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In honour of Prof. Ekhaguere at 70

Review of academic research works of Professor G.O.S. Ekhaguere: 1976–2017

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Abstract. The contributions of Professor G.O.S. Ekhaguere between 1976 and 2017 can broadly be classified into 4 groups as follows: (a) contributions to mathematical physics, (b) contributions to non-commutative stochastic analysis, (c) contributions to ∗-algebras and (d) contributions to mathematical finance. All the contributions are significant, breaking new grounds at the frontiers, lead to new enquiries, questions and applications to physical problems. The groups are not mutually exclusive. Results in some of the groups are often applied or employed in other groups.

Keywords: mathematical physics, non-commutative stochastic analysis, ∗-algebras, mathematical finance.

1. Contribution to Mathematical Physics

(a) In 1977, Professor G.O.S. Ekhaguere (GOSE, hereinafter) worked on the Markov properties of stochastic processes due to Nelson and Wong. The study furnished important applications to Mathematics and Physics problems. Further details can be found in **Journal of Mathematical Physics, 18(1977)2104-2107** and reviewed by T. Neabrunn for the AMS published MR.

(b) In 1978, GOSE did not only develop the theory of superselection rules, but also established a wide class of inequivalent irreducible ∗-representations of the canonical commutations relations of the electromagnetic field. He employed the method of C^*-algebra for the representation. These contributions were published in **J. Mathematical Physics, Volume 19,1751-1757 2**, reviewed for the MR by Y. Kato.

He Continued his study of superselection rules in **J. Mathematical Physics, 25 (1984), 678-683** and characterized a subclass of the ∗-representation consisting of positive ∗-representation. He exhibited new superselection sectors.

1.1 *Gaussian fields of Markov types*

In 1979, GOSE established necessary and sufficient conditions for a class of Guassian generalized field to have a Markov property. He showed that Wong's notion of Markov property is weaker than that of Nelson in certain cases. More so, he showed that his results have applications to quantum field theory by employing the theory of Markovian generalized stochastic fields. These eesults were published in **Physics A 99(3) 545-568 1979**, reviewed by Koichiro Matsuno for the MR.

Furthermore he established a theorem in the same year 1979, that new Markov fields may be obtained from old ones by the use of multiplicative measurable operators. He employed the Gudder-Marchand formulation of noncommutative integration. Results were published in **J. Maths Physic. 20(8) 1679-1683 (1979)**, reviewed by Paul Benioff for the MR.

In 1980, GOSE established a characterization of Markovian homogenous multicomponent Gaussian fields. He gave a necessary and sufficient condition for Markov property. Results were published in **Communications in Maths Physics, 63-77 (1980)**.

In 1982, GOSE formulated a noncommutative stochastic process over complete locally convex ∗-algebra and discussed quantum fields as examples. The results were published in **J. Physics A 15(11) 3453-3463** and reviewed by H. Araki for the MR.

1.2 *Central limit theorems*

In 1985, GOSE established some central limit theorems in probability space. He extended the results of Urbanik to the case where the random variables are densely defined self adjoint linear operators on a separable Hilbert space. A 50 page-long contribution was published in **Publications Research Institute Maths Science (1985) V21(3) 541-591**.

2. Contributions to non-commutative stochastic analysis

(a) In 1982, GOSE made a very significant contribution to the theory of non-commutative stochastic integration. He developed stochastic integration with respect to certain Martingales of non-commuting measurable operators and showed by some calculations that his formulation extends the classical Ito integration. The results were published in **J. Nigerian Maths Society Vol 1 (1982), 11-23**.

(b) In 1985, GOSE reformulated some results of Hudson and Patasarathy in the language of Op*-algebra. He established a notion of differentiability within the context of the algebra. Then he established a chain rule for stochastic differentials of suitable integrands. Results were published in **Lecture Notes in Physics 262, Springer, 1986** and reviewed by David Applebaum.

GOSE also established the existence and some properties of solutions of quantum integral equations in **COMO, 1985, 453-455**.

(c) In 1990, GOSE established a major and complicated theorem on the functional Ito formula in quantum stochastic calculus. A noncommutative analogue of the Ito formula for Boson quantum stochastic integrals was developed. His algebraic approach allowed the validity of results for unbounded operators. He introduced an operator algebra of unbounded linear operators on certain Hilbert spaces and considered the locally convex completions of the algebra. Results were published in **J. Maths Physics 31(1990) 2921-2929** and reviewed by Koichiro Matsuno for the MR. It should be noted that the paper was submitted in 1986 but published after 4 years under editorial review.

(d) In 1994, GOSE established stochastic integration in $*$-algebras without Doob-Meyer decomposition theorems. He defined the integration with respect to square integrable Martingales in unital $*$-algebras. He generalized some of his previous results in this direction. Earlier, he had worked on decomposition theorems and established a Doob-Mayer decomposition theorems. The theory was shown to be applicable to algebras generated by annihilation and creation operators on symmetric Fock spaces. Results were published in **J. Nigerian Maths Soc 13 (1994), 9-22 and 81-101** and reviewed by Stanislaw Goldstein.

(e) In 1992, he began to publish series of papers on quantum stochastic differential inclusions of the form

$$\begin{aligned} dx(t) &\in E(t,x(t))d\wedge_\pi(t) + F(t,x(t))dA_f(t) \\ &\quad + G(t,x(t))dA_g^+(t) + H(t,x(t))dt \\ x(t_0) &= x_0, \quad \text{almost all} \quad t \in [t_0,T] \end{aligned}$$

which is understood in integral form as:

$$\begin{aligned} x(t) \in x_0 + \int_{t_0}^t & (E(s,x(s))d\wedge_\pi(s) + F(s,x(s))dA_f(s) \\ & + G(s,x(s))dA_g^+(s) + H(s,x(s))ds) \end{aligned}$$

where the integral is in the sense of Hudson and Pathasarathy.

GOSE first generalised the standard Fock space quantum stochastic calculus to multivalued stochastic integrands. His approach have applications to stochastic controls and stochastic differential equations with discontinuous coefficients. In addition, GOSE established the existence of solutions to Lipschitzian QSDI and those of their convexifications.

In 1995, GOSE proved that a QSDI of hypermaximal monotone type has a unique adapted solution. As examples, he exhibited a large class of QSDI of hypermaximal monotone type arising as perturbations of certain quantum stochastic equations by some multivaled processes.

In 1996, GOSE developed the theory of Quantum stochastic evolutions in parallel with the classical theory. His 3 papers in this area were published in **International Journal on Theoretical Physics, Vol 31 (1992) 2003-2027, Vol 34 (1995) 323-353, Vol 35 (1996) 1909-1946** and reviewed by 2 world class mathematicians K.R Pathasarathy and Camillo Trapani for the AMS MR.

In the year 2007, by endowing the space $\mathcal{A}$ of quantum stochastic processes consisting of a class of linear maps from a preHilbert space to its completion, with seven different types of topologies generated by diverse families of seminorms, GOSE established properties of topological solutions of noncommutative stochastic differential equations in integral form given by:

$$\begin{aligned} dx(t) &= E(t,x(t))d\wedge_\pi(t) + F(t,x(t))dA_f(t) \\ &\quad + G(t,x(t))dA_g^+(t) + H(t,x(t))dt \\ x(t_0) &= x_0, \quad \text{almost all} \quad t \in [t_0, T] \end{aligned}$$

Results were published in **Stochastic Analysis and Applications, 25, 961-991 (2007)**, and reviewed for the MR by Raja Bhat.

3. Contributions to ∗-algebras

In 1988, GOSE worked on the Dirichlet forms of partial ∗-algebras. Some results on C*-algebras were extended to partial ∗-algebras.

He defined on $L^2(\mathcal{A},\mathcal{B},\tau)$ over the triple consisting of a partial ∗algebra and Ideal of $\mathcal{A}$ and τ a sesquilinear form satisfying some assumptions. GOSE established Dirichlet forms on $L^2(\mathcal{A},\mathcal{B},\tau)$ which are sesquilinear forms whose domain is closed under the actions of Lipschitzian maps. He examined relationship between between Markovian operators and Dirichlet forms. These results were published in **Maths Proceedings, Cambridge Philosophical Society, 104, (1988) 129-140** and reviewed by Camillo Trapani.

In 1989, GOSE worked on unbounded partial Hilbert algebras. He introduced this notion and studied some properties and examples. Results were published in **J. Maths Physics 30, (1989), 1957-1964** and reviewed by Konrad Schmudgen.

In 1991, GOSE worked on Partial W*-dynamical systems and on completely positive conjugate - bilinear maps on partial ∗-algebras. He introduced and studied the partial W*-dynamical systems $(M, \{\Phi_t\}, t \in \mathbb{R}_+)$, where Φ_t is a semigroup of a completely positive conjugate bilinear map on M. He solved the dilation problems and described the associated Markov process. He established major results on the generalization to a particular partial ∗-algebra of Stinesprings concerning completely positive maps on C^*-algebras and on generalization of Radon-Nikodym for ∗ algebras. Results were published in **J. Maths Physics, 32 (1991), 2951-2958**.

In 1993, GOSE worked on non-commutative mean ergodic theorem for partial W^* -dynamical semigroups. He furnished applications to statistical mechanics and quantum field theory and proved the Mean Ergodic Theorem for semigroups of maps on the algebra. The result which generalized Watanabe's Mean Ergodic Theorem was published in **Internal. Journal of Theoretical Physics 32 (1993), 1187-1196** and was reviewed by A.I. Danilenko.

In the year 2001, GOSE established an algebraic representation theory of partial algebras. He employed the notion of operator set. Results were published in the **Annals of Henri Poincare 2, 377-385** and reviewed by Camillo Trapani.

In the year 2007, GOSE examined Bitraces on Partial O*-algebras. He studied some properties of ∗-representations determined by bitraces. He furnished the notion of partial W*-algebras as generalization of W*-algebras. Results were published in **International Journal of Mathematics and Mathematical Sciences, 2007**, and reviewed by Francesco Tschinke for the MR.

In the year 2008, GOSE embarked on characterizations of partial algebras. He established that every locally convex algebra is an inductive limit of locally convex partial algebras. He identified partial algebras that can be represented as partial algebras of unbounded operators. Results were published in **J. Mathematical Analysis and Applications, 337, 1295-1301**.

In the year 2015, GOSE carried out a study on partial W*-dynamical systems and their dilations. He introduced the concepts of a partial O* -algebras and a partial W^*-algebras whose elements are linear operators on a Hilbert space. He described the infinitesimal generators of a ∗-biautomorphism groups and ∗-biderivations of a partial W^*-algebra. He established a relationship between the generators of ∗-biderivatives. Results were published in **Contemporary Maths 645, AMS, Provedence, RI** and reviewed by Chul Ki ko for the MR.

4. Contributions to mathematical finance

In the year 2004, GOSE examined some aspects of the mathematical foundations of the theory of contingent claims in financial markets. He described the classical theory of the pricing of contingent claims in an ideal financial markets and subsequently highlighted some ways of relaxing assumptions of an ideal market in the case of an imperfect or real world financial markets. The article appeared in **Publications of the ICMCS 1, 197-214**.