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In honour of Prof. Ekhaguere at 70

A review of some recent results on quantum stochastic differential equations and inclusions

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Abstract. Building upon the fundamental foundation laid by Ekhaguere in his pioneering work concerning the existence and some qualitative properties of Lipschitzian quantum stochastic differential inclusions rendered in the framework of the Hudson-Parthasarathy formulation (H-P) of quantum stochastic calculus (QSC), this paper reviews our important results in this field. The review covers the period 1992 to the year 2010 and it concerns both the theoretical and numerical aspects of quantum stochastic differential equations (QSDE) and inclusions (QSDI) driven by quantum martingales and semi-martingales in the weak and strong sense. It is well known that the H-P formulation of QSC sufficiently generalise the Ito calculus.

Keywords: quantum stochastic differential equations, inclusions, martingales, Ito calculus.

1. Introduction

1.1 *Classical Stochastic Differential Equations (SDE)*

A Stochastic Differential Equation (SDE) is a generalisation of a deterministic ordinary differential equation incorporating random phenomenon into its formulation and parameters. This makes the equations more useful for modelling real life problems much more realistically than their deterministic counterparts. It also involves much more complex analysis using specially developed calculus called stochastic calculus (e.g. Ito Calculus pioneered in 1949 by K. Ito (1915-2008), a Japanese mathematician) as distinct from the Newtonian calculus which is applicable to deterministic equations.

This review is primarily intended to stimulate the research interest of younger generation of mathematicians in the field and for them to appreciate the generalisation of the classical stochastic calculus defined on finite dimensional Euclidean spaces to the recently developed non commutative operator valued quantum stochastic calculus defined on infinite dimensional locally convex spaces with diverse analytical and computational benefits. As shown in our works below, by a suitable choice of parameters, Ito Stochastic differential equations can be recovered from the generalised quantum stochastic differential equations whose solutions are operator processes.

Systems in many branches of science, engineering, industry and government are often perturbed by various types of environmental noise arising from some uncertainties and random errors in measurements. If we consider the simple population growth model:

$$\frac{dN}{dt}(t) = a(t)N(t) \tag{1.1}$$

with initial condition $N(0) = N_0$, where $N(t)$ is the size of the population at time t and $a(t)$ is the relative growth. It might happen that $a(t)$ is not completely known or subject to some random fluctuation so that it may be written in the form:

$$a(t) = r(t) + \sigma(t)\xi.$$

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Here ξ stands for the noise process. Equation (1.1) then becomes:

$$\frac{dN}{dt}(t) = r(t)N(t) + \sigma(t)N(t)\xi,$$

which is given in integral form by:

$$N(t) = N_0 + \int_0^t r(s)N(s)ds + \int_0^t \sigma(s)N(s)\xi(s)ds. \tag{1.2}$$

The question is: What is the mathematical interpretation for the noise term involving $\xi(t)$ and what is the integration

$$\int_0^t \sigma(s)N(s)\xi(s)ds? \tag{1.3}$$

It turns out that a reasonable mathematical interpretation for the 'noise' term $\xi(t)$ is the so called white noise which is formally regarded as the derivative of a Brownian motion (a Wiener process) $B(t)$. Hence we have $\xi(t) = \frac{dB}{dt}(t)$ or $\xi(t)dt = dB(t)$ and therefore, we have the integral that appears in Equation (1.2) given by

$$\int_0^t \sigma(s)N(s)\xi(s)ds = \int_0^t \sigma(s)N(s)dB(s). \tag{1.4}$$

If the Brownian motion $B(t)$ were differentiable, then the integral would have no problem at all. Unfortunately, we know that $B(t)$ is nowhere differentiable hence the integral cannot be defined in the ordinary way. The stochastic nature of the Brownian motion was used by K. Ito in 1949 to establish the integral now known as Ito stochastic integral. This led to the study of a class of stochastic differential equations driven by a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ of the form:

$$dX(t) = E(t, X(t))dt + F(t, X(t))dB(t),\ X(0) = X_0,\ t \in [0, T], \tag{1.5}$$

where the coefficients $E,\ F$ belong to the appropriate spaces of real or vector valued stochastic processes.

Let it be known that many generalisations and extensions of Ito integral have appeared in the literature. These include generalisation to stochastic integrals driven by semimartingales and several formulations of the non commutative quantum stochastic integrals driven by operator valued processes on certain topological spaces.

2. Some applications of SDE to real life problems

In order to motivate and stimulate multidisciplinary research collaborations, we will proceed to list some of the recent areas where SDE (1.5) has been applied for modelling and solving real life problems. These are:

(a) Population dynamics, protein kinetics and genetics

The simplest deterministic model of population growth is the exponential equation

$$\frac{dN}{dt}(t) = AN(t), \quad A > 0. \tag{2.1}$$

Allowing the vagaries of environment, A can be modelled to vary randomly as $A+\sigma\xi(t)$, for some zero mean process $\xi(t)$. Incorporating a finite supportable carrying capacity K, Equation (2.1) becomes

$$\frac{dN}{dt}(t) = A(K - N(t))N(t) = \lambda N(t) - N^2(t) \tag{2.2}$$

where $AK = \lambda$.

On randomizing the parameter λ in Equation (2.2) to $\lambda + \sigma\xi(t)$, we obtain an SDE of the form

$$dN(t) = \left[\lambda N(t) - N^2(t)\right] dt + \sigma N(t) dW(t) \tag{2.3}$$

which is explicitly solvable.

A frequently studied deterministic model of multi-species interaction is the Voltera-Lotka system:

$$\frac{dN_i}{dt}(t) = N_i(t)\left(A_i + \sum_{j=1}^{d} b^{i,j} N_j(t)\right), \; i = 1, 2 \cdots d$$

in the case of d different species. Randomising the growth parameters A_i as $A_i + \sigma_i\xi_i(t)$ leads to a system of SDE with independent Wiener processes given by:

$$dN_i(t) = N_i(t)\left(A^i + \sum_{j=1}^{d} b^{i,j} N_j(t)\right) dt + \sigma_i N_i(t) dW_i(t). \tag{2.4}$$

Explicit solutions are not known for equation (2.4), so approximate solutions are usually obtained numerically.

Protein kinetics: Stochastic counterparts of many ordinary differential equations modelling chemical kinetics such as the Brusselor equations can be derived by randomizing coefficients. For example, the kinetics of the proportion X of one of two possible forms of certain proteins can be modelled by an ODE of the form

$$\frac{dX}{dt}(t) = \alpha - X - \lambda X(1 - X) \tag{2.5}$$

where $0 \leq X \leq 1$ and the other form has proportion $Y = 1 - X$. For random fluctuations of the interaction coefficients λ of the form $\lambda + \sigma\xi(t)$, with white noise $\xi(t)$, we have the stochastic version of (2.5) given by the Stratonovich SDE:

$$dX(t) = (\alpha - X(t) + \lambda X(t)(1 - X(t))dt + \sigma X(t)(1 - X(t))dW(t). \tag{2.6}$$

The solutions of (2.6) are not known explicitly but remain in the interval $[0, 1]$.

The Ito equation equivalent to (2.6) given by

$$dX(t) = AX(t)dt + \sigma X(t)(1 - X(t))dW(t) \tag{2.7}$$

has been applied to genetics with $X(t)$ representing the proportion at time t of one of the two possible alleles of a certain gene. A discrete time Markov process can be constructed to model the changes from generation to generation in the alleles proportion due to natural selection.

(b) Experimental psychology and neuronal activity

The coordination of human movement particularly of periodically repeated movement has been extensively investigated by experimental psychologists with the objective of gaining deeper understanding into neurological control mechanism. The neurological system is extremely complicated, yet in some situations, a single characteristic appears to dominate and a satisfactory phenomological SDE model can be constructed to describe its dynamics.

Neuronal activity: Many stochastic models have been proposed to describe the spontaneous firing activity of a single neutron. These are usually based on jump processes and allow arbitrary large hyperpolarisation values for the membrane potential. A model incorporating several features of neutronal activities has been derived in the form of SDE.

(c) Investment finance and option pricing

Investment finance: SDE have been used to model share price dynamics in models of investment finance. Merton (1973) considered an investor who chooses between two different types of investment, one risky and the other safe (riskless). At each instant of time, the investor must select the fraction f of his wealth that he put into risky asset with the remaining fraction $1 - f$ going into the safe one. If his current consumption rate $c \geq 0$, then his wealth $X(t)$ satisfies the SDE

$$dX(t) = (\{(1 - f)a + fb\}X(t) - c)dt + f\beta X(t)dW(t). \tag{2.8}$$

If the investor has perfect information about his current wealth, Markovian feed back controls of the form: $U(t, X(t)) = (f(t, X(t)), C(t, X(t))$ provides a natural way for choosing his current investment and consumption rate.

Option pricing: Suppose that the price $X(t)$ of a risky asset evolves according to the Ito SDE in integral form given by:

$$X(t) = X_0 + \int_0^t b(s, X(s))dW(s), \quad t \in [0, T]$$

an European call option with strike price K gives the right to buy the stock at time T at a fixed price K. The resulting payoff is then given by

$$f(X(T)) = (X(T) - K)^+.$$

Suppose that we apply a dynamical portfolio strategy or hedging strategy (c_t, η_t) where η_t is the amount of riskless asset of constant value 1 say, and the amount c_t is the risky asset. Then the value $V(t)$ of the portfolio at time t is

$$V(t) = c_t X(t) + \eta_t$$

An important problem is to determine the fair price of the option. By the Nobel prize winning Blacks-Scholes formula, we have

$$V(0) = E(f(X(T)).$$

The corresponding self financing hedging strategy in quite general situation leads to a perfect replication of the claim

$$V(T) = f(X(T)).$$

Please note that the Black - Scholes equations have been applied to real option valuation of assets and valuation of flexibility or opportunity for real investments in the energy sector such as oil and gas, electricity as well as in the real sector.

(d) Turbulent diffusion and radio − astronomy
SDE have long been used to model turbulent diffusion and related issues, dating back to Langevin's equations for Brownian motion. If $X(t) \in \mathbb{R}^3$ represents the position of a fluid particle at time t and $V(t)$ its velocity, a simple model for the Lagrangian dynamics of such a particle is of the form:

$$dX(t) = V(t)dt, \qquad dV(t) = -\frac{1}{T}V(t)dt + \sigma dW(t) \tag{2.9}$$

where T is a large relaxation time for the process $V(t)$. Variation of the last equations have been considered with coloured noise. In some instances, Poisson processes have been used as the driving processes. Other areas of research activities where SDE have found applications are:

(e) Helicopter Rotor and Satellite Orbit Stability,
(f) Biological Waste Treatment, Hydrology and Indoor Air Quality,
(g) Seismology and Structural Mechanics,
(h) Fatigue Cracking, Optical Bistability and Nemantic Liquid Crystals,
(i) Blood Clotting Dynamics and Cellular Energetics,
(j) Josephson Junctions, Communications and Stochastic Annealing.

3. Some contributions to classical stochastic differential equations

Our contributions in this field concern some qualitative and numerical aspects of classical stochastic differential equations driven by Brownian motions. We have the following results concerning oscillatory behaviour of solutions of stochastic delay differential equations.

1. Together with A.O. Atonuje, we published some results (Atonuje and Ayoola, 2007) concerning the non-contribution to the oscillatory behaviour of solutions of stochastic delay differential equations (SDDE) of the form:

$$dX(t) = -\sum_{j=1}^{n} a_i(t)X(t-r_i)dt + \mu X(t)dB(t),\ t \geq 0, \qquad X(t) = \nu(t),\ t \in [\tilde{t} - \rho,\ 0]. \tag{3.1}$$

We were able to prove that even when non-oscillatory solutions exist in the corresponding deterministic delay differential equation, the presence of noise perturbation stimulates an oscillation subject to certain conditions on the delay terms
.
2. We have also shown that in the absence of the noise term, non -oscillatory solutions can occur for the deterministic case but with the presence of noise, all solutions of SDDE oscillate almost certainly whenever the feedback intensity is negative (Atonuje and Ayoola, 2007, 2008a, b). Delay and noise play complementary roles in the oscillatory behaviour of the solution of the SDDE (3.1).

3. **Finite Element Solutions of Stochastic Partial Differential Equations**
In another ocassion, I. N. Njoseh and myself published some results (Njoseh and Ayoola, 2008a) on the finite element method for a strongly damped stochastic wave equation driven by a space - time noise of the form:

$$U_{tt} + \alpha AU_t + AU = dW \text{ in } \Omega,\ t > 0, \qquad U(\cdot,t) = 0,\ \text{on } \delta\Omega,\ t > 0, \qquad U(0) = \phi,\ U_t(0) = \nu,\ \text{in } \Omega \tag{3.2}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary $\delta\Omega$ and $A = -\triangle$ self adjoint operator, W a Wiener process. We provided some error estimates of optimal order for semi-discrete and fully discrete finite elements schemes by using L_2 - projections of the initial data as starting values.

4. We also carried out a finite element analysis of the stochastic Cahn- Hilliard Equation (Njoseh and Ayoola, 2008b) of the form:

$$U_t - \triangle(-\triangle U + f(U) = a(U)W_{tx} \text{ on } \Omega$$

with initial condition

$$U(0,\cdot) = u_0, \text{ in } \Omega \ \text{ and } \frac{\partial U}{\partial n} = \frac{\partial}{\partial n}\triangle U = 0, \text{on } \partial\Omega, \tag{3.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^d$. The equation is a semi-linear parabolic of fourth order. It has been used to model phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably.

4. Our contributions to quantum stochastic differential equations and inclusions

Quantum stochastic calculus is a differential calculus incorporating various noises in quantum world. The first quantum stochastic calculus was introduced by **R. L. Hudson and K.R Parthasarathy (Hudson and K.R Parthasarathy, 1984)**, for Boson noises. This is roughly speaking, a sort of Ito calculus for the most fundamental noises in quantum theory. The study of stochastic calculi for several types of noises such as Boson, Fermi, free, Boolean, monotonic, etc is still a hot topic.

Quantum stochastic differential equations (QSDE) are stochastic differential equations for operator processes driven by quantum noises. In addition to reducing to classical SDE in special cases, they are applied in the study of quantum information, quantum open systems, quantum measurement, in the study of quantum Markov processes and dilations of quantum Markov semi-groups.

Quantum information theory treats any problem related to transmission of information through quantum systems, to storing, encoding, decoding information in quantum systems. Foundations of quantum mechanics are relevant and measurement theory is involved in modelling decoding and error procedures, at least.

Quantum Markov semi-groups are the natural mathematical objects for modelling the irreversible evolution of open quantum systems. This is governed by the so-called master equation whose solutions are given by a Markov semi-group. These are mathematically viewed as non-commutative generalisation of classical Markov semi-groups acting on a commutative algebra (a function space).

Quantum measurement theory is a very important topic in quantum probability. It deals with the issues of measurements of observables inside quantum mechanics. It has applications in open system theory, quantum optics, operator theory, quantum probability and quantum and classical stochastic processes.

Quantum probability is a transversal subject which finds its fundamental axioms in quantum physics and has deep connections with domains such as quantum mechanics, quantum field theory, quantum optics and scattering theory (Attal, 1998). Sometimes, quantum probability is regarded as part of functional analysis (C^*-algebra,, von Neumann algebra theory, non-commutative geometry, quantum groups, etc).

Building upon the fundamental foundation laid by G.O.S. Ekhaguere in his pioneering paper (Ekhaguere, 1992) concerning the existence and some qualitative properties of Lipschitzian quantum stochastic differential inclusions of the form (4.3) below, our recent research activities in this field concern both the theoretical and numerical aspects of quantum stochastic differential equations (QSDE) and inclusions (QSDI) driven by quantum martingales and semi-martingales in the weak and strong sense within the framework of the Hudson-Parthasarathy (1984) (H-P) formulations of quantum stochastic calculus (QSC). It is well known that the H-P formulation of QSC sufficiently generalise the Ito calculus. Quantum stochastic calculus employs the principles of quantum probability which is a non-commutative generalisation of classical theory of probability. Random variables (or observables in the language of physics) are represented by self adjoint operators in a complex Hilbert space and probabilities, or states are represented by unit vectors in the Hilbert spaces. Thus, problems in classical stochastic calculus can be reformulated in quantum forms based on the fact that given any family $S = S_t,\ t \in T)$ of commuting self adjoint operators in a complex Hilbert space H, which are collectively cyclic, (i.e there exists a unit vector $\phi \in H$ for which the set $\{e^{ixS_t}\phi : x \in \mathbb{R}\}$ is total in H), it can be shown that there exists a probability space $(\Omega,\ \mathcal{F}\ P)$, is a family of real valued measurable functions (classical stochastic processes) $(X_t : t \in T)$ on Ω and a Hilbert space isomorphism

$$D_S : H \to L^2(\Omega, \mathcal{F},\ P)$$

such that the vacuum vector (unit vector in H) is mapped to the function identically 1 and the operator S_t becomes the operator of multiplication by X_t (Hudson, 2001).

A quantum stochastic process consisting of commuting self - adjoint operators is completely equivalent to a classical stochastic process (Attal, 1998). In general, quantum stochastic calculus concerns non-commuting self adjoint operators thereby containing the classical theory as a special case. It should therefore, be noted that several benefits have been achieved by interpreting classical probability in non-commutative quantum form. Such benefits include a better understanding of classical stochastic flows and some parts of Wiener space analysis and Wiener chaos expansions where a fundamental chaotic representation property of the Azema martingales have been discovered (Mayer, 1993; Hudson, 2001). However, it is well known that the subject of quantum probability is far from being reduced to a simple non commutative extension of classical probability theory. It is not exclusive to finding non-commutative analogues of the classical theorems. Its connection with quantum field theory is very deep as earlier stated.

4.1 *Fundamental concepts and structures*

In what follows, we employed the formulations of the Hudson-Parthasarathy quantum stochastic calculus described briefly. Let D be an inner product space and H, the completion of D. We denote by $L^+(D,\ H)$, the set

$$\{X : D \to H\ \ /X \text{ is a linear map with } D \subseteq DomX^*\},$$

where X^* is the operator adjoint of X. We remark that $L^+(D,\ H)$ is a linear space under the usual notions of addition and scalar multiplications of operators.

If H is a Hilbert space, then the Boson Fock space $\Gamma(H)$ determined by H is the Hilbert space direct sum given by :

$$\Gamma(H) = \bigoplus_{n=0}^{\infty} H^{(n)}$$

where $H^{(0)} = \mathbb{C}$. For $n \geq 1$, $H^{(n)}$ is the subspace of the n-fold Hilbert space tensor product of H with itself comprising all symmetric tensors:

$$H^{(n)} = (H \otimes \cdots \otimes H)_{\text{sym}}$$

For each $f \in H$, an element of the form:

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f$$

is called an exponential or coherent vectors in $\Gamma(H)$. Here $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$. The element $e(0)$ in $\Gamma(H)$ is called the vacuum vector.

It is well established that the exponential vectors enjoy a number of properties. Its linear span is dense in the Fock space, the set of exponential vectors is linearly independent among other properties. These properties aid the use of exponential vectors for the development of the H-P QSC calculus.

For an arbitrary inner product space and its completion $\mathcal{R}$, and γ some fixed Hilbert space we write

$$\mathcal{A} = L^+(\underline{\otimes}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))),$$

where $L^2_\gamma(\mathbb{R}_+)$ is the Hilbert space of square integrable, γ-valued maps on $\mathbb{R}_+$ and $\underline{\otimes}$ denotes the algebraic tensor product. Many other relevant spaces are similarly defined and employed. Motivated by the first and second fundamental formula of H-P (Hudson and Parthasarathy, 1984), Ekhaguere (1992) equipped the linear space $\mathcal{A}$ with a locally convex topology generated by a family of seminorms

$$\{x \to \|x\|_{\eta\xi} = | < \eta, x\xi > |, \ x \in \mathcal{A}, \eta, \xi \in \underline{\otimes}.\}$$

The space $\mathcal{A}$ is similarly equipped with another locally convex topology generated by the family of semi norms:

$$\{x \to \|x\|_\xi = \|x\xi\|, \xi \in \underline{\otimes}\}$$

following the second fundamental formula of H-P. The completions of these spaces are employed in our research activities in this field. Thus, we define a quantum stochastic process as an $\tilde{\mathcal{A}}$ valued map on some intervals contained in the positive segment of the real number line.

For the coefficients of the QSDE (resp. QSDI) $E,\ F,\ G,\ H : [0,T] \times \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ (resp.$E,\ F,\ G,\ H : [0,T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$) belonging to appropriate spaces and with the solutions $X(t)$ enjoying suitable properties, the following is a brief account of our contributions.

4.2 *Our research contributions*

(1) **Numerical schemes for solving quantum stochastic differential equations.** We proposed, developed and multi-step schemes for solving numerically Lipschitzian quantum stochastic differential equation (LQSDE) (Ayoola, 2000a) of the form:

$$dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA^+_f(t) + G(t, X(t))dA_g(t) + H(t, X(t))dt \qquad (4.1)$$

in an interval $[t_0, T]$ with initial condition $X(t_0) = X_0$. The driving processes $\Lambda_\pi,\ A^+_f,\ A_g$ are the stochastic integrators in the Boson Fock space quantum stochastic calculus. Convergence of the discrete schemes to the exact solutions and error estimates were obtained for explicit scheme of class A in the locally convex space of solutions. Results in (Ayoola, 2000a) contain the Euler-Maruyama schemes for Ito stochastic differential equations as a special case and numerical examples were given. Explicit and exact solutions of LQSDE (7.61) are rarely available, making the search for approximate solutions a necessary and worthwhile endeavour. Prior to the publication of Ayoola (2000a), very little, if any at all, was known about the features of numerical solutions of LQSDE (4.1). As the LQSDE is a non-commutative generalisation of the classical Ito stochastic differential equation (Ito SDE), driven by Brownian motion, the implementation of the multi-step schemes and other discrete schemes developed in my subsequent works completely eliminated the need for the computation of random increments by random number generators as obtained in the implementation of stochastic Taylor schemes for simulation of sample paths and functional of solutions of classical Ito SDE. This paper, (Ayoola, 2000a) has opened further research directions concerning the refinement of the schemes in several ways, as well as the study of numerical stability associated with the multi-step schemes. The convergence and stability of a general multi-step scheme for (4.1) was considered in (Ayoola, 2000b). For the *Mathematical Reviews* database published by the American Mathematical Society (AMS), the reviews of the articles (Ayoola, 2000a,b) were respectively written in the years 2001 and 2004. The first was written by one of the founding fathers of quantum stochastic calculus, **Emeritus Professor K. R. Parthasarathy** of the Indian Statistical Institute, and the other written by **Professor Henri Schurz** of Southern Illinois University, Carbondale, USA. The assigned review numbers are respectively:: MR 2001e:81065 and MR: 2004:81072.

(2) **One step Schemes for solving quantum stochastic differential equations (QSDE)** The paper: Ayoola (2000a) was concerned with the development, analysis and applications of several one-step schemes for computing weak solutions of LQSDE (4.1). The work was accomplished in the framework of Hudson and Parthasarathy formulation of quantum stochastic calculus and subject to the matrix elements of solution being sufficiently differentiable. The results here concern non-commutative generalisation of the usual Euler scheme, Runge-Kutta schemes and an integral scheme for computing solutions of the LQSDE. The paper contains results for the Ito SDE as a special case with Ito processes as multiplication valued operators in a simple Fock space. The schemes exhibit important implementation benefits as in Ayoola (2000a,b). The article in Ayoola (2001) is 40 page long and contains the main existence results of Ayoola (1998b) as appendix, as well as some numerical experiments to illustrate the main features of the different schemes and their error estimates. The one step schemes here also generalise discrete schemes reported in Ayoola (1999a) and Ayoola (1999b). Extension of the results here to the case of continuous time Euler approximation scheme and a computational scheme under Caratheodory conditions was undertaken in Ayoola (2002b). The findings in Ayoola (2001a) has created further research questions involving extensions to LQSDE (4.1) of various improvements already established for classical discrete schemes in the finite dimensional setting. The mathematical review of my article, Ayoola (2001a) was written by **Professor Rolando Rebolledo Berroe** of Zentralblatt Mathematics Database, Germany with review number: Zbl 0998.60056 and the Abstract is listed in the *AMS Mathematical Review* with number MR 2002f:65017.

(3)**Kurzweil equations associated with QSDE.** In Ayoola (2001b), we introduced and studied Kurzweil equations associated with LQSDE (4.1) and I established the non-commutative quantum extensions of classical Kurzweil integrals and some technical results. In addition, we proved the interesting equivalence between LQSDE (4.1) in integral form and the Kurzweil equation of the form:

$$\frac{d}{d\tau} < \eta, X(\tau)\xi >= D\Phi(X(\tau), t)(\eta, \xi)$$

on $[t_0, T]$ and for $t \in [t_0, T]$, for a suitable map Φ and $\eta,\ \xi$ belonging to an appropriate class. Investigation of approximate solutions of LQSDE (4.1) by utilising established results on Kurzweil integrals and equations was afforded by the equivalence results. It was shown in the paper that the associated Kurzweil equation may be used to obtain reasonably high accurate solutions of the LQSDE. This paper extends established relationship between Lebesgue and Kurzweil integrals to quantum stochastic integrals. This particular study generalised some numerical results in Ayoola (2000a,b) since the results in Ayoola (2001b) hold under pure Caratheodory conditions where the matrix elements of solutions need not be differentiable more than once. The results also generalised several analogous results for classical initial value problems to the non commutative quantum setting involving unbounded linear operators on a Hilbert space. Further research problems have thus been opened by the paper of Ayoola (2001b) concerning the issue of variational stability of LQSDE (4.1). The review of the article Ayoola (2001b) was written in the year 2002 by **Professor Debashish Goswami** of Indian Statistical Institute for the *AMS Mathematical Reviews* with review number MR 2002g: 81078.

(4) **Construction of approximate attainability set for QSDI** We presented similarly, a numerical method for constructing with a specified accuracy, attainability set $R^{(T)}(x_0)(\eta, \xi)$ (Ayoola (2001c) defined by

$$R^{(T)}(x_0)(\eta, \xi) = \{< \eta, \Phi(T)\xi >: \Phi(\cdot) \in S^{(T)}(x_0)\} \tag{4.2}$$

for the Lipschitzian quantum stochastic differential inclusion (LQSDI) in integral form:

$$X(t) \in x_0 + \int_{t_0}^{t} (E(s, X(s))d\lambda_\pi(s) + F(s, X(s))dA_f(s)$$

$$+ G(s, X(s))dA_g^+(s) + H(s, X(s)) ds, \ t \in [t_0, T]. \tag{4.3}$$

where $S^{(T)}(x_0)$ is the set of solutions to LQSDI (4.3).

An algorithm is described for numerically approximating the attainability set within any prescribed accuracy. Results in this paper generalised an analogous classical result of Komarov and Pevchikh to non-commutative quantum stochastic differential inclusion (4.3). Attainability sets are important for several characterisation of the set of trajectories of LQSDI (4.3). In (Ayoola (2008b), I established the existence of solutions of QSDI (4.3) satisfying a general Lipschitz condition. The Lipschitz condition of Ayoola (2001c) is a special case and extension of the numerical algorithm of this paper to general case is still open. The AMS review of (Ayoola (2001c) was written by **Professor Volker Wihstutz** of North Carolina State University, Charlotte for the *AMS Mathematical Reviews* with review number MR 2002f:65018.

(5)**Lagrangian quadrature for computing solutions of QSDE**
Investigations in Ayoola (2002a) concerned the analysis of the Lagrangian quadrature schemes for computing weak solutions of LQSDE (1.1) with matrix elements that are sufficiently smooth. Results concerning the convergence of Lagrangian schemes to exact solutions were obtained. Precise estimates for an error term were given in the case when the nodes of approximations are chosen to be roots of the Chebyshev polynomials. Some important features of the quadrature schemes are the conversion of LQSDE (4.1) to solvable algebraic equations in term of the nodal values and that the nodes need not be equally spaced. This paper established the possibility of applying numerous results in linear and computer algebra for investigating numerical solutions to LQSDE (4.1). Numerical experiments were performed by solving associated linear systems taking into consideration, computational complexity of the algorithm and round off errors.
The AMS review of this paper was written by **Professor Vassili N. Kolokoltsov** of Warwick University, UK, for the *AMS Mathematical Reviews* with review number MR 2003e:60121.

(6) **Topological properties of solution sets of QSDI** In Ayoola (2008c), we established a continuous mapping of the space of the matrix elements of an arbitrary nonempty set of quasi solutions of Lipschitzian QSDI (4.3) into the space of the matrix elements of its solutions. As a corollary, we furnished a generalisation of my previous selection result in Ayoola (2004b). In particular, when the coefficients of the inclusion are integrally bounded, it was shown that the space of the matrix elements of solutions is an absolute retract, contractible, locally and integrally connected in an arbitrary dimension. As usual, we employ the Hudson and Parthasarathy formulation of quantum stochastic calculus. The AMS review of this paper was written by **Professor Vassili N. Kolokoltsov** of Warwick University, UK for the *AMS Mathematics Reviews* with assigned number MR 2008k: 81174.

(7) **Existence of strong solutions of LQSDE.** In Ayoola (2002c), the existence, uniqueness and stability of strong solutions of LQSDE (4.1) were established. The locally convex topology on the space of quantum stochastic processes in this case is generated by a family of semi-norms induced by the norm of the Fock space. The second fundamental formula of Hudson and Parthasarathy concerning the estimate of the square of the norm of the values of stochastic processes on exponential vectors facilitates the existence results by method of successive approximations. Results here generalise analogous results concerning classical SDE driven by Brownian motion. Convergence in the sense of this paper generalise the root mean square convergence of successive approximation in the case of classical Ito process considered as quantum stochastic process in a simple Fock space. The study of Ayoola and Gbolagade (2005) happened to be a continuation of

Ayoola (2002c) concerning the existence and stability of solutions of QSDE satisfying a general Lipschitz condition in the strong topology. Ayoola and Gbolagade (2005) established a class of Lipschitzian QSDE where the coefficients are merely continuous on the locally convex space of the quantum observables.

The AMS Mathematical Reviews of paper Ayoola (2002c) and Ayoola and Gbolagade (2005) were written respectively by **Professor Vassili N. Kolokoltsov** of Warwick University, UK and **Professor Debashism Goswami** of Indian Statistical Institute for the *AMS Mathematical Reviews* with review numbers respectively given by MR 2003b: 60081 and MR 2005m: 81179.

(8) **Exponential formula for the reachable set of QSDI.** Ayoola (2003a) was my second major work on quantum stochastic differential inclusions (QSDI) (4.3). The paper was a continuation of my previous work Ayoola (2001c) concerning the QSDI, where the coefficients are assumed to have suitable regularity properties. The basic set-up of the paper is that of multi-valued functions with appropriately defined multi-valued stochastic integrals. By endowing the family of closed subsets of the locally convex space of quantum observables with a Hausdorff topology, the paper established the following exponential formula:

$$R^{(T)}(x_0) = \lim_{N\to\infty} \left(I + \frac{T}{N}P\right)^N (x_0) \tag{4.4}$$

where $R^{(T)}(x_0)$ is the reachable set of QSDI (3), I is the identity multifunction.

Repeated composition of multi-functions is understood in some sense and the limit in Equation (4.4) is interpreted as the Kuratowski limit of sets. Equation (4.4) has a direct consequence for the convergence, to the exact value, of discrete approximations to the reachable set. The basic motivation for considering QSDI (4.3) concerns the need to develop a reasonable numerical scheme for solving QSDE (4.1) with discontinuous coefficients since many of such interesting QSDE can be reformulated as QSDI with regular coefficients.

The AMS review of Ayoola (2003a) was written by **Professor Debashish Goswami** of Indian Statistical for the *AMS Mathematical Reviews* with review number MR 2004e:81073.

(9) **Discrete approximation of solutions of QSDI.** Ayoola (2003b) was a continuation of my study of discrete approximation of QSDI (4.3). This paper is concerned with the error estimates involved in the solution of a discrete version of QSDI (4.3). The main results relied on some properties of the averaged modulus of continuity for multi-valued sesquilinear forms associated with QSDI (4.3). The paper established a sharp estimate for the Hausdorff distance between the set of solutions of QSDI (4.3) and the set of solutions of its discrete approximations. This paper however, extended the result of Dontchev and Farkhi (1989) concerning classical differential inclusions to the present non-commutative quantum setting involving inclusions in certain locally convex spaces.

The AMS review of this paper was written by **Professor Habib Querdiane** of the University of Tunis El Manar, Tunisia for the *AMS Mathematics Reviews* with review number MR 2005a: 60109.

(10) **Existence of continuous selections of solution and reachable sets.** Ayoola (2004b) established the existence of continuous selections of solution set of Lipschitzian QSDI (4.3). The paper precisely proved that if

$$S^{(T)}(a)(\eta,\xi) =: \{t \to < \eta, X(t)\xi >,\ X \in S^{(T)}(a)\}$$

is a set of absolutely continuous complex valued functions associated with the set of solutions $S^{(T)}(a)$ of QSDI (4.3), then the multifunction $< \eta, a\xi > \to S^{(T)}(a)(\eta,\xi)$ admits a continuous selection for all $a \in A$ given that the set of matrix elements $A(\eta,\xi)$ of A is compact in the field of complex numbers. As corollaries to the main result, we proved that the solutions set map, as well as the reachable sets of QSDI (4.3) admitted some continuous representations. A search in the AMS Mathematical Reviews and the ISI Web of Science databases showed that Ayoola (2004a) was the first known selection result concerning QSDI (4.3) in the framework of the Hudson -Parthasarathy formulation of quantum stochastic calculus. Consequently, the paper opened further research questions in respect of the refinement, generalisation and applications of the selection results in parallel with the classical cases of differential inclusions in finite dimensional Euclidian spaces. Ayoola (2008a) was a follow up publication, where I showed that a continuous selection from the set of solutions exists directly defined on the space of stochastic processes with values in the space of adapted weakly absolutely continuous solutions. As a corollary, the reachable set multifunction admits a continuous selection. Ayoola and Adeyeye (2007) also extended the selection results in Ayoola (2004a) as an interpolation to cover a finite number of trajectories. The AMS Mathematical reviews of Ayoola (2004a), Ayoola and Adeyeye (2007) and Ayoola (2008a) were written by **Professor C. R. Belton** of Mathematics Department, Lancaster University, UK and **Professor Vitonofrio Crismale** of Muhammad Ibn University Saudi Arabia, for the *AMS Mathematical Reviews* with review numbers MR 2005i: 81076, MR 2008m:65012 and MR 2011d:81177 respectively.

(11) **Mayer problem for quantum stochastic control.** In the framework of Ekhaguere's formulation of the multi-valued analog of H-P quantum stochastic calculus, Ogundiran and Ayoola (2010), concerns some results on quantum stochastic control. In particular, we studied the regularity properties of the value function inherited from the multi-valued quantum stochastic processes involved. We showed that under the assumption of directional differentiability of the value function, the associated Mayer problem had at least one optimal solution. Our theory covered earlier work on quantum stochastic control by Belavkin and by Andreas Boukas. The AMS Review of this paper was written by **Professor Andreas Boukas** of Southern Illinois University, Carbondale, USA, with Review number 2011d:81180.

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