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On M-loop as a special class of G-loops

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Abstract. M-Loops are considered as part of a generalised Moufang loop. The concept of isotopy-isomorphy is considered for a loop, which therefore yielded a loop that is isomorphic to its isotope. This special class of loop investigated is an M-loop. Isotopy played a central role in loop theory. Both isomorphism and isotopism served as equivalence relations for any non-empty set of loops. In this work, the M-loops that are isomorphic to each of their loops isotopes are classified as G-loops.

Keywords: G-loop, M-loop, isomorphy, isotopy.

1. Introduction

It is true that G-loops are loops that are isomorphic to each of their loops isotopes. Two algebraic structures are said to be isomorphic if one can be obtained from other by relabeling the element. A property is said to be isotopic invariant if whenever it holds in the domain loop i.e.,$(G, *)$ then it must hold in co-domain loop i.e., $(H, \circ)$ which is an isotope of the formal. Therefore, the property in consideration is said to be a universal property,hence the loop is called a universal loop.

A map α:Q→H is an isomorphism of a groupoid $(Q, *)$ onto a groupoid $(H, \circ)$ if α is a bijection and $(x*y)\alpha = x\alpha \circ y\alpha \; \forall \; x$,$y$ in Q. For a pair of groupoids $(Q, *)$ and $(H, \circ)$, then $(Q, *)$ is isomorphic to $(H, \circ)$,provided that there is at least one one isomorphism of $(Q, *)$ onto $(H, \circ)$.

If $(Q, *)$ and $(H, \circ)$ are two groupoids $(Q, *)$ is an isotope of the quasigroup $(H, \circ)$ if there are permutations α, β, γ of Q such that for all x,y in Q, $x\alpha*y\beta = (x \circ y)\gamma$. If $(Q, *)$ and $(H, \circ)$ are isotopic, then Q and H have the same cardinality. We shall consider a special kind of isotope known as f,g - isotope as a standard loop isotope in this work. The isotope $(Q, *)$ such that $x * y = (x/g)\circ(f \setminus y)$ is called the f,g-isotope of $(Q, *)$.

The isotopic invariance of varieties of loops especially those that fall in the class of Bol-Moufang type have been of interest to researchers in loop theory in the recent past. Loops such as Bol loops, Moufang loops, central loops and extra loops are most popular loops of Bol-Moufang type whose isotopic invariance have been considered. In 1970, Basarab [1] worked on the existing work of Osborn [13] of 1961 on universal weak inverse property loops ($WIPLs$) by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs in 1967. Osborn [13],while investigating the universality of WIPLs discoversd that a universal WIPL $(G, .)$ satisfies the identity

$$yx.(zEy.y) = (y.xz).y \qquad \forall x, y, z \in G \tag{1.1}$$

where $Ey = LyLy^{\lambda} = R^{-1}_{y^{\rho}}R^{-1}_{y} = LyRyL^{-1}_{y}R^{-1}_{y}$ and y^{λ} and y^{ρ} are respectively the left and right inverse elements of y.

Eight years after Osborn's [13] 1960 work on $WIPL$, in 1968, Huthnance Jr [8] studied the theory of generalized Moufang loops. He named a loop satisfying (1.1) a **generalized Moufang loop** and later on the same thesis called them **M-Loops**. Few publications that have considered G-loops includes Solarin [17], Chiboka [6], Basarab [2], Robinson [15], Belousov[3], Wilson [18], Osborn [13]. Jaiyeola and Adeniran worked on G-Osborn loops [10].

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2. Preliminaries

DEFINITION 2.1 *A groupoid $(G, *)$ together with a binary operation $*$ is called a groupoid if it is closed under the operation.*

DEFINITION 2.2 *A groupoid $(G, *)$ is called a quasigroup if on the set G there exist operations "$\backslash$" and "/" such that in algbra $(G, *, \backslash, /)$ the following identities are fulfilled:*

$$x *(x \backslash y) = y$$
$$(y/x) *x=y$$
$$x \backslash (x * y) = y$$
$$(y * x) / x = y$$

DEFINITION 2.3 *A loop $(G, *, \backslash, /, e)$ is a set G together with three binary operations $*, \backslash, /$ and nullary operation e such that:*

i. $x*(x \backslash y)=y$, $(y/x)*x = y$ for all x,y in G.
ii. $x\backslash(x * y)=y$, $(y * x)/x=y$ for all x, y in G.
iii. $x\backslash x=y/y$ or $e*x=x$ for all x,y in G.
Therefore, a quasigroup $(G, *)$ which has an identity element is a loop.

Consider $(G, *)$ and $(H, \circ)$ to be two distinct quasigroups or loops. Let α,β and γ be three bijections,that map G onto H. The triple $T = (\alpha, \beta, \gamma)$ is called an isotopism of $(G, *)$ onto $(H, \circ)$ if and only if $x\alpha \circ y\beta = (x * y)\gamma$, $\forall x,y \in G$. So, $(H, \circ)$ is called a groupoid (quasigroup, loop) isotope of $(G, \circ)$. If $\gamma = I$ is the identity map on G so that $H = G$, then the triple $T=(\alpha, \beta, I)$ is called a *principal isotopism* of $(G, *)$ onto $(G, \circ)$ and $(G, \circ)$ is called a *principal isotope of* of $(G, *)$. Eventually, the equation of relationship now becomes $x*y=x\alpha\circ y\beta$, $\forall$ $x, y\in G$. But if $\alpha=R_g$ and $\beta=L_f$ where $R_x : G \to G$, the right translation is defined by $yR_x = y * x$ and $L_x : G \to G$, the left translation is defined by $yL_x = x * y$ for all $x, y \in G$, for some $f, g \in G$, the relationship now becomes

$$x * y = xR_g \circ yL_f \text{ , for all } x, y \in G \text{ or}$$
$$x \circ y = xR_g^{-1} * yL_f^{-1} \text{ for } x, y \in G.$$

With the new form, the triple T $=(R_g, L_f, I)$ is called an $f, g-$*principal isotopism* of $(G, *)$ onto $(G, \circ)$, f and g are called *translation elements* of G or at times written in the pair form (g, f), while $(G, \circ)$ is called an $f, g-$*principal isotopism* of $(G, *)$.

DEFINITION 2.4 *A loop $(G, *)$ is a G-loop if it is isomorphic to all of its loop isotopes.*

This terminology was first employed by Belousov [3]. Bruck [4] has raised the question as to the necessary and sufficient condition for a loop to be a G-loop. He noted that associativity is sufficient but not necessary since the multiplicative loop of any alternative division ring has this property. The concept of a G-loop is a generalization of the concept of a group in the sense that every group is isomorphic to all its loop isotopes.

3. Discussion and findings

DEFINITION 3.1 *A loop $(G, *)$ is said to be an M-loop if it satisfies the identical relation*

$$(x * y) * (z * x^\alpha) = [x * (y * z)]x^\alpha \tag{3.1}$$

where x^α is the image of x under some single-valued map α of G into itself.

If

$$\alpha : x \to x^k$$

where k is a positive integer, the law (3.1) is called an M_k-law. Pflugfelder [11] showed that every

M-loop is a Moufang loop. Hence the class of M-loops is just the class of Moufang loops. Chein and Pflugfelder [5] investigated the class of M-loops which satisfy an $M_k - law$ for some $k > 1$, but for which $x^{k-1} = 1$ does not hold for all x in the loop.

DEFINITION 3.2 *If $(G, *)$ is an M-loop, but not an M_k-loop for any $k > 1$, then G is called a strictly Moufang loop.*

Chein and Pflugfelder [5] proved that every loop which satisfies an M_k-law for some $k \neq 1$ (mod 3) is a G-loop.

LEMMA 3.3 *(Sharma [16]) If the loop $(G, *)$ satisfies any one of the following M-loop identities, then $(G, *)$ satisfies the following three identities*

$$(xy)\ (zx^{\alpha}) = (x.yz)\ x^{\alpha}$$
$$(yx.z)\ x^{\alpha} = y\ (x.zx^{\alpha})$$
$$(x^{j\alpha j}y.x)\ z = x^{j\alpha j}\ (y.xz)$$

for all x,y,z $\in G$, where x^{α} is the image of x under some mapping α of the loop $(G, .)$ into itself, and j is the inverse permutation of $(G, .)$ defined by
$xj = x^{-1}$.

DEFINITION 3.4 *Let Π_R and Π_L be non-empty subsets of the symmetric group on the set G defined by $\Pi_R = \{R(a)|a \in G\}$ and $\Pi_L = \{L(a)|a \in G\}$. Then $(G, .)$ is a conjugacy closed loop if and if $R(x)^{-1}R(y)R(x) \in \Pi_R$ and $L(x)^{-1}L(y)L(x) \in \Pi_L$ for all x,y in G.*

Goodaire and Robinson [8] have established some characterizations, and some algebraic properties of conjugacy closed loops.

THEOREM 3.5 *(Chiboka[5]) Let $(G, .)$ be an extra loop. Then $(G, .)$ is a G-loop.*

THEOREM 3.6 *Let $(G, .)$ be an extra loop, then $(G, .)$ is an M-loop.*

Proof. $(G, .)$ is conjugacy closed, so $R(x)^{-1}R(y)R(x) \in \Pi_R$ and $L(x)^{-1}L(y)L(x) \in \Pi_L$ for all $x, y \in G$. Now, $zR(x)^{-1}R(y)R(x) = za$ for all z and some $a \in G$.Since $(G, .)$ is an inverse property loop, we have $(zx^{-1}.y)x = za\ z = x$, then $yx = xa \Longrightarrow a = x^{-1}.yx$ so, $(zx^{-1}.y)x = z(x^{-1}.yx)$ for all $x, y, z \in G$. If we replace x with t^{-1}, we get $(zt.y)t^{-1} = z(t.yt^{-1})$ for all $t, y, z \in G$, and Lemma 3.1, with α the inverse mapping $(G, .)$ is an M-Loop. ■

COROLLARY 3.7 *Let $(G, .)$ be a central loop, then $(G, .)$ satisfies $(yx.z)x^{-1} = y(x.zx^{-1})$ for all $x, y, z \in G$, hence $(G, .)$ is an M-loop.*

Proof. Let C be a central algebra. Then $R(y)R(x)R(y)^{-1} = R(y.xy)^{-1}$ for all $x, y \in C$, we have $yR(x)R(z)R(x)^{-1} = yR(x.zx^{-1})$ for all $y \in G$, so $(yx.z)x^{-1} = y(x.zx^{-1})$. By Lemma 3.1, with α the inverse mapping, $(G, .)$ is an M-loop. ■

THEOREM 3.8 *Let $(G, .)$ be a loop and α is a mapping of G onto itself such that $R(x)R(y)R(x^{\alpha})$ $\in \Pi_R$ and $L(x)L(y)L(x^{\alpha})$ or $L(x^{\alpha})L(y)L(x) \in \Pi_L$, then $(G, .)$ is an inverse property loop.*

Proof. Suppose $R(x)R(y)R(x^{\alpha}) \in \Pi_R$, then there exists $a \in G$ such that $(zx.y)x^{\alpha} = za$ for all $x, y, z \in G$. In particular, $z = 1$ gives $xy.x^{\alpha} = a$ for all $x, y \in G \Longrightarrow (zx.y)x^{\alpha} = z(xy.x^{\alpha})$ for all $x, y, z \in G$. Substituting $y = x^p$, we have $(zx.x^p)x^{\alpha} = z(xx^p.x^{\alpha}) = zx^{\alpha} \Longrightarrow zx.x^p = z$ for all $x, y, z \in G$, hence $(G, .)$ satisfies the right inverse property. If $L(x)L(y)L(x^{\alpha}) \in \Pi_L$, then $x^{\alpha}(y.xz) = bz$ for all x, y, z and some $b \in G$. Now, $z = 1$ gives $x^{\alpha}.yx = b$ for all $x, y \in G$ so $x^{\alpha}(y.xz) = (x^{\alpha}.yx)z$ for all $x, y, z \in G$. Setting $y = x^{\lambda}$ gives $x^{\alpha}(x^{\lambda}.xz) = (x^{\alpha}.x^{\lambda}x)z = x^{\alpha}z\ x^{\lambda}.xz = z$ for all $x, z \in G$, hence $(G, .)$ satisfies the left inverse property. Similarly, if $L(x^{\alpha})L(y)L(x) \in \Pi_L$, then $x(y.x^{\alpha}z) = bz$ for all $x, y, z \in G$ and some $b \in G$. But $z = 1$ gives $x.yx^{\alpha} = b$ for all $x, y \in G \Longrightarrow x(y.x^{\alpha}z) = (x.yx^{\alpha})z$ setting $y = (x^{\alpha})^{\lambda}$ gives $x[(x^{\alpha})^{\lambda}.x^{\alpha}z] = [x.(x^{\alpha})^{\lambda}x^{\alpha}]z \Longrightarrow x[(x^{\alpha})^{\lambda}.x^{\alpha}z] = xz\ (x^{\alpha})^{\lambda}.x^{\alpha}z = z$ for all $x, z \in G$. Consequently, $(G, .)$ has the left inverse property. ■

THEOREM 3.9 *Let $(G, .)$ be an M-loop with α the inverse mapping, then $(G, .)$ is conjugacy closed.*

Proof. From Lemma 3.1, with $x^\alpha = x^{-1}$ we have

$$(yx.z)x^{-1} = y(x.zx^{-1}) \tag{3.2}$$

and

$$((x^j)^{-1})^j[y.xz] = [((x^j)^{-1})^j y.x]z \tag{3.3}$$

Equation (3.11) implies that $yR(x)R(z)R(x^{-1}) = yR(x)R(z)R(x)^{-1} \implies yR(x.zx^{-1})$ $R(x)R(z)R(x)^{-1} \in \Pi_R$ for all $x, z \in G$ Equation 3.12 implies that $zL(x)L(y)L(x^{-1}) = zL(x)L(y)L(x)^{-1} \implies zL(z^{-1}y.x)$ $L(x)L(y)L(x)^{-1} \in \Pi_L$ for all $x, y \in G$. Hence, $(G, .)$ is conjugacy closed.

THEOREM 3.10 *Let $(G, .)$ be a conjugacy closed loop. The following statements are equivalent.*

(i) *$(G, .)$ is flexible;*
(ii) *$(G, .)$ has the inverse property;*
(iii) *$(G, .)$ is an M-loop.*

Proof. Let A and B be autotopisms of G for all $x \in G$. Since $(G, .)$ is conjugacy closed, we have $A =< L(x)R(x)^{-1}, L(x), L(x) >$ and $B =< R(x), R(x)L(x)^{-1}, R(x) >$ If $(G, .)$ is flexible, then $xy.x = x.yx$ for all $x, y \in G$, so $yL(x)R(x) = yR(x)L(x) \implies L(x)R(x) = R(x)L(x)$ for all $x \in G$. Now, $BA = < R(x), R(x)L(x)^{-1}, R(x) >< L(x)R(x)^{-1}L(x), L(x) > = < R(x)L(x)R(x)^{-1}, R(x)L(x)^{-1}L(x), R(x)L(x) > = < L(x), R(x), R(x)L(x) > = < L(x), R(x), L(x)R(x) >$ is an autotopism of G for all $x \in G$, hence $(G, .)$ is a Moufang loop. Thus, (i) $\implies$ (iii). Being an M-loop, (i) $\implies$ (ii) since every M-loop has the inverse property. If $(G, .)$ is an inverse property loop, then $R(x)^{-1} = R(x^{-1})$ and $L(x)^{-1} = L(x^{-1})$ for all $x \in G$, so $zR(x)R(y)R(x)^{-1} = zR(x.yx^{-1}) \implies (zx.y)x^{-1} = z(x.yx^{-1})$. This is equivalent to (3.11) with $x^\alpha = x^{-1}$, hence by Sharma [13], $(G, .)$ is an M-loop. Thus (ii) $\implies$ (i) and (iii). It is known that every M-loop is both flexible and has the inverse property, but an inverse property loop is not necessarily an M-loop. Thus, (ii) of Theorem 3.5 provides a condition for an inverse property loop to be an M-loop. ∎

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