisms yields the Grothendieck group K_0R of the ring R, which is sometimes called the $0th$ K - group of R. We also have the first and second algebraic K - groups of a ring R, defined by $K_1R = GL(R)/E(R)$ and $K_2R = ker(\phi : St(R) \longrightarrow E(R) \subset GL(R))$ respectively([10]). The study of the groups $K_iR, i = 0, 1, 2$ constitutes the classical algebraic K - theory([10]). In topological K - theory, the group completion $K_0(\xi_{\mathbb{F}}(X))$ of the category $\xi_{\mathbb{F}}(X)$ of $\mathbb{F}$-vector bundles over paracompact spaces X and vector bundle morphisms gives rise to $K_0(\xi_{\mathbb{R}}(X)) = KO(X)$ and $K_0(\xi_{\mathbb{C}}(X)) = KU(X)$ when $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ respectively (page 38 of [6], page 89 of [14]).

In this paper, it has been proved that the topological K-group $KO(F(n + 1))$ of the complete flag manifold $F(n + 1)$ of length n is isomorphic to the algebraic K - group K_0A where A is a Noetherian subring of the ring of real valued continuous functions on $F(n + 1)$, provided that the Krull dimension of A and the length of $F(n + 1)$ satisfy a certain condition. Thus identifying a topological K - group with an algebraic K - group. It is also proved that the ring of all $\nu \times \nu$ matrices with entries from the Grothendieck group of a commutative ring R is isomorphic to the ring of all endomorphisms on the first algebraic K - group of an R - algebra S over the Grothendieck group of R. This relates K_0R and K_1S. In the sequel, some preliminary definitions and examples are followed by some isomorphism results.

2. Preliminary definitions and examples

Definition 1 *Let $\mathbb{F}$ denote the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Suppose $n_1, \cdots, n_s$ are fixed positive integers such that $n_1 + \cdots + n_s = n$. A 'flag' or more precisely a "$(n_1 \cdots n_s)$-flag over $\mathbb{F}$" is a collection σ of mutually orthogonal subspaces $(\sigma_1, \cdots, \sigma_s)$ in $\mathbb{F}^n$ such*

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that $dim_{\mathbb{F}}\sigma_i = n_i$. The space of all such flags forms a compact smooth manifold called the generalized real flag manifold (respectively complex flag manifold) for $\mathbb{F} = \mathbb{R}$ (respectively $\mathbb{F} = \mathbb{C}$) denoted by $G_{\mathbb{F}}(n_1, \cdots, n_s)$ or simply G([9]).

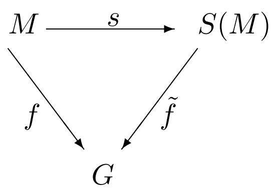
Example 1 *Examples of flag manifolds are the following*

- (i) *The flag manifold $G_{\mathbb{R}}(n_1, n_2)$ is called the Grassmannian of n_1 planes in Euclidean $n_1 + n_2$ space and is denoted by $G_{n_1}(\mathbb{R}^{n_1+n_2})$.*
- (ii) *The flag manifold $G_{\mathbb{R}}(1, n_2)$ is called the projective space in Euclidean $n_2 + 1$ space and is denoted by $\mathbb{R}P^{n_2}=G_1(\mathbb{R}^{n_2+1})$.*
- (iii) *Let E be a complex vector space of dimension n. A flag in E is a sequence of subspaces $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$, where $dimE_i = i$. The set of all such flags is called a complete flag manifold of length n and is denoted by $F(E)$, $W(0, 1, \cdots, n)$ or $F(n + 1)$.*
- (iv) *Given integers $q_1, q_2, \cdots, q_m$ such that $0 \leq q_1 \leq q_2 \leq \cdots \leq q_m$, a $(q_1, \cdots, q_m)$– flag is defined as a nested system*

$$S : S_{q_1} \subset S_{q_2} \subset \cdots \subset S_{q_m}, dimS_{q_i} = q_i,$$

of subspaces of S_n, the complex n-projective space. The set of all such flags is called an incomplete flag manifold in S_n and is denoted by $W(q_1, \cdots, q_m)$, where $q_1, \cdots, q_m$ are called the flag dimensions of $W(q_1, \cdots, q_m)$.
- (v) *$G_{\mathbb{F}}(\underbrace{1, 1, \cdots, 1}_{rtimes}, n - r)$ is called the incomplete flag manifold of length r.*

Definition 2 *Let M be an abelian monoid, there is associated with M an abelian group $S(M)$ and a homomorphism of the underlying monoids $s : M \longrightarrow S(M)$, having the following universal property: For any abelian group G, and any homomorphism of the underlying monoids $f : M \longrightarrow G$, there is a unique group homomorphism $\tilde{f} : S(M) \longrightarrow G$ such that the following diagram*



is commutative. The group $S(M)$ is called the group completion of M (See page 52 of [8]).

Let C be an additive category. Suppose $E \in ob(C)$, $\dot{E}$ the isomorphism class of E, then the set of all such isomorphism classes $\dot{E}$ denoted by $\phi(C)$ constitute an abelian monoid. The group completion of this monoid $\phi(C)$ is called the Grothendieck group of C, and is denoted by $K(C)$(see page 54 of [8]).

Example 2 (i) *Let A be an arbitrary ring with identity, and let $P(A)$ be the category with finitely generated projective A - modules as objects and the A-linear maps as morphisms. The Grothendieck group $K(P(A))$ of $P(A)$ is denoted by $K(A)$. A problem in "algebraic K-theory" is to compute $K(A)$ for interesting rings A.*

(ii) *Let $\xi(X)$ be the category of vector bundles over a compact space X. We denote the Grothendieck group $K(\xi(X))$ of $\xi(X)$ by $K(X)$. If the basic field $\mathbb{F}$ is $\mathbb{R}$ (respectively $\mathbb{C}$) we write $K_{\mathbb{R}} = KO(X)$ (respectively $K_{\mathbb{C}} = KU(X)$) for the group $K(X)$. A problem in "topological K - theory" is to compute $K(X)$ for interesting spaces X. It is in this way that topological K - theory arises as a special case of algebraic K - theory.*

Definition 3 (i) *The quotient group $GL(R)/E(R)$ is called the first algebraic K - theory group of the ring R with identity and is denoted by K_1R (See page 27 of [10]).*

(ii) *The second algebraic K - theory group of a ring R is defined as the kernel of an epimorphism $\phi : St(R) \longrightarrow E(R) \subset GL(R)$ (See page 29 of [10]).*

3. Isomorphism theorems

We prove the following isomorphism theorems.

THEOREM 1 *Let A be a Noetherian subring of the ring $C^0(F(n+1);\mathbb{R})$ of continuous functions $f : F(n+1) \longrightarrow \mathbb{R}$ and $K_0(A)$ the Grothendieck group of A. If A is of Krull dimension N, then there is an isomorphism $KO(F(n+1)) \cong K_0(A)$ provided $n = \frac{-1+\sqrt{8N+1}}{2}$.*

Proof. Since A is an affine ring of dimension N ([7]) and $F(n+1)$ is a finite CW-complex ([5]); it follows that $KO(F(n+1)) \cong K_0(A)$ provided

$$\text{Krull dim(A)=topl dim(F(n+1))}$$

(see corollaries 1 and 2 in section 3 of [12] and proposition 2.2 of [11]). It is known that $F(n+1)$ is of dimension $\frac{1}{2}n(n+1)$ ([13]). It follows that the desired result is obtained if $n = \frac{-1+\sqrt{8N+1}}{2}$. For $\frac{1}{2}(\frac{-1+\sqrt{8N+1}}{2})(\frac{-1+\sqrt{8N+1}}{2}+1) = N$.

■

THEOREM 2 *Let R be a commutative ring and S an R - algebra. Suppose that K_1R is a free module of rank ν. Then $End_{K_0R}(K_1S) \cong M_\nu(K_0R)$.*

Proof. It is known that there is a natural external product operation $K_0R \otimes K_1S \longrightarrow K_1(R \otimes S)$, and since R is commutative with S an R - algebra, there is a natural product operation $K_0(R) \otimes K_1(S) \longrightarrow K_1(S)$, making $K_1(S)$ into a module over the ring $K_0(R)$ (see Corollary 1.6.1 on page 204 of [14]). Since $K_1(S)$ is a free module of rank ν, it follows by Theorem 8.1.1 on page 310 of [2] that $End_{K_0(R)}(K_1S)$ is isomorphic to the ring of all row-finite $\nu \times \nu$ matrices over $K_0(R)$. ■

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