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In honour of Prof. Ekhaguere at 70

The effect of financial crisis in an investment portfolio with jump models: instigators and way forward

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Abstract. The financial crisis of 2008 has been a major crisis in nations economies and has posed alot of treat to investors' portfolio in recent time. This paper intend to give detail discussion on the causes of this crisis and provide a way forward in other to avoid future occurrence of the crisis. Some of the contributing factors of this crisis are improper debt and poor portfolio management and jump risks in an investment portfolio. In other to avoid future occurrence and bring confidence to investors, we considers the optimal net debt ratio in the presence of collateral security, optimal investment strategies and consumption plan of an investor who faces both diffusion and jump risks. The asset of the investor is divided into two parts: assets in financial market and fixed assets. Assets in financial market are invested into a market that is made up of a riskless asset and multiple risky assets. The fixed asset has a price that follows a jump-diffusion process. Also consider in this paper, is the impact of labor force with the production rate function of the labor market being stochastic. The objectives of this paper are to (i) maximize the total expected discounted utility of consumption in the infinite time horizon, (ii) determine the optimal net debt ratio for an investor under an economy that faces financial crisis, and (iii) determine the optimal investment strategies of an investor who invest in an economy that is exposed to both diffusion and jump risks. Furthermore, the optimal consumption and investment strategies as well as optimal net debt ratio under power utility function were obtained.

Keywords: Net debt ratio, collateral security, jump-diffusion risks, power utility, financial crisis.

1. Introduction

The subprime mortgage crisis which commenced in 2006 when there was a steep rise in home foreclosures and later became out of control in 2007, triggered the global financial crisis in 2008. This has resulted to so many questions on what led to the crisis and how should the stakeholders address these problems in order to avoid future occurrence. This crisis has resulted to so many challenges. Some of which are as follows: the consumption rate dropped drastically and the housing market went down, the number foreclosures rose and the stock market became vulnerable. Furthermore, the problems of subprime crisis and home foreclosure has resulted to disagreement among consumers, lenders, financial institutions and government. The mortgage crisis started with the bursting of the U.S. housing bubble in 2001 and reached its peak in 2005. Some stakeholders believed that an indiscriminate increase in the valuations of real properties was one the factors that contributed tremendously to the crisis. That is, wrong valuation of financial instruments posed a lot of threat to the financial system. The rapid increase in the valuation of real property was due to decreases in home prices and mortgage debt. The International Monetary Fund (IMF) reported in its semiannual Global Financial Stability Report released on April 8, 2008, that the fall in U.S. housing prices and increase in debt on the residential mortgage market could lead to losses of $565 billion, excluding losses from other categories of loans and securities. It was further asserted that in U.S., the total potential losses was at about $945 billion, see [1].

One of the causes of financial crisis was that many firms were unable to pay their debt. But, [2] asserted that the primary cause of the recession was the credit crisis resulting from the bursting of the housing bubble. He enumerated four primary causes of the housing bubble to include: low mortgage interest rates, low short-term interest rates, relaxed standards for mortgage loans, and irrational exuberance. He concluded that the combination of the above factors led to severity of the housing bubble and the resulting credit crisis.

The increase in demand for housing in U.S. due to high expectation of high return in future led to increase in mortgage value. Many property speculators took advantage of low interest rate by purchasing houses with high values with the mind to resell at high price in future date. [3] stated that the low interest rates and rises in housing prices induced a substantial demand for mortgages and an unsustainable excess debt. He asserted that financial institutions were busy chasing high returns thereby overusing the leverage tool and lowering the quality of mortgages. He further asserted that high leverage can make the financial market vulnerable. When the housing bubble collapse, there was high default rates on subprime, unemployment crisis, decline in productivity, collapse of many businesses, reduction in consumer income and families experienced hardship. The mortgage loans became vulnerable especially the loans made to higher-risk borrowers with lower income or lesser credit history, which led to high level debt. These exposed lenders to more risks. In other to minimize these risks that lenders are exposed to, it is vital to permits a borrower to offer his or her home or any other valuable asset(s) as collateral in case of failure to repay the loan, which is referred to as equity loan.

One of the major factor that led to the increase in home ownership rates and housing bubble was the rise of subprime borrowing in U.S. The U.S. home ownership rate increased from 64% in 1994 and rise to an all-time high peak of 69.2% in 2004. As we know in economics principle, the higher the demand of housing will lead to price hike. The demand for housing led to the rise of housing prices thereby increases consumer spending and home values to 124%between 1997 and 2006, see [1]. As a result of this, U.S. household debt increase with over 30% in 2007. Borrowers enjoy taking loan but the challenge is to pay back. Most financial institutions give out loan to costumers without backed up security. Some of these borrowers may default which may result to increase in debt rate. The major challenge is how to recover these loans and minimize the debt ratio. Some of these debt may result to bad debt due to the following factors: if (i) the loan is mismanaged; (ii) the borrower dies and no members of the family are able to pay back the loan; and (iii) the investment of the borrower is adversely affected by natural disaster, government policies and other uncontrollable factors. Hence, the need for a back-up security on the loan given, which we referred to in this paper, as collateral security. Introduction of collateral security will go a long way in reducing the incident of bad debts in an investment portfolio. The accumulation of these bad debts may have resulted to the global financial crisis in 2008.

In order to reduce the investment risks in an investment portfolio there is the need for investment diversification. In this paper, we consider an investor's problem who chooses to invest his or her asset in financial market and fixed assets. The underlying assets in the financial market are riskless and risky assets (stocks). One of the aims of the investor is to maximize the total expected discounted utility of consumption over an infinite time horizon in the presence of financial crisis under three control variables: optimal net debt ratio; optimal portfolio strategies and optimal consumption plan. The investor is risk averse and chooses constant relative risk averse (CRRA) utility function. In this paper, the fixed asset price and stock prices follow a jump-diffusion process with time dependent drifts. The loan risks are reduced by ensuring that the borrower present a collateral security on the loan, hence collateral security is introduce in our model. [3] considered the optimal debt ratio and consumption plan for an economy that faces financial crisis. The impact of labour market condition was put into consideration. He assumed the production rate function to be stochastic and influence by the government policy and unanticipated risks. His aim was to maximize the total expected discounted utility of consumption in the infinite time horizon under optimal debt ratio and consumption plans. [4] studied a stochastic optimal control model, optimal debt ratio management and estimation. They considered productivity of capital, asset return, interest rate, and market regime switches. They maximize the utility of terminal wealth by choosing the optimal debt ratio. They further considered hidden Markov chain technique and nonlinear filtering technique in order to estimate the actual situation of the strategy.

Some work have been done in the area of optimal stochastic control and optimal debt management. [5] considered stochastic optimal control and dynamic programming technique to derived an optimal debt. He asserted that the deviation of the actual debt from the optimal, will serve as a warning signal of a financial crisis. [6] explained how mathematical techniques of stochastic optimal control can be applied to mortgage crisis. The optimal debt ratio was derived while the drift term is allowed to be probabilistic and subject to some economic constraints. He asserted that the crisis occurred because the market failed to put into consideration pertinent economic constraints in the dynamic stochastic optimization. He further derived the excess debt ratio. [7] applied mathematical model of economic system to the development of financial crisis. He proposed that the severity of financial crisis can be determined by means of superposition of the fluctuation on connected markets that possesses a resonance behavior.

In the area of jump-diffusion process, see [8], [9], [10], [11]. [12] considered an insurer who faces an external jump-diffusion risk that is negatively correlated with the capital returns in a regime switching model, investment and liability ratio policies were selected in such away as to maximize the investor's expected utility of terminal wealth. [13] proposed a model that captured the dynamics of asset returns with periods of financial crises which are characterized by contagion in the presence of jump processes. They further proposed filtered values of the jump intensities as a measure of market stress and they examined the out-of-sample forecasting abilities. [14] considered optimal consumption and investment with bounded downside risk for an investor with power utility functions in the presence of jump-diffusion process. They suggested that there is the need to impose on the investor's portfolio a stricter constraint which depends on the probability of having negative jumps in the assets in the entire time period. [15] studied the optimal portfolio selection problem in jump-diffusion models in the presence of a large number of assets. They derived a closed form solution for the optimal portfolio weights by solving a set of ordinary differential equations.

The remainder of the paper is organized as follows. The financial models, dynamics of asset returns, the asset value, debt process and collateral security are presented in section 2. Section 3 deals with the dynamics of the net wealth process and the income rate of the investor. The optimal controls, value functions, optimal investment, optimal net debt ratio, optimal consumption and the verification theorem are presented in section 4. Section 5 concludes the paper.

2. The Models

This paper consider two classes of assets: the assets in the financial market and fixed assets. Before describing these assets, we first define our probability space.

2.1 *Our probability space*

In this subsection, we begin by describing our financial models. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\bar{\mathbf{W}}(t) = (\mathbf{W}_S(t), \mathbf{W}_P(t), \mathbf{W}_\beta(t))' = (\mathbf{W}(t), \mathbf{W}_\beta(t))'$ defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where

$$\mathbf{W}(t) = (\mathbf{W}_S(t)\mathbf{W}_P(t)),$$

$\mathcal{F}_t = \sigma(\bar{\mathbf{W}}(s) : s \leq t)$ and $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0, T]\}$, is a $n + m + d$-dimensional Brownian motions with respect to stock market risks, return from fixed asset and an income growth risks, respectively at time, t and $\mathbf{W}(t)$ is a $m + n$-dimensional Brownian motions with respect to stock market source of risks $\mathbf{W}_S(t)$, n-dimensional Brownian motions, price of fixed asset source of risks $\mathbf{W}_P(t)$, an m-dimensional Brownian motions and the income growth rate source of risk $\mathbf{W}_\beta$(t) is the an d-dimensional Brownian motions at time t. $\mathbf{P}$ denotes the real world probability measure, the sign "$'$", denotes transpose.

2.2 *Dynamics of assets return in financial market*

Consider Merton's problem of maximizing the infinite-horizon of expected utility of consumption by investing the amount $R(t)$ in a market that is characterized by a set of risky assets and a riskless asset. In other words, the investor selects the amounts to be invested in the n risky assets with prices $S(t) = [S_1(t), \ldots, S_n(t)]'$ and a riskless asset with price $S_0(t)$ at time $t \in [0, \infty)$ as well as his or her consumption path.

The stock prices follow the exponential Lévy process. The dynamics of the underlying assets are as follows:

$$dS_0(t) = r(t)S_0(t)dt, \tag{1}$$

$$dS_i(t) = S_i(t_-)\left(\mu_i(t)dt + \sum_{j=1}^{n}\sigma_{i,j}dW_{j,S}(t) + \sum_{l=1}^{m_S}J_{i,l}dY_{l,S}(t)\right), i = 1,\ldots,n \tag{2}$$

with a rate of interest $r(t) \geq 0$, the expected growth rate of stocks $\mu_i(t) > 0$, $i = 1,\ldots,n$ and volatilities of stocks $\{\sigma_{i,j}\}_{i,j=1}^{n}$. $\mathbf{W}_S(t) = [W_{1,S}(t),\ldots,W_{n,S}(t)]'$ is an n-dimensional standard Brownian motions. $\mathbf{Y}_S(t) = [Y_{1,S}(t),\ldots,Y_{m_S,S}(t)]'$ is an m_S-dimensional Lévy pure jump process with Lévy measure $\lambda_l\nu_l(dz_l)$, where $\lambda_l \geq 0, l = 1,\ldots,m_S$ is a fixed parameter and the measure ν satisfies $\int_{\mathcal{R}}\min(1,|z_l|)\nu(dz_l) < \infty, l = 1,\ldots,m_S$, so the jumps have finite variation. The constant $J_{i,l}$ is asset $i's$ scaling of $l's$ jump. We assume that $r(t)$ is bounded so that there exists $\kappa_r > 0$ such that

$$\forall r(t) \in [0,T]\times\mathcal{R}, |r(t)| \leq \kappa_r. \tag{3}$$

We assume that the Lévy pure jump processes $\mathbf{Y}_S(t)$ and the Brownian motions $\mathbf{W}_S(t)$ are mutually independent and that the sum of the jumps has support on $(-1,\infty)$ to guarantee the positivity of S_i.

We now re-write (2) as follows:

$$dS(t) = S(t_-)\left(\mu(t)dt + \sigma d\mathbf{W}_S(t) + \mathbf{J}_l d\mathbf{Y}_S(t)\right), \tag{4}$$

where $\mu(t) = [\mu_1(t),\ldots,\mu_n(t)]'$, $\Sigma = \sigma\sigma'$ and

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \ldots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \ldots & \sigma_{n,n} \end{pmatrix},$$

$$\mathbf{J}_l = \begin{pmatrix} J_{1,l} & \ldots & J_{1,m_S} \\ \vdots & \ddots & \vdots \\ J_{n,1} & \ldots & J_{n,m_S} \end{pmatrix}.$$

We assume that Σ is a nonsingular matrix and positive definite matrix. The amount of fund invested in stock, $S(t)$ at time, t is denoted by $\Delta(t) = [\Delta_1(t),\ldots,\Delta_n(t)]$ and the remainder $\Delta_0(t) = R(t) - \Delta(t)\mathbf{I}$ is invested in riskless asset, where $\mathbf{I} = [1,\ldots,1]' \in \mathcal{R}^n$.

The wealth dynamics of assets in the financial market is given as follows:

$$dR(t) = \Delta_0\frac{dS_0(t)}{S_0(t)} + \Delta(t)\frac{dS(t)}{S(t_-)}, R(0) = R_0 \in \mathcal{R}_+. \tag{5}$$

Using (1) and (4) on (5), we have

$$\begin{aligned} &dR(t) = (r(t)R(t_-) + \Delta(t)(\mu(t) - r(t)\mathbf{I}))dt + \Delta(t)\sigma d\mathbf{W}_S(t) + \Delta(t)\mathbf{J}_l d\mathbf{Y}_S(t), \\ &R(0) = R_0 \in \mathcal{R}_+. \end{aligned} \tag{6}$$

2.3 *The fixed asset price*

Here, we consider the fixed asset and its prices process at time t. Let the fixed asset be equal to the product $P(t)Q(t)$, where $P(t)$ is the price of the fixed asset and $Q(t)$ the quantity of the fixed assets at time t. We assume that the fixed asset price $P(t)$ in the financial market satisfies the jump-diffusion process

$$dP(t) = P(t_{\text{-}})\left(\alpha(t)dt + \sigma_P d\mathbf{W}_P(t) + \mathbf{J}_P d\mathbf{Y}_P(t)\right), \tag{7}$$

where $\alpha : \mathcal{R} \times [0,T] \longrightarrow \mathcal{R}$ is the return rate of the fixed asset and is time dependent because it changes overtime due to some macro- and micro-economic factors such as inflation, government policies, natural effects, economic growths, e.t.c. T the terminal time, the vector $\sigma_P = [\sigma_1, \dots, \sigma_m]$ is the corresponding volatility, $\mathbf{W}_P(t) = [W_{1,P}(t), \dots, W_{m,P}(t)]'$ is an m-dimensional Brownian motions and the constant vector $\mathbf{J}_P = [J_{1,P}, \dots, J_{m_P,P}]$ is scaling factor of the fixed asset price jump. $\mathbf{Y}_P(t) = [Y_{1,P}(t), \dots, Y_{m_P,P}(t)]'$ is an m_P-dimensional Lévy pure jump process with Lévy measure $\lambda_{i,P}\nu_{i,P}(dz_{i,P})$, where $\lambda_{i,P} \geq 0, i = 1, \dots, m_P$ is a fixed parameter and the measure ν satisfies $\int_{\mathcal{R}} \min(1, |z_i|)\nu(dz_l) < \infty, i = 1, \dots, m_P$, so the jumps have finite variation. The constant $J_{i,P}$ is asset $i's$ scaling of $i's$ jump. We assume that α is bounded so that there exists $\kappa_\alpha > 0$ such that

$$\forall \alpha(t) \in [0,T] \times \mathcal{R}, |\alpha(t)| \leq \kappa_\alpha. \tag{8}$$

We assume that the Lévy pure jump process $\mathbf{Y}_P(t)$ and the Brownian motion $\mathbf{W}_P(t)$ are mutually independent and that the sum of the jumps has support on $(-1, \infty)$ to guarantee the positivity of P.

2.4 *The asset value*

Let $K(t)$ be the asset values at time t which is the sum of assets in financial market (AFM) $R(t)$ and fixed asset $P(t)Q(t)$. It then follows that the asset value is

$$K(t) = R(t) + P(t)Q(t), \tag{9}$$

where $R(t)$ satisfies (6) and $P(t)$ satisfies (7). It therefore follows that the change in the asset value is

$$dK(t) = dR(t) + P(t)dQ(t) + dP(t)Q(t) = dR(t) + P(t)dQ(t) + \frac{dP(t)}{P(t_{\text{-}})}(K(t) - R(t)). \tag{10}$$

(10) now becomes

$$dK(t) = dR(t) + dP(t)Q(t) = dR(t) + \frac{dP(t)}{P(t)}(K(t) - R(t)) + P(t)dQ(t). \tag{11}$$

Substituting (6) and (7) into (11), we have

$$\begin{aligned} dK(t) &= [r(t)R(t_{\text{-}}) + \Delta(t)(\mu(t) - r(t)\mathbf{I}) + (K(t) - R(t))\alpha(t)]dt + P(t)dQ(t) \\ &+(\Delta(t)\sigma, \sigma_P(K(t) - R(t)))d\mathbf{W}(t) + (\Delta(t)\mathbf{J}_l, \mathbf{J}_P(K(t) - R(t)))d\mathbf{Y}(t), \end{aligned} \tag{12}$$

where $\mathbf{W}(t) = [\mathbf{W}_S(t), \mathbf{W}_P(t)]'$, $\mathbf{Y}(t) = [\mathbf{Y}_S(t), \mathbf{Y}_P(t)]'$, and $\mathbf{Y}_S(t)$ and $\mathbf{Y}_P(t)$ are assume to be independent. That is

$$\rho_S^Y \equiv Cov[\mathbf{Y}_S(t), \mathbf{Y}_S(t)]; \rho_P^Y \equiv Cov[\mathbf{Y}_P(t), \mathbf{Y}_P(t)]; 0 \equiv Cov[\mathbf{Y}_P(t), \mathbf{Y}_S(t)]. \tag{13}$$

It is imperative to note that the change of total asset value is composed of two parts. The quantity, which is the investment part of AFM $[r(t)R(t) + \Delta(t)(\mu(t) - r(t)\mathbf{I})]dt + P(t)dQ(t) + \Delta(t)\sigma d\mathbf{W}_S(t) + \Delta(t)\mathbf{J}_l d\mathbf{Y}_S(t)$ is change due to the change in assets in the financial market at time t. The quantity

$(K(t)-R(t))[\alpha(t)dt+\sigma_P d\mathbf{W}_P(t)+\mathbf{J}_P d\mathbf{Y}_P(t)]$ is due to the price change $dP(t)$ which brings about the capital gain or loss of the fixed asset value at time t.

In next subsection, we consider the debt process and collateral security which give rise to the net debt process in the investment.

2.5 *The debt process*

Suppose the investor decides to borrow some amount of money $L(t)$ with interest rate $r(t)$ at time t. It is also assume that the investor consumes continuously at the rate $C(t)$ at time t such that

$$dC(t) = C(t)dt. \tag{14}$$

Let $I(t)$ be the income process with a growth rate $\beta(t)$, define as the product of the income growth rate and asset value, see [3]. Then change in income is given by

$$dI(t) = \beta(t)K(t)dt. \tag{15}$$

We can now define our change in debt dynamics using (14) and (15) as follows:

$$dL(t) = r(t)L(t)dt + dC(t) + P(t)dQ(t) - dI(t). \tag{16}$$

2.6 *The collateral security*

The amount borrowed is assume to be back up with a collateral security. Let $G(t)$ be the value of the security on the debt $L(t)$ at time t such that

$$dG(t) = \xi(t)G(t)dt,$$

where $\xi(t)$ is the expected growth rate of the security. We must express (16) in terms of $G(t)$. Since the amount (or asset) $G(t)$ is to be removed from the loan $L(t)$, it then follows that the following holds:

$$d(L(t)-G(t)) = r(t)L(t)dt + dC(t) + P(t)dQ(t) - dI(t) - dG(t). \tag{17}$$

Let $Z(t) = L(t) - G(t)$ be the net debt at time t. We now re-write (17) as follows:

$$dZ(t) = r(t)Z(t)dt + (r(t)-\xi(t))G(t)dt + P(t)dQ(t) + C(t)dt - \beta(t)K(t)dt, \tag{18}$$

which is the dynamics of our net debt at time t. One of the aims of this paper is to optimize the net debt $Z(t)$ not $L(t)$, since the later has to a great extent protected by $G(t)$.

3. The dynamics of the net wealth process

DEFINITION 1 *Let $X(t)$ be the net wealth process of the economy at time t defined as the difference between the asset values $K(t)$ and debt $L(t)$ at time t. Mathematically,*

$$X(t) = K(t) - L(t).$$

PROPOSITION 1 *Suppose $X(t)$ is the net wealth process of the economy, then*

$$\begin{array}{l}\frac{dX(t)}{X(t_-)} = [r(t)(1-p(t)) + \pi(t)(\mu(t) - r(t)\boldsymbol{I})\\ +(1+z(t)+g(t))(\alpha(t)+\beta(t)) - c(t)]dt\\ +(\pi(t)\sigma, (1+z(t)+g(t))\sigma_P)d\boldsymbol{W}(t)\\ +(\pi(t)\boldsymbol{J}_l, \boldsymbol{J}_P(1+z(t)+g(t)))d\boldsymbol{Y}(t),\\ X(0) = x_0 > 0.\end{array} \tag{19}$$

Proof. Given $X(t)$ as the net wealth process of the economy. By definition 1, we have that

$$X(t) = K(t) - L(t) = R(t) + P(t)Q(t) - L(t). \tag{20}$$

Finding the differential of both sides of (20), we have the dynamics of the net wealth process as follows:

$$dX(t) = dK(t) - dL(t) = dK(t) - d(Z(t) + G(t)). \tag{21}$$

Substituting (12) and (18) into (21), we have the following:

$$\begin{array}{l}dX(t) = [r(t)(R(t_-) - Z(t) - G(t)) + \Delta(t)(\mu(t) - r(t)\mathbf{I}) + (K(t) - R(t))(\alpha(t)\\ +\beta(t)) - C(t)]dt + (\Delta(t)\sigma, \sigma_P(K(t) - R(t)))d\mathbf{W}(t) + (\Delta(t)\mathbf{J}_l, \mathbf{J}_P(K(t) - R(t)))d\mathbf{Y}(t),\\ X(0) = x_0 > 0.\end{array} \tag{22}$$

But $K(t_-) = X(t_-) + Z(t) + G(t)$, substituting it into (22), we have

$$\begin{array}{l}dX(t) = [r(t)(R(t_-) - Z(t) - G(t)) + \Delta(t)(\mu(t) - r(t)\mathbf{I})\\ +(X(t_-) + Z(t) + G(t) - R(t))(\alpha(t) + \beta(t)) - C(t)]dt\\ +(\Delta(t)\sigma, (X(t_-) + Z(t) + G(t) - R(t))\sigma_P)d\mathbf{W}(t)\\ +(\Delta(t)\mathbf{J}_l, \mathbf{J}_P(X(t_-) + Z(t) + G(t) - R(t)))d\mathbf{Y}(t),\\ X(0) = x_0 > 0.\end{array} \tag{23}$$

It is observed that $R(t_-) - Z(t) - G(t) = X(t) - P(t)Q(t)$. Substituting it into (23), we have the following

$$\begin{array}{l}\frac{dX(t)}{X(t_-)} = [r(t)(1-p(t)) + \pi(t)(\mu(t) - r(t)\mathbf{I})\\ +(1+z(t)+g(t))(\alpha(t)+\beta(t)) - c(t)]dt\\ +(\pi(t)\sigma, (1+z(t)+g(t))\sigma_P)d\mathbf{W}(t)\\ +(\pi(t)\mathbf{J}_l, \mathbf{J}_P(1+z(t)+g(t)))d\mathbf{Y}(t)\\ X(0) = x_0 > 0,\end{array} \tag{24}$$

where
$p(t) = \frac{P(t)Q(t)}{X(t_-)}$ is the proportion of wealth invested in fixed asset at time t,
$\pi(t) = \frac{\Delta(t)}{X(t_-)}$ is the proportion of wealth invested in risky assets at time t,
$z(t) = \frac{Z(t)}{X(t_-)}$ is the net debt ratio at time t,
$g_G(t) = \frac{G(t)}{X(t_-)}$ is the collateral security ratio at time t,
$g_R(t) = \frac{R(t)}{X(t_-)}$ is the proportion of wealth invested in the financial market at time t,
$g(t) = g_G - g_R$ is the ratio of the "net income spread (NIS)" at time t,
$c(t) = \frac{C(t)}{X(t_-)}$ is the consumption ratio of the investor at time t. □

Note that $G(t)$ is the using asset while $R(t)$ holding securities.

We assume that $g(t)$ and $p(t)$ are bounded such that

$$\forall g(t), p(t) \in [0,T] \times \mathcal{R}, |g(t)| \leq \kappa_g, |p(t)| \leq \kappa_p, \tag{25}$$

where κ_g and κ_p are constants.

3.1 *The income rate of the investor*

We assume that income growth rate is risky and is a stochastic process. It is affected by economic, scientific, human, environmental and political factors. All these factors can bring about random shocks on the income growth rate at time t. The rate of unemployment was identified by [3] as a major factor that influence the income rate. He asserted that when the unemployment rate ω is low, the economy can be seen to be expanding, and production rate will be higher than expected, while increase in unemployment rate ω can bring about recession, and the production rate will be lower than expected. Here, we assume the income growth rate $\beta(t)$ follows the dynamics

$$\begin{aligned} d\beta(t) &= [f(\beta(t)) + \beta(t)\eta(\omega)]dt + \sigma_\beta d\mathbf{W}_\beta(t), \\ \beta(0) &= \beta_0, \end{aligned} \tag{26}$$

where $f(\beta(t)) : \mathcal{R} \longrightarrow \mathcal{R}$ is the expected production rate and is a solution to the ordinary differential equation

$$df(\beta(t)) = \epsilon f(\beta(t))dt, f(\beta(0)) = 1, \tag{27}$$

ϵ is a constant, $\eta(\omega) : \mathcal{R} \longrightarrow \mathcal{R}$ is a continuous function in ω and represents the effect of unemployment rate to production process. According to [3], if $\eta(\omega) > 0$, it implies that the economy is expanding, if $\lambda_1(\omega) < 0$, it implies that the economy in recession and if $\eta(\omega) = 0$, it implies that the economy is in critical position. The vector $\sigma_\beta = [\sigma_{1,\beta}, \ldots, \sigma_{d,\beta}]$ is the volatility of the income rate, $\mathbf{W}_\beta(t) = [W_{1,\beta}(t), \ldots, W_{d,\beta}(t)]'$ is an d-dimensional Brownian motions of the income growth rate. The vector $\rho = (\rho_1, \rho_2)$ is the instantaneous correlation between the growth rate of asset value risks (for stock and fixed asset price risks) and income growth rate risks. From now on, we assume that $n = m = d$.

4. The Optimal Controls and Value Function

In this section, we consider the admissible strategies, optimal controls and value function.

4.1 *The admissible strategies*

For admissible net debt ratio $z(t)$, we assume that for all $T \in (0, \infty)$,

$$E\int_0^T z(t)^2 dt < \infty. \tag{28}$$

For admissible portfolio strategy $\pi(t)$, $t \in (0, \infty)$, we have

$$E\int_0^T \pi(t)\pi(t)' dt < \infty. \tag{29}$$

For admissible consumption strategy $c(t)$, $t \in (0, \infty)$, we have

$$E\int_0^T c(t)^2 dt < \infty. \tag{30}$$

A strategy $u(\cdot) = \{(z(t), \pi(t), c(t)) : t \geq 0\}$ which is progressively measurable with respect to $\{\mathbf{W}(t), \mathbf{W}_\beta(t) : 0 \leq s \leq t\}$ is referred to as admissible strategy. Denote the collection of all admissible strategies by $\mathcal{A}$. It then follows that the set of all admissible strategy $\mathcal{A}$ can be defined as

$$\begin{aligned}&\mathcal{A} = \{u(t) = (z(t), \pi(t), c(t)) \in \mathcal{R} \times \mathcal{R}^n \times \mathcal{R} : E \int_0^T z(t)^2 dt < \infty;\\ &E \int_0^T \pi(t)\pi(t)' dt < \infty; E \int_0^T c(t)^2 dt < \infty\}.\end{aligned} \tag{31}$$

4.2 *The Optimal Controls and Value Function*

In this section, we consider the optimal controls and the corresponding value function. Setting $X(t) = x$ and $\beta(t) = \beta$, we define the value function as follows:

$$L(x, \beta) := \sup_{u \in \mathcal{A}} F(x, \beta; u), \tag{32}$$

where

$$F(x, \beta; u) = E_{x,\beta} \int_0^\infty e^{-\delta t} V(c(t)x(t)) dt, \tag{33}$$

$\delta > 0$ is the discounted rate and $V(c(t)x(t))$ utility of consumption.

The investor's problem at time t is to select the net debt rate, portfolio weight and consumption rate processes $\{z(s), \pi(s), c(s)\}_{t \le s \le \infty}$ which maximize the infinite horizon, expected utility of consumption

$$\begin{cases} L(X(t), \beta(t)) = \max_{\{z(s), c(s), \pi(s): t \le s \le \infty\}} E_t \left[\int_t^\infty e^{-\delta s} V(c(s)x(s)) ds \right], \\ s.t. \\ d \begin{bmatrix} \beta \\ X \end{bmatrix} = \bar{m} dt + \bar{M} \begin{bmatrix} d\mathbf{W}_\beta \\ d\mathbf{W} \end{bmatrix} + \Phi \begin{bmatrix} 0 \\ d\mathbf{Y} \end{bmatrix}, \\ X(0) = x_0, \beta(0) = \beta_0, \end{cases} \tag{34}$$

where

$$\bar{m} = \begin{bmatrix} f(\beta(t)) + \beta(t)\eta(\omega) \\ \\ X[r(t)(1 - p(t)) + \pi(t)(\mu(t) - r(t)\mathbf{I}) \\ +(1 + z(t) + g(t))(\alpha(t) + \beta(t)) - c(t)] \end{bmatrix},$$

$$\bar{M} = \begin{bmatrix} \sigma_\beta \\ X(\pi(t)\sigma, (1 + z(t) + g(t))\sigma_P) \end{bmatrix},$$

$$\Phi = \begin{bmatrix} 0 \\ X(\pi(t)\mathbf{J}_l, \mathbf{J}_P(1 + z(t) + g(t))) \end{bmatrix}.$$

We now consider an investor who chooses power utility function. Using stochastic dynamic programming approach and Itó lemma for semimartingale processes, the Hamilton-Jacobi-Bellman equation characterizing the optimal solutions to the investor's problem is

$$\begin{aligned}&\mathcal{L}^u(L(x, \beta)) = \tfrac{1}{2} x^2 (\pi\sigma + (1 + z + g)\sigma_P)(\pi\sigma + (1 + z\\ &+g)\sigma_P)' L_{xx} + \tfrac{1}{2}\sigma_\beta \sigma_\beta' L_{\beta\beta} + x(\rho_1 \pi\sigma + \rho_2(1 + z + g)\sigma_P)\sigma_\beta' L_{x\beta}\\ &+x[r(1 - p) + \pi(\mu - r\mathbf{I}) + (1 + z + g)(\alpha + \beta) - c] L_x\\ &+[f(\beta) + \beta\eta(\omega)] L_\beta + \lambda_S \int_{\mathcal{R}} [L(x + x\pi \mathbf{J}_l \mathbf{I}, \beta) - L(x, \beta)] \nu(dz)\\ &+\lambda_P \int_{\mathcal{R}} [L(x + x\mathbf{J}_P \mathbf{I}(1 + z + g), \beta) - L(x, \beta)] \nu_P(dz_P),\end{aligned} \tag{35}$$

with the transversality condition $\lim_{t\to\infty} E[L(t,X(t),\beta(t))] = 0$, where $L_x = \frac{\partial L}{\partial x}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{xx} = \frac{\partial^2 L}{\partial x^2}$, $L_{x\beta} = \frac{\partial^2 L}{\partial x \partial \beta}$, $L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}$. The standard time-homogeneity argument for infinite-horizon problems gives that

$$\begin{aligned} L(X(t),\beta(t)) &= \max_{\{z(s),c(s),\pi(s):t\le s\le\infty\}} E_t\left[\int_t^\infty e^{-\delta(s-t)}V(c(s)x(s))ds\right] \\ &= \max_{\{z(t-u),c(t-u),\pi(t-u):t\le s\le\infty\}} E_t\left[\int_0^\infty e^{-\delta u}V(c(t+u)(x(t+u))du\right] \\ &= \max_{\{z(u),c(u),\pi(u):0\le u\le\infty\}} E_0\left[\int_0^\infty e^{-\delta u}V(c(u)(x(u))du\right] \\ &\equiv U(X(t),\beta(t)), \end{aligned}$$

which is independent of t. The third equality in the above argument makes use of the fact that the optimal control is indeed Markovian. It then follows that $L(X(t),\beta(t)) = e^{-\delta t}U(X(t),\beta(t))$ and (35) reduces to the following equation for the time-homogeneous value function U:

$$\begin{array}{l} \mathcal{L}^u(U(x,\beta)) = V(c(t)(x(t)) - \delta U(x,\beta) + \frac{1}{2}x^2\pi\sigma(\pi\sigma)'U_{xx} + x^2(1+z+g)\pi\sigma\sigma_P' U_{xx} \\ +\frac{1}{2}x^2(1+z+g)^2\sigma_P\sigma_P'U_{xx} + \rho_1 x\pi\sigma\sigma_\beta' U_{x\beta} + (1+z+g)\rho_2 x\sigma_P\sigma_\beta' U_{x\beta} \\ +\frac{1}{2}\sigma_\beta\sigma_\beta' U_{\beta\beta} + x[r(1-p) + \pi(\mu - r\mathbf{I}) + (1+z+g)(\alpha+\beta) - c]U_x \\ +[f(\beta) + \beta\eta(\omega)]U_\beta + \lambda_S\int_{\mathcal{R}}[U(t,x+x\pi\mathbf{J}_l\mathbf{I},\beta) - U(t,x,\beta)]\nu(dz) \\ +\lambda_P\int_{\mathcal{R}}[U(t,x+x\mathbf{J}_P\mathbf{I}(1+z+g),\beta) - U(t,x,\beta)]\nu_P(dz_P), \end{array} \tag{36}$$

with the transversality condition $\lim_{t\to\infty} E[e^{-\delta t}U(X(t),\beta(t))] = 0$.

The maximization problem in (36) separates into one for $z(t)$, with first-order condition

$$\begin{array}{l} x^2 z\pi\sigma\sigma_P' U_{xx} + \frac{1}{2}x^2(2z + 2zg + z^2)\sigma_P\sigma_P' U_{xx} + z\rho_2 x\sigma_P\sigma_\beta' U_{x\beta} \\ +xz(\alpha+\beta)U_x + \lambda_P\int_{\mathcal{R}}[U(t,x+x\mathbf{J}_P\mathbf{I}(1+z+g),\beta) - U(t,x,\beta)]\nu_P(dz_P), \end{array} \tag{37}$$

one for $c(t)$:

$$\frac{\partial V(c(s)x(s))}{\partial c} = xU_x \tag{38}$$

and one for $\pi(t)$:

$$\begin{array}{l} \frac{1}{2}x^2\pi\sigma(\pi\sigma)'U_{xx} + x^2(1+z+g)\pi\sigma\sigma_P' U_{xx} + \rho_1 x\pi\sigma\sigma_\beta' U_{x\beta} \\ +x\pi(\mu - r\mathbf{I})U_x + \lambda_S\int_{\mathcal{R}}[U(t,x+x\pi\mathbf{J}_l\mathbf{I},\beta) - U(t,x,\beta)]\nu(dz). \end{array} \tag{39}$$

For a given wealth x, the optimal consumption choice is therefore

$$c^*(t) = \left[\frac{\partial V}{\partial c}\right]^{-1}(xU_x). \tag{40}$$

In order to determine the optimal debt ratio, optimal consumption plan, optimal portfolio weights, wealth and value function, we have to be more specific about the utility function V.

4.3 *Power utility*

Consider an investor with the following power utility, $V(cx) = \frac{(cx)^{1-\gamma}}{1-\gamma}$ for $cx > 0$ and $V(cx) = -\infty$ for $cx < 0$ with CRRA coefficient $\gamma \in (0,1)\cup(1,\infty)$. We now state formally the value function satisfies the HJB equation (36):

$$\max_u\{\mathcal{L}^u U(x,\beta) - \delta U(x,\beta) + V(cx)\} = 0. \tag{41}$$

The discounted rate $\delta > 0$ represents the preference rate of consumption. Now, for an arbitrary admissible strategy $u = (z, \pi, c)$, the objective function $F(\cdot)$ follows:

$$F(x, \beta, u) = E_{x,\beta} \int_0^\infty e^{-\delta t} V(c(t)x(t))dt. \tag{42}$$

We will assume a solution to (36) in the form

$$U(x, \beta) = \frac{x^{1-\gamma} e^{h(\beta)}}{1-\gamma}, \tag{43}$$

so that

$$\begin{aligned} \frac{\partial U(x,\beta)}{\partial x} &= (1-\gamma)U(x,\beta)/x, \frac{\partial^2 U(x,\beta)}{\partial x^2} = -\gamma(1-\gamma)U(x,\beta)/x^2, \\ \frac{\partial U(x,\beta)}{\partial \beta} &= h_\beta U(x,\beta), \frac{\partial U(x,\beta)}{\partial x \beta} = h_\beta(1-\gamma)U(x,\beta)/x, \\ \frac{\partial^2 U(x,\beta)}{\partial \beta^2} &= [(h_\beta)^2 + h_{\beta\beta}]U(x,\beta). \end{aligned} \tag{44}$$

Considering the three control variables z, π and c, (36) becomes

$$\begin{aligned} 0 = \max_{z,\pi,c}\{ &\tfrac{1}{2}x^2\pi\sigma(\pi\sigma)'U_{xx} + x^2(1+g)\pi\sigma\sigma'_P U_{xx} + zx^2\pi\sigma\sigma'_P U_{xx} + \rho_1 x\pi\sigma\sigma'_\beta U_{x\beta} \\ &+ x\pi(\mu - r\mathbf{I})U_x + \lambda_S \textstyle\int_{\mathcal{R}}[U(x + x\pi\mathbf{J}_l\mathbf{I}, \beta) - U(x,\beta)]\nu(dz) \\ &+ \tfrac{1}{2}x^2(2z + 2zg + z^2)\sigma_P\sigma'_P U_{xx} + z\rho_2 x\sigma_P\sigma'_\beta U_{x\beta} + xz(\alpha+\beta)U_x \\ &+ \lambda_P \textstyle\int_{\mathcal{R}}[U(x + x\mathbf{J}_P\mathbf{I}(1+z+g), \beta) - U(x,\beta)]\nu_P(dz_P) \\ &- xcU_x + V(cx)\} + \tfrac{1}{2}x^2(1 + 2g + g^2)\sigma_P\sigma'_P U_{xx} \\ &+ (1+g)\rho_2 x\sigma_P\sigma'_\beta U_{x\beta} + \tfrac{1}{2}\sigma_\beta\sigma'_\beta U_{\beta\beta} + x[r(1-p) + (1+g)(\alpha+\beta)]U_x \\ &+ [f(\beta) + \beta\eta(\omega)]U_\beta - \delta U(x,\beta) = 0. \end{aligned} \tag{45}$$

Using (44) on (45), we have the following:

$$\begin{aligned} 0 = \max_{z,\pi,c}\{ &-\tfrac{1}{2}\pi\sigma(\pi\sigma)'\gamma(1-\gamma)U(x,\beta) - (1+g)\pi\sigma\sigma'_P\gamma(1-\gamma)U(x,\beta) \\ &- z\pi\sigma\sigma'_P\gamma(1-\gamma)U(x,\beta) + \rho_1\pi\sigma\sigma'_\beta h_\beta(1-\gamma)U(x,\beta) \\ &+ \pi(\mu - r\mathbf{I})(1-\gamma)U(x,\beta) + \lambda_S \textstyle\int_{\mathcal{R}}[(1 + \pi\mathbf{J}_l\mathbf{I})^{1-\gamma}U(x,\beta) - U(x,\beta)]\nu(dz) \\ &- \tfrac{1}{2}(2z + 2zg + z^2)\sigma_P\sigma'_P\gamma(1-\gamma)U(x,\beta) \\ &+ z\rho_2\sigma_P\sigma'_\beta h_\beta(1-\gamma)U(x,\beta) + z(\alpha+\beta)(1-\gamma)U(x,\beta) \\ &+ \lambda_P \textstyle\int_{\mathcal{R}}[(1 + \mathbf{J}_P\mathbf{I}(1+z+g)))^{1-\gamma}U(x,\beta) - U(x,\beta)]\nu_P(dz_P) \\ &- c(1-\gamma)U(x,\beta) + V(cx)\} - \tfrac{1}{2}(1 + 2g + g^2)\sigma_P\sigma'_P\gamma(1-\gamma)U(x,\beta) \\ &+ (1+g)\rho_2\sigma_P\sigma'_\beta h_\beta(1-\gamma)U(x,\beta) + \tfrac{1}{2}\sigma_\beta\sigma'_\beta[(h_\beta)^2 + h_{\beta\beta}]U(x\beta) \\ &+ [r(1-p) + (1+g)(\alpha+\beta)](1-\gamma)U(x,\beta) \\ &+ [f(\beta) + \beta\eta(\omega)]h_\beta U(x,\beta) - \delta U(x,\beta). \end{aligned} \tag{46}$$

It therefore follows that the optimal consumption plan c^* is

$$c^* = \left(\frac{\partial V}{\partial c}\right)^{-1} ((1-\gamma)U(x,\beta)). \tag{47}$$

Now, dividing (46) through by $-(1-\gamma)U(x,\beta) < 0$, so that max becomes min, we have

$$\begin{aligned} 0 = \min_{z,\pi,c}\{ &\tfrac{1}{2}\pi\sigma(\pi\sigma)'\gamma + (1+g)\pi\sigma\sigma'_P\gamma + z\pi\sigma\sigma'_P\gamma - \rho_1\pi\sigma\sigma'_\beta h_\beta \\ &- \pi(\mu - r\mathbf{I}) - \tfrac{\lambda_S}{1-\gamma}\textstyle\int_{\mathcal{R}}[(1 + \pi\mathbf{J}_l\mathbf{I})^{1-\gamma} - 1]\nu(dz) + \tfrac{1}{2}(2z + 2zg + z^2)\sigma_P\sigma'_P\gamma \\ &- z\rho_2\sigma_P\sigma'_\beta h_\beta - z(\alpha+\beta) - \tfrac{\lambda_P}{1-\gamma}\textstyle\int_{\mathcal{R}}[(1 + \mathbf{J}_P\mathbf{I}(1+z+g)))^{1-\gamma} - 1]\nu_P(dz_P) \\ &+ c - \tfrac{V(cx)}{(1-\gamma)U(x,\beta)}\} + \tfrac{1}{2}(1 + 2g + g^2)\sigma_P\sigma'_P\gamma - (1+g)\rho_2\sigma_P\sigma'_\beta h_\beta \\ &- \tfrac{1}{2(1-\gamma)}\sigma_\beta\sigma'_\beta[(h_\beta)^2 + h_{\beta\beta}] - [r(1-p) + (1+g)(\alpha+\beta)] \\ &- \tfrac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]h_\beta + \tfrac{\delta}{1-\gamma}. \end{aligned} \tag{48}$$

From (64), we deduce our optimal investment strategies, optimal net debt ratio and optimal consumption strategy of the investor.

4.4 *Optimal investment policies*

The optimal policy for the portfolio weights strategy π is

$$\pi^{'*} = arg\min_{\pi} f_1(\pi), \tag{49}$$

where the functions

$$f_1(\pi) = \tfrac{1}{2}\pi\sigma(\pi\sigma)'\gamma + (1+g)\pi\sigma\sigma_P'\gamma + z\pi\sigma\sigma_P'\gamma - \rho_1\pi\sigma\sigma_\beta' h_\beta - \pi(\mu - r\mathbf{I}) + m\lambda_S\psi_1(\pi\mathbf{J}_l\mathbf{I}) \tag{50}$$

and

$$\psi_1(\pi\mathbf{J}_l\mathbf{I}) = -\frac{1}{1-\gamma}\int_{\mathcal{R}}[(1+\pi\mathbf{J}_l\mathbf{I})^{1-\gamma} - 1]\nu(dz) \tag{51}$$

are both convex.

Proposition 2 *The optimal investment strategies of the investor is*

$$\begin{aligned}\pi^{'*} = &\left(1 - \tfrac{(\boldsymbol{J}_l\boldsymbol{I})'\Sigma^{-1}\boldsymbol{J}_l\boldsymbol{I}}{\boldsymbol{I}'\boldsymbol{J}_l'\Sigma^{-1}\boldsymbol{J}_l\boldsymbol{I}}\right)[\tfrac{1}{\gamma}\Sigma^{-1}(\mu - r\boldsymbol{I}) + \tfrac{1}{\gamma}\rho_1 h_\beta\Sigma^{-1}\sigma\sigma_\beta' \\ &-(z+g+1)\Sigma^{-1}\sigma\sigma_P'] + a\tfrac{\Sigma^{-1}\boldsymbol{J}_l\boldsymbol{I}}{\boldsymbol{I}'\boldsymbol{J}_l'\Sigma^{-1}\boldsymbol{J}_l\boldsymbol{I}}.\end{aligned} \tag{52}$$

Proof. By the first order conditions for the portfolio process, we have that

$$\begin{aligned}&\tfrac{\partial f_1(\pi)}{\partial\pi} = \sigma(\pi\sigma)'\gamma + (1+g)\sigma\sigma_P'\gamma + z\sigma\sigma_P'\gamma - \rho_1\sigma\sigma_\beta' h_\beta - (\mu - r\mathbf{I}) \\ &+m\lambda_S\mathbf{J}_l\mathbf{I}\dot{\psi}_1(\pi\mathbf{J}_l\mathbf{I}) = 0,\end{aligned} \tag{53}$$

Define the scalar $a = \pi\mathbf{J}_l\mathbf{I}$. After multiplying (53) by $\frac{1}{\gamma}\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}$, we have that a must satisfy

$$\begin{aligned}&a + (1+g)\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\sigma\sigma_P' + z\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\sigma\sigma_P' - \tfrac{\rho_1}{\gamma}\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\sigma\sigma_\beta' h_\beta - \tfrac{1}{\gamma}\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}(\mu - r\mathbf{I}) \\ &+\tfrac{m\lambda_S}{\gamma}\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}\dot{\psi}_1(a) = 0,\end{aligned} \tag{54}$$

then use (53) to solve for π:

$$\begin{aligned}&\pi'^* = \tfrac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{I}) + \tfrac{\rho_1}{\gamma}\Sigma^{-1}\sigma\sigma_\beta' h_\beta - z\Sigma^{-1}\sigma\sigma_P' - (1+g)\Sigma^{-1}\sigma\sigma_P' \\ &-\tfrac{m\lambda_S}{\gamma}\Sigma^{-1}\mathbf{J}_l\mathbf{I}\dot{\psi}_1(a).\end{aligned} \tag{55}$$

From (54), we have the following

$$\begin{aligned}&\dot{\psi}_1(a) = \tfrac{\gamma}{m\lambda_S}\tfrac{1}{\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}[-a - (1+g)\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\sigma\sigma_P' - z\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\sigma\sigma_P' \\ &+\tfrac{\rho_1}{\gamma}\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\sigma\sigma_\beta' h_\beta + \tfrac{1}{\gamma}\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}(\mu - r\mathbf{I})].\end{aligned} \tag{56}$$

Substituting (56) into (55), we have the following

$$\pi'^{*} = \underbrace{\tfrac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{I})\left(1 - \frac{(\mathbf{J}_l\mathbf{I})'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}{\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}\right)}_{\phi_1} + \underbrace{\tfrac{1}{\gamma}\rho_1 h_\beta \Sigma^{-1}\sigma\sigma_\beta'\left(1 - \frac{(\mathbf{J}_l\mathbf{I})'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}{\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}\right)}_{\phi_2}$$
$$\underbrace{-(z+g+1)\Sigma^{-1}\sigma\sigma_P'\left(1 - \frac{(\mathbf{J}_l\mathbf{I})'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}{\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}\right)}_{\phi_3} + \underbrace{a\frac{\Sigma^{-1}\mathbf{J}_l\mathbf{I}}{\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}}_{\phi_4}. \quad (57)$$

□

It shows that it is optimal to invest in a portfolio that comprises of four components:

(1) a speculative portfolio ϕ_1 proportional to the market price of risk corresponding to the risky assets through the relative risk averse index $\frac{1}{\gamma}$,
(2) an income risk hedging portfolio ϕ_2 proportional to the diffusion term of the income process through the cross derivative of function h with respect to the income rate β and the relative risk averse index $\frac{1}{\gamma}$,
(3) a hedging portfolio against debt risk ϕ_3 proportional to the diffusion term of the fixed asset price through the net debt ratio z and NIS g,
(4) a hedging portfolio value ϕ_4 of the risky assets and a a control parameter that measure how much of this portfolio value the investor is willing to take, risk-free funds holding the riskless assets only.

(57) is the investor's optimal portfolio strategies with jump-diffusion process.

It then follows that

$$\pi'^{*} = D_1 + \tfrac{a\Sigma^{-1}\mathbf{J}_l\mathbf{I}}{\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}, \quad (58)$$

where

$$D_1 = \left(1 - \tfrac{(\mathbf{J}_l\mathbf{I})'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}{\mathbf{I}'\mathbf{J}_l'\Sigma^{-1}\mathbf{J}_l\mathbf{I}}\right)\left[\tfrac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{I}) + \tfrac{1}{\gamma}\rho_1 h_\beta\Sigma^{-1}\sigma\sigma_\beta' - (z+g+1)\Sigma^{-1}\sigma\sigma_P'\right]. \quad (59)$$

4.5 *Optimal net debt ratio policy*

The optimal policy for the net debt ratio z is

$$z^* = arg\min_z f_2(z), \quad (60)$$

where the functions

$$f_2(z) = z\pi\sigma\sigma_P'\gamma + \tfrac{1}{2}(2z + 2zg + z^2)\sigma_P\sigma_P'\gamma - z\rho_2\sigma_P\sigma_\beta' h_\beta - z(\alpha+\beta) + \psi_2(\mathbf{J}_P\mathbf{I}z) \quad (61)$$

and

$$\psi_2(\mathbf{J}_P\mathbf{I}(1+z+g)) = -\frac{1}{1-\gamma}\int_{\mathcal{R}}[(1+\mathbf{J}_P\mathbf{I}(1+z+g)))^{1-\gamma} - 1]\nu_P(dz_P) \quad (62)$$

are both convex.

PROPOSITION 3 *The optimal net debt ratio of the investor is*

$$z^* = \left(1 - \tfrac{(\boldsymbol{J}_P\boldsymbol{I})^2}{\boldsymbol{I}'\boldsymbol{J}_P\boldsymbol{J}_P'\boldsymbol{I}}\right)\left[\tfrac{\pi\sigma\sigma_P'}{\sigma_P\sigma_P'} + b - \tfrac{\rho_2\sigma_P\sigma_\beta' h_\beta}{\gamma\sigma_P\sigma_P'} - \tfrac{(\alpha+\beta)}{\gamma\sigma_P\sigma_P'}\right]. \quad (63)$$

Proof. By the principles of the first order conditions, we have that

$$\begin{array}{l}\frac{\partial f_2(z)}{\partial z} = \pi\sigma\sigma'_P\gamma + (1+g+z)\sigma_P\sigma'_P\gamma - \rho_2\sigma_P\sigma'_\beta h_\beta - (\alpha+\beta) \\ +m_b\lambda_P\mathbf{J}_P\mathbf{I}\dot{\psi}_2(\mathbf{J}_P\mathbf{I}(1+z+g)) = 0.\end{array} \tag{64}$$

Furthermore, we define the scalar $b = \mathbf{J}_P\mathbf{I}(1+z+g)$. After multiplying (64) by $\frac{\mathbf{J}_P\mathbf{I}}{\gamma\sigma_P\sigma'_P}$, we have that b must satisfy

$$\frac{\pi\sigma\sigma'_P}{\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I} + b - \frac{\rho_2\sigma_P\sigma'_\beta h_\beta}{\gamma\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I} - \frac{(\alpha+\beta)}{\gamma\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I} + \frac{m_b\lambda_P\mathbf{I}'\mathbf{J}_P\mathbf{J}'_P\mathbf{I}}{\gamma\sigma_P\sigma'_P}\dot{\psi}_2(b) = 0. \tag{65}$$

It then follows that

$$\dot{\psi}_2(b) = \frac{\gamma\sigma_P\sigma'_P}{m_b\lambda_P\mathbf{I}'\mathbf{J}_P\mathbf{J}'_P\mathbf{I}}[-\frac{\pi\sigma\sigma'_P}{\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I} - b + \frac{\rho_2\sigma_P\sigma'_\beta h_\beta}{\gamma\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I} + \frac{(\alpha+\beta)}{\gamma\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I}]. \tag{66}$$

Substituting (66) into (64), we have the following

$$z^* = \left(1 - \frac{(\mathbf{J}_P\mathbf{I})^2}{\mathbf{I}'\mathbf{J}_P\mathbf{J}'_P\mathbf{I}}\right)\left[\frac{\pi\sigma\sigma'_P}{\sigma_P\sigma'_P} + b - \frac{\rho_2\sigma_P\sigma'_\beta h_\beta}{\gamma\sigma_P\sigma'_P} - \frac{(\alpha+\beta)}{\gamma\sigma_P\sigma'_P}\right]. \tag{67}$$

□

(67) is the optimal debt ratio for an investor in the presence of jump process. The control parameter b measure how much of the net debt that the investor (or the economy) is willing to tolerate.

PROPOSITION 4 *The NIS ratio g is*

$$\begin{array}{l}g = \frac{\boldsymbol{I}'\boldsymbol{J}_P\boldsymbol{J}'_P\boldsymbol{I} - \boldsymbol{J}_P\boldsymbol{I}(\boldsymbol{I}'\boldsymbol{J}_P\boldsymbol{J}'_P\boldsymbol{I}) + (\boldsymbol{J}_P\boldsymbol{I})^2}{\boldsymbol{J}_P\boldsymbol{I}(\boldsymbol{I}'\boldsymbol{J}_P\boldsymbol{J}'_P\boldsymbol{I})}b \\ -\left(1 - \frac{(\boldsymbol{J}_P\boldsymbol{I})^2}{\boldsymbol{I}'\boldsymbol{J}_P\boldsymbol{J}'_P\boldsymbol{I}}\right)\left[\frac{\pi\sigma\sigma'_P}{\sigma_P\sigma'_P} - \frac{\rho_2\sigma_P\sigma'_\beta h_\beta}{\gamma\sigma_P\sigma'_P} - \frac{(\alpha+\beta)}{\gamma\sigma_P\sigma'_P}\right] - 1.\end{array} \tag{68}$$

Proof. Using the fact that $b = \mathbf{J}_P\mathbf{I}(1+z^*+g)$, it then follows that

$$z^* = \frac{b-(1+g)\mathbf{J}_P\mathbf{I}}{\mathbf{J}_P\mathbf{I}}. \tag{69}$$

Equating (67) and (69) and then simplify the result follows. □

Again, the economy will be free from debt if $\lim_{b\longrightarrow(1+g)\mathbf{J}_P\mathbf{I}} z* = 0$. It is observe that the NIS ratio g will play a vital role in minimizing the debt risk in the investment portfolio.

PROPOSITION 5 *The economy is free from debt if*

$$\lim_{b\longrightarrow(1+g)\boldsymbol{J}_P\boldsymbol{I}} z* = 0.$$

Proof. By definition of limit for a function, given $\epsilon_z > 0$, we can find $\delta_z > 0$ such that

$$|z^*| < \epsilon_z \ \forall \ |b - (1+g)\mathbf{J}_P\mathbf{I}| < \delta_z.$$

But from (67), we have that

$$\left|\frac{b-(1+g)\mathbf{J}_P\mathbf{I}}{\mathbf{J}_P\mathbf{I}}\right| = \frac{|b-(1+g)\mathbf{J}_P\mathbf{I}|}{|\mathbf{J}_P\mathbf{I}|} < \frac{\delta_z}{|\mathbf{J}_P\mathbf{I}|} = \epsilon_z.$$

It then follows that $\delta_z = |\mathbf{J}_P\mathbf{I}|\epsilon_z$. □

We now have the following corollary.

COROLLARY 1 *If $\lambda_S = 0$ and $\lambda_P = 0$, then*

$$\pi'^{*} = \tfrac{1}{\gamma}\Sigma^{-1}(\mu - r\boldsymbol{I}) + \tfrac{\rho_1}{\gamma}\Sigma^{-1}\sigma\sigma'_{\beta}h_{\beta} - z^{*}\Sigma^{-1}\sigma\sigma'_{P} - (1+g)\Sigma^{-1}\sigma\sigma'_{P} \tag{70}$$

and

$$z^{*} = -(1+g) - \tfrac{\pi'^{*}\sigma\sigma'_{P}}{\sigma_P\sigma'_P} + \tfrac{\rho_2\sigma_P\sigma'_{\beta}h_{\beta}}{\sigma_P\sigma'_P\gamma} + \tfrac{(\alpha+\beta)}{\sigma_P\sigma'_P\gamma}. \tag{71}$$

Corollary 1 tells us that in the pure diffusive case, $\lambda_S = 0$, we obtain the familiar Merton solution

$$\pi'^{*} = \tfrac{1}{\gamma}\Sigma^{-1}(\mu - r\mathbf{I}) + \tfrac{\rho_1}{\gamma}\Sigma^{-1}\sigma\sigma'_{\beta}h_{\beta} - z^{*}\Sigma^{-1}\sigma\sigma'_{P} - (1+g)\Sigma^{-1}\sigma\sigma'_{P}, \tag{72}$$

and if the fixed asset price is a pure diffusive case, we have that $\lambda_P = 0$, which implies that the net debt ratio will be

$$z^{*} = -(1+g) - \tfrac{\pi'^{*}\sigma\sigma'_{P}}{\sigma_P\sigma'_P} + \tfrac{\rho_2\sigma_P\sigma'_{\beta}h_{\beta}}{\sigma_P\sigma'_P\gamma} + \tfrac{(\alpha+\beta)}{\sigma_P\sigma'_P\gamma}. \tag{73}$$

COROLLARY 2 *Suppose Proposition 5 holds, then Corollary 1 becomes*

$$\pi'^{*} = \tfrac{1}{\gamma}\Sigma^{-1}(\mu - r\boldsymbol{I}) + \tfrac{\rho_1}{\gamma}\Sigma^{-1}\sigma\sigma'_{\beta}h_{\beta} - (1+g)\Sigma^{-1}\sigma\sigma'_{P}$$

and

$$g = -1 - \tfrac{\pi'^{*}\sigma\sigma'_{P}}{\sigma_P\sigma'_P} + \tfrac{\rho_2\sigma_P\sigma'_{\beta}h_{\beta}}{\sigma_P\sigma'_P\gamma} + \tfrac{(\alpha+\beta)}{\sigma_P\sigma'_P\gamma}.$$

PROPOSITION 6 *Suppose Corollary 1 holds, then*

$$h_{\beta} = \Lambda\left(\frac{1}{\gamma}\Sigma^{-1}(\mu - r\boldsymbol{I}) + (1+g+z)\left(\frac{\sigma_P\sigma_P}{\sigma\sigma'_P} - \Sigma^{-1}\sigma\sigma'_P\right) - \frac{\alpha+\beta}{\gamma\sigma\sigma'_P}\right),$$

where $\Lambda = \frac{\gamma\sigma\sigma'_P}{\rho_2\sigma_P\sigma'_{\beta} - \rho_1(\sigma\sigma'_P)'\Sigma^{-1}\sigma\sigma'_{\beta}}$.

Proof. Using (72) and (73), we see that the result follows immediately. □

Proposition 4 tells us that if the net debt ratio z is known, the derivative function h_{β} can be determined.

4.6 *Optimal consumption policy*

In this subsection, we give the optimal consumption plan of the investor and is given in the following proposition.

PROPOSITION 7 *The optimal consumption plan of the investor is*

$$C^{*} = x^{\frac{\gamma-1}{\gamma}}e^{-\frac{h(\beta)}{\gamma}}.$$

Proof. For the optimal consumption policy, with $\left[\frac{\partial V}{\partial C}\right]^{-1}(y) = y^{-\frac{1}{\gamma}}$ and $(1-\gamma)U(x,\beta) = x^{1-\gamma}e^{h(\beta)}$ in (47), we obtain the following:

$$C^{*} = (\partial U/\partial C)^{-1})((1-\gamma)U(x,\beta)) = x^{\frac{\gamma-1}{\gamma}}e^{-\frac{h(\beta)}{\gamma}}. \tag{74}$$

□

It is observe that if $h(\beta) > 0$, consumption rate will increase as γ decreases and decreases as γ increases. We also observe that as $h(\beta)$ becomes very large for all other parameters remain fixed, consumption tends to zero and consumption becomes very large as $h(\beta)$ tends to zero. Clearly, if $h(\beta) < 0$, consumption will continue to increase as $h(\beta)$ increases. If $\gamma \longrightarrow +\infty$ and $h(\beta) \longrightarrow -\infty$, $C^* \longrightarrow 0$ and vice versa.

4.7 *Explicit form of the HJB equation*

We now give the explicit form of our HJB equation in this subsection.

PROPOSITION 8 *The explicit form of our HJB equation (64) is of the form*

$$\frac{1}{1-\gamma}\sigma_\beta\sigma'_\beta h_{\beta\beta} + [k_2 + \frac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]]h_\beta - k_1(x,\beta) = 0.$$

Proof. Using (57), (74) and (67) on our HJB equation (64), we have

$$\tfrac{1}{2(1-\gamma)}\sigma_\beta\sigma'_\beta[(h_\beta)^2 + h_{\beta\beta}] + [k_2 + \tfrac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]]h_\beta - k_1(x,\beta) = 0, \tag{75}$$

where

$$\begin{aligned} k_1(x,\beta) &= x^{\frac{\gamma-1}{\gamma}} e^{-\frac{h(\beta)}{\gamma}} + \tfrac{\gamma}{2}\left[D'_1\Sigma D_1 + \tfrac{2aD'_1\mathbf{J}_l\mathbf{I}}{\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\mathbf{J}_l\mathbf{I}} + \tfrac{a^2(\Sigma^{-1}\mathbf{J}_l\mathbf{I})'\mathbf{J}_l\mathbf{I}}{(\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\mathbf{J}_l\mathbf{I})^2}\right] \\ &+ \left[D'_1 + \tfrac{a(\Sigma^{-1}\mathbf{J}_l\mathbf{I})'}{\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\mathbf{J}_l\mathbf{I}}\right]\left(\tfrac{b}{\mathbf{J}_P\mathbf{I}}\sigma\sigma'_P\gamma - (\mu - r\mathbf{I})\right) \\ &-\lambda_S\psi_1(a) - \lambda_P\psi_2(b) + \tfrac{b}{2\mathbf{J}_P\mathbf{I}}\left(1 + g + \tfrac{b}{\mathbf{J}_P\mathbf{I}}\right)\sigma_P\sigma'_P\gamma \\ &+\tfrac{b(\alpha+\beta)}{\mathbf{J}_P\mathbf{I}} + \tfrac{\gamma}{2}(1 + 2g + g^2)\sigma_P\sigma'_P + \tfrac{\delta}{1-\gamma} - r(1-p). \end{aligned} \tag{76}$$

$$k_2 = \rho_1\left[\left[D'_1 + \tfrac{a(\Sigma^{-1}\mathbf{J}_l\mathbf{I})'}{\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\mathbf{J}_l\mathbf{I}}\right]\sigma\sigma'_\beta + \tfrac{b}{\mathbf{J}_P\mathbf{I}}\sigma_P\sigma'_\beta\right]. \tag{77}$$

Solving the following ODE $h_{\beta\beta} = (h_\beta)^2$, we have a solution of the form:

$$\begin{cases} h(\beta(t)) = G_1(t) - \ln[G_2(t) + E(t)\beta] \\ h(\beta(T)) = 0, \end{cases} \tag{78}$$

where $G_1(t)$, $G_2(t)$, $E(t) \in \mathcal{R}$ and $G_1(t) + E(t)\beta > 0$. (75) now becomes

$$\tfrac{1}{1-\gamma}\sigma_\beta\sigma'_\beta h_{\beta\beta} + [k_2 + \tfrac{1}{1-\gamma}[f(\beta) + \beta\eta(\omega)]]h_\beta - k_1(x,\beta) = 0. \tag{79}$$

□

Since $(\gamma, \mu, r, \Sigma, J_l, J_P)$ are all not constants (where $\mu(t)$ and $r(t)$ are time-dependent), the objective functions f_1 and f_2 are time-dependent. It is obvious that any optimal solution will indeed be time-dependent. Again, the objective functions are state dependent as well. It then follows that any optimal solution will be state dependent. In other words, any optimal portfolio strategy $\pi'^*(t)$ will be a time-dependent and it follows that π'^* is dependent of time and state. Finally, the objective functions f_1 and f_2 are strictly convex, goes to $+\infty$ in all directions, so that a unique minimizer will always be obtained.

Finally, we have to check to ensure that the transversality condition is satisfied. We now substitute the optimizers x^* and c^* into (34), and then taking expectations, one finds

$$L(t, x^*(t), \beta(t)) = E_t\left[\int_t^\infty e^{-\delta s}V(c^*(s)x^*(s))ds\right]. \tag{80}$$

Taking the limit as t tends to infinity, there will exponential decay of (80). Substituting π'^*, z^* and c^* into (24), we found that an investor with power utility who selects this optimal portfolio, optimal debt ratio and consumption plan will achieve a wealth process $x^*(t) = X^*(t)$ that follows a geometric Lévy process.

The following proposition gives an explicit solutions for the scalars a and b.

PROPOSITION 9 *Suppose the Lévy measure generate both asymmetric positive and negative jumps with the following solvency constraints* $|a| < 1$ *and* $|b| < 1$, *then*

(i)
$$a = \frac{1}{6}\left[-2A - \frac{(22^{\frac{1}{3}}(A^2 - 3(-1 + B(\lambda_{S-} + \lambda_{S+}))))}{(2A^3 + 27B(\lambda_{S-} - \lambda_{S+}) - 9A(2 + B(\lambda_{S-} + \lambda_{S+})) + \sqrt{\Phi})^{\frac{1}{3}}}\right] - \frac{1}{6}\left[4^{\frac{1}{3}}\left(2A^3 + 27B(\lambda_{S-} - \lambda_{S+}) - 9A(2 + B(\lambda_{S-} + \lambda_{S+})) + \sqrt{\Phi}\right)^{\frac{1}{3}}\right],$$

(ii)
$$b = -\frac{A_1}{3} + \frac{2^{\frac{1}{3}}(A_1^2 + 3(1 + B_1(\lambda_{P-} + \lambda_{P+})))}{3[-2A_1^3 + 27B_1(\lambda_{P-} - \lambda_{P+}) - 9A_1(-2 + B_1(\lambda_{P-} + \lambda_{P+})) + \sqrt{\Psi}]^{\frac{1}{3}}} + \frac{1}{32^{\frac{1}{3}}}\left([-2A_1^3 + 27B_1(\lambda_{P-} - \lambda_{P+}) - 9A_1(-2 + B_1(\lambda_{P-} + \lambda_{P+})) + \sqrt{\Psi}]^{\frac{1}{3}}\right)$$

Proof. (i) Since we allow the Lévy measure to generate asymmetric positive and negative jumps, see [9]. So, we consider a Lévy measure such that

$$m\lambda_S\nu(dz) = \begin{cases} \lambda_{S+}dz/z, \text{ if } z \in (0,1] \\ -\lambda_{S-}dz/z, \text{ if } z \in [-1,0) \end{cases} \tag{81}$$

with $\lambda_{S+} > 0$ and $\lambda_{S-} > 0$. Setting $\gamma = 2$, we have that $m\lambda_S\psi(a) = -\lambda_{S+}\log(1+a) - \lambda_{S-}\log(1-a)$ and (54) becomes a cubic equation in a:

$$f'_S(a) = a + A - B[-\lambda_{S+}(1+a)^{-1} + \lambda_{S-}(1-a)^{-1}] = 0, \tag{82}$$

with solvency constraint $|a| < 1$, where $A = (1+g)\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\sigma\sigma'_P + z\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\sigma\sigma'_P - \frac{\rho_1}{2}\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\sigma\sigma_{\beta'}h_\beta - \frac{1}{2}\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}(\mu - r\mathbf{I})$ and $B = \frac{1}{2}\mathbf{I}'\mathbf{J}'_l\Sigma^{-1}\mathbf{J}_l\mathbf{I}$. The optimal solution to (54) is indeed solvable in closed form under the solvency constraint:

$$a = \tfrac{1}{6}\left[-2A - \frac{(22^{\frac{1}{3}}(A^2-3(-1+B(\lambda_{S-}+\lambda_{S+}))))}{(2A^3+27B(\lambda_{S-}-\lambda_{S+})-9A(2+B(\lambda_{S-}+\lambda_{S+}))+\sqrt{\Phi})^{\frac{1}{3}}}\right] - \tfrac{1}{6}\left[4^{\frac{1}{3}}\left(2A^3 + 27B(\lambda_{S-} - \lambda_{S+}) - 9A(2 + B(\lambda_{S-} + \lambda_{S+})) + \sqrt{\Phi}\right)^{\frac{1}{3}}\right], \tag{83}$$

where $\Phi = (-4(A^2 - 3(-1 + B(\lambda_{S-} + \lambda_{S+})))^3 + (2A^3 + 27B(\lambda_{S-} - \lambda_{S+}) - 9A(2 + B(\lambda_{S-} + \lambda_{S+})))^2)$.

(ii) Similarly, we consider for the net debt ratio:

$$m_P\lambda_P\nu(dz_P) = \begin{cases} \lambda_{P+}dz_P/z_P, \text{ if } z_P \in (0,1] \\ -\lambda_{P-}dz_P/z_P, \text{ if } z_P \in [-1,0) \end{cases} \tag{84}$$

with $\lambda_{P+} > 0$ and $\lambda_{P-} > 0$. Setting $\gamma = 2$, we have that $m_P\lambda_P\psi(b) = -\lambda_{P+}\log(1+b) - \lambda_{P-}\log(1-b)$ and (65) becomes a cubic equation in b:

$$f'_P(b) = b + A_1 + B_1[-\lambda_{P+}(1+b)^{-1} + \lambda_{P-}(1-b)^{-1}] = 0, \tag{85}$$

where $A_1 = \frac{\pi\sigma\sigma'_P}{\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I} + (1+g)\mathbf{J}_P\mathbf{I} - \frac{\rho_2\sigma_P\sigma'_\beta h_\beta}{2\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I} - \frac{(\alpha+\beta)}{2\sigma_P\sigma'_P}\mathbf{J}_P\mathbf{I}$ and $B_1 = \frac{(\mathbf{J}_P\mathbf{I})^2}{2\sigma_P\sigma'_P}$.

$$\begin{aligned} b = -\tfrac{A_1}{3} + \tfrac{2^{\frac{1}{3}}(A_1^2+3(1+B_1(\lambda_{P-}+\lambda_{P+})))}{3[-2A_1^3+27B_1(\lambda_{P-}-\lambda_{P+})-9A_1(-2+B_1(\lambda_{P-}+\lambda_{P+}))+\sqrt{\Psi}]^{\frac{1}{3}}} \\ +\tfrac{1}{32^{\frac{1}{3}}}\left([-2A_1^3+27B_1(\lambda_{P-}-\lambda_{P+})-9A_1(-2+B_1(\lambda_{P-}+\lambda_{P+}))+\sqrt{\Psi}]^{\frac{1}{3}}\right) \end{aligned} \tag{86}$$

where $\Psi = (((-3+A_1)(2A_1(3+A_1)+9B_1\lambda_{P-})+9(3+A_1)B_1\lambda_{P+})^2 - 4(A_1^2+3(1+B_1(\lambda_{P-}+\lambda_{P+})))^3)$. ■

Proposition 9 (i): we observed that if

$$(1+g)\sigma\sigma'_P + z\sigma\sigma'_P - \frac{\rho_1}{2}\sigma\sigma_{\beta'}h_\beta - \frac{1}{2}(\mu - r\mathbf{I}) = -\frac{1}{2}(\lambda_{S+} - \lambda_{S-})\mathbf{J}_l\mathbf{I},$$

it implies that the investor has no exposure to jump risk. This is due to the fact that jumps in one investment are used to hedge or offset jump risk in another investment.

Proposition 9 (ii): we observed that if

$$(1+g) + \frac{\pi\sigma\sigma'_P}{\sigma_P\sigma'_P} - \frac{\rho_2\sigma_P\sigma'_\beta h_\beta}{2\sigma_P\sigma'_P} - \frac{(\alpha+\beta)}{2\sigma_P\sigma'_P} = -\frac{(\lambda_{P+} - \lambda_{P-})\mathbf{J}_P\mathbf{I}}{2\sigma_P\sigma'_P},$$

it implies that the investor net debt ratio has no exposure to jump risk.

5. Conclusion

In this section, we will give the concluding remarks of the results obtained in his paper. We obtained the optimal net debt ratio in the presence of collateral security, optimal investment strategies and consumption plan of an investor who is exposed to both diffusion and jump risks. The maximization of the total expected discounted utility of consumption in the infinite time horizon under power utility function was considered. The optimal consumption and investment strategies as well as optimal net debt ratio under power utility function were obtained. We found that the optimal consumption depend on the optimal wealth, income growth and the coefficient of CRRA utility function. Furthermore, it was found that if $h(\beta) > 0$, consumption rate will increase as γ decreases and decreases as γ increases. It was also found that as $h(\beta)$ becomes very large, consumption tends to zero and consumption becomes very large as $h(\beta)$ tends to zero. We also found that if $h(\beta) < 0$, consumption will continue to increase as $h(\beta)$ increases and if $\gamma \longrightarrow +\infty$ and $h(\beta) \longrightarrow -\infty$, $c^* \longrightarrow 0$ and vice versa. We further found that the jump risks in the investment portfolios are hedge if

$$(1+g)\sigma\sigma'_P + z\sigma\sigma'_P - \frac{\rho_1}{2}\sigma\sigma_{\beta'}h_\beta - \frac{1}{2}(\mu - r\mathbf{I}) = -\frac{1}{2}(\lambda_{S+} - \lambda_{S-})\mathbf{J}_l\mathbf{I}$$

and net debt ratio has no exposure to jump risk if

$$(1+g) + \frac{\pi\sigma\sigma'_P}{\sigma_P\sigma'_P} - \frac{\rho_2\sigma_P\sigma'_\beta h_\beta}{2\sigma_P\sigma'_P} - \frac{(\alpha+\beta)}{2\sigma_P\sigma'_P} = -\frac{(\lambda_{P+} - \lambda_{P-})\mathbf{J}_P\mathbf{I}}{2\sigma_P\sigma'_P}$$

.

We found that it is optimal to invest in a portfolio that comprises of four components: first, a speculative portfolio ϕ_1 which is proportional to the market price of risk corresponding to the risky assets through the relative risk averse index $\frac{1}{\gamma}$; second, an income risk hedging portfolio ϕ_2 proportional to the diffusion term of the income process through the cross derivative of function h with respect to the income rate β and the relative risk averse index $\frac{1}{\gamma}$; third, a hedging portfolio ϕ_3 of the fixed asset proportional to the diffusion term of the fixed asset price through the net debt ratio z and NIS rate g; last, a hedging portfolio value ϕ_4 of the risky assets. A control parameter that measure how much of this portfolio value the investor is willing to take was obtained. Also, the control parameter that measure how much of the net debt that the investor (or the economy) is willing to tolerate was obtained.

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