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Unsteady Magnetohydrodynamic Flow Model Of A Fourth Grade Fluid In A Porous Medium

Aroloye S.J, Fenuga O.J and Abiala I.O

Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria.

Abstract

Analysis is carried out to study the time dependent flow of an incompressible electrically conducting fourth grade fluid over an infinite porous plate in the presence of suction/injection, Darcy's resistance and magnetic field. A magnetic field is applied in the transverse direction to the flow. The governing nonlinear higher order partial differential equation for this flow is modelled. Translation symmetries in variable t and y are employed to construct two different classes of the exact close–form travelling wave solutions of the model equation. The physical behaviour and the properties of various interesting flow parameters on the structure of the velocity are presented graphically and discussed. Also, the importances of the rheological effects are mentioned. It is found that suction and injection have opposite role on flow velocity. The velocity field as well as the boundary layer thickness, decrease with an increase in the suction parameter. It also found that the magnetic field control the boundary layer thickness. It is depicted that the shock wave solution obtained is going to be very helpful in carrying out further analysis of the shock wave behaviour associated with the non-Newtonian fluid flow models.

Keyword: Fourth grade fluid. Lie symmetry approach.MHD flow, travelling wave solutions. Porous Medium.

1. INTRODUTION

Fluids such as water, air and kerosene are described as Newtonian fluids. These fluids are essentially modelled by the Navier-Stoke equations, which generally describe a linear relationship between the stress and the strain rate. In nature, there are many fluids which do not obey the Newtonian law of viscosity and the Navier-Stoke equations are inadequate for such fluids. These fluids are termed as non-Newtonian fluids. A distinguishing feature of non-Newtonian fluids is that they exhibit both viscous and elastic properties and the relationship between the stress and the strain rate is non-linear. A few examples of non-Newtonian fluids are, mayonnaise, toothpaste, egg whites, liquid soaps, multi grade engine oil, polymers solution, greases, coal slurries, paints, drilling mud , hydrocarbon oils, geological materials, polymers solutions, synovial, shampoo drilling muds, blood, paint, certain oil and greases, liquid forms, food products and many other emulsion. The understanding of the complex behaviour and properties of non-Newtonian fluids is crucial these days.The problems dealing with the flow of non-Newtonian fluids have several technical applications in industry, science and engineering. Such applications include extractions of crude oil from petroleum products, oil and gas well drilling, food stuff, extrusion of molten plastic, paper, polymer industry, textile industry and so on.

One of the widely accepted models amongst non-Newtonian fluids is the class of Rilvlin-Ericksen fluids of differential type [1] which can be classified as second, third and fourth grade fluids. Rivlin-Ericksen fluids [1] have secured special attention in order to describe the several features of non-Newtonian fluids such as rod climbing, shear thinning, shear thickening and normal stress effects. Due to the complex physical structure of non-Newtonian fluids, these fluids are classified according to the constitutive properties which characterise them. Literature surveys revealed that much focus has been given to the flow problems of a second-grade fluid [2-4]. A second grade fluid model, despite its simplicity, has drawbacks in that although it can predict the normal stress differences, it fails to recognise the shear thinning and shear thickening properties if the shear viscosity is assumed to be constant. This impediment is remedied by the third-grade fluid [5-8] and the fourth –grade fluid models. Very little attention has been given in literature to the studies related to the fourth grade fluid models [9, 10]. Currently, only the fourth grade fluid seems to define the properties of non-Newtonian flow phenomena most generally. The fourth-grade fluid model is known to capture the interesting non-Newtonian flow properties such as shear thinning and shear thickening that many other non-Newtonian models do not exhibit. This model is also capable of describing the normal stress effects that lead to phenomena like "die-swell'' and rod-climbing'' [11].

Therefore, some experiments may be well described by fourth grade fluid .The mathematical constitutive model of a fourth grade fluid represents a more realistic description of the behaviours of non-Newtonian fluids. A fourth grade fluid model represents a further attempt towards the study of the flow properties of non- Newtonian fluid. This model is known to capture the non-Newtonian effects at once. With these facts in mind, we have considered an unsteady fourth grade fluid model in this study. In general, the governing equations for the flow problems of fourth grade fluids are up to fifth-order nonlinear equations. Literatures survey shows that very limited studies are reported and these investigations further narrow down when we speak about exact closed form solutions of these problems. However, there are few investigations available in the literature in which researcher have utilized various approaches to construct solutions of fourth grade fluid flow problems.

Correspondence Author: Aroloye S.J., E-mail: saroloyr@unilag.edu.ng, Tel: +2348073547811

Wan and Wu [12] have tackled the problem for the unsteady flow of fourth grade fluid due to an oscillating plate using numerical methods. Siddiqui et al. [13] have obtained an optimal homotopy type solution for the thin film flow of a fourth grade fluid down a vertical cylinder by using the homotopy perturbation method (HPM). Hayat and co-workers[14-17] studied the fourth grade fluid problems in different types of flow situations by using the homotopy analysis method (HAM). The steady flow of a fourth grade fluid past a porous plate was treated by Marinca et al.[18] with the help of the optimal homotopy asymtoptic method(OHAM). Hayat et al. [19] investigated effect of magnetic field on the flow of a fourth order fluid. He found that velocity decreased when the magnitude of the magnetic field increased .Aziz and Mahomed [20] obtained analytical and numerical solution for the unsteady flow of a fourth grade fluid on a porous plate. Shock wave solution for a nonlinear partial differential equation arising in the study of non-Newtonian fourth grade fluid model was investigated by Aziz et al. [21]. Hayat et al.[22] studied travelling wave solution to Stokes's problem for a fourth grade fluid. Aziz and Mahomed [23]. provided exact closed –form solutions for a nonlinear partial differential equation arising in the study of a fourth grade fluid model. Hayat et al. [24] examined some solutions for the flow of a fourth grade fluid in a porous space. Despite all of these works in recents years, the exact closed-form solutions for the problems dealing with the flow of fourth grade fluids are still rare in the literature.

Lie theory of differential equation [25,26] was inaugurated and utilized in the solution of differential equations by the Norwegian mathematician Marious Sophus Lie in the 1870s.The main motive of the Lie symmetry analysis formulated by Lie is to find one or several parameters local continuous transformation group leaving the equations invariant. Thus, the symmetry method for differential equations provides a rigorous way to classify them according to their invariance properties. This allows us to obtain group invariant and partially invariant solutions of differential equations in a tractable manner. The Lie symmetry approach has thus become of great importance in the theory and applications of differential equations and widely applied by several authors to solve difficult nonlinear problems particularly dealing with the flows of non Newtonian fluids [27-31]. The main goal of this research work is to investigate combined effects of unsteadiness, suction/injection, magnetic field, Darcy's resistance on time-dependent flow of an incompressible fourth grade fluid over a porous plate within a porous medium. Such flows have application in transpiration cooling, gaseous diffusion, boundary layer control, chromatography, filtration, flow in packed columns and so on. In this study, the governing nonlinear problem was first modelled then transformed to ordinary differential equation and solved exactly by employing the Lie Symmetry Approach. Backward and forward wave travelling solutions are provided for the model equation. Further, the influence of the various emerging parameters of the flow model is discussed through graphical analysis of the obtained results

2. Mathematical Modelling

In Figure 1, unsteady MHD flow of incompressible and electrically conducting fourth grade fluid occupies region $y > 0$ in a porous medium is considered. A Cartesian frame of reference is chosen such that x −axis is along the direction of fluid flow and parallel to the porous plate and $y-$ axis is perpendicular to it. At $= 0$, the fluid is bounded by an infinite porous plate. A transverse uniform magnetic field $\mathbf{B} = (0, B_0, 0\ )$ is applied at surface of the plate. Due to porous character of the plates there is a cross-flow of the fluid with a constant velocity V_0. The velocity field of the fluid flow is functions of y and t only .The plate is coinciding with the plane at $y = 0$. Unsteady motion of the conducting fluid through a porous medium is governed by the conservation laws of momentum and mass, that is

Continuity equation $\nabla.\mathbf{V} = 0$ (1)

Momentum equation $\frac{d\mathbf{V}}{dt} = \nabla.\mathbf{T} + \mathbf{J}\times\mathbf{B} + \mathbf{R}$ (2)

Where $\frac{d}{dt}$ is the material time derivative defined as $\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}.\nabla)$

In which ρ denotes fluid density, $\times$ denotes the vector product, $\mathbf{T}$ is the Cauchy stress tensor for an incompressible fourth grade fluid, $\mathbf{J}$ is the current density, B is the magnetic induction and $\boldsymbol{R} = (R_x, R_y, R_z)$is the Darcy's resistance due to the porous media in the x, y, z respectively .As such , for the flow model under investigation, we seek a velocity field $\mathbf{V}$ of the form

$\mathbf{V} = (u(y,t), -V_0, 0)$ (3)

Where $u(y,t)$ represent the fluid velocity field in the x direction, $V_0 > 0$ is the suction velocity and $V_0 < 0$ corresponds to the injection velocity.This velocity field is identically satisfies the continuity equation (1). Thus, the disturbance in the fluid is a function of y and t only

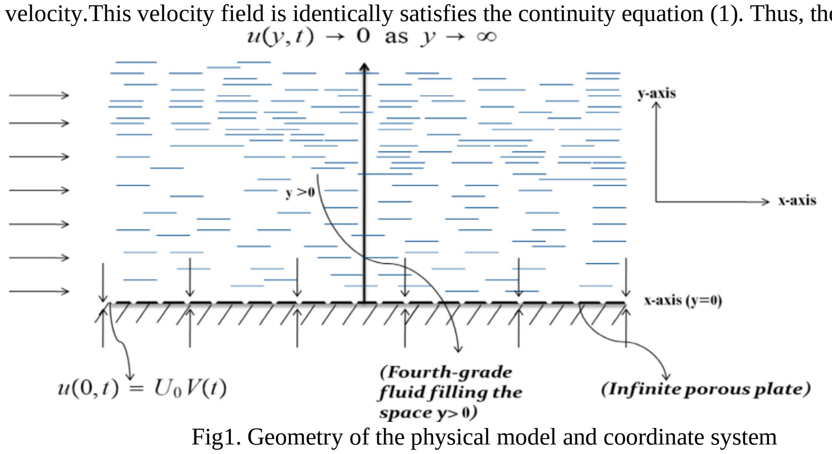

Fig1. Geometry of the physical model and coordinate system

The current density is given by $\mathbf{J}$ and $\mathbf{B}$ and gives the total magnetic field such that $\mathbf{B} = \mathbf{B_0} + \mathbf{b}$ where $\mathbf{B_0}$ is the applied magnetic field and $\mathbf{b}$ is the induced magnetic field In the analysis, the fluid is electrically conducting, and there is an applied magnetic field in the transverse direction to the flow. In the low magnetic Reynolds number flow, the induced electric and magnetic field can be neglected and thus the magnetic force $\mathbf{J}\times\mathbf{B}$ becomes

$\mathbf{J}\times\mathbf{B} = \sigma(\mathbf{V}\times\mathbf{B})\times\mathbf{B} = -\sigma B_0^2\mathbf{V}$ (4)

Where σ is the electrical conductivity of the fluid

The Cauchy stress tensor for a fourth grade fluid satisfies the constitutive equation,

$$\mathbf{T} = -P\mathbf{I} + \mu\mathbf{A_1} + \alpha_1\mathbf{A_1} + \alpha_2\mathbf{A_1^2} + \beta_2(\mathbf{A_2}\mathbf{A_1} + \mathbf{A_1}\mathbf{A_2}) + \beta_3(\mathrm{tr}\mathbf{A_1^2})\mathbf{A_1} + \gamma_1\mathbf{A_4} + \gamma_2(\mathbf{A_3}\mathbf{A_1} + \mathbf{A_1}\mathbf{A_3})$$
$$+\beta_1\mathbf{A_3} + \gamma_3\mathbf{A_2^2} + \gamma_4(\mathbf{A_1}\mathbf{A_1^2} + \mathbf{A_1^2}\mathbf{A_1}) + \gamma_5(\mathrm{tr}\mathbf{A_2})\mathbf{A_2} + \gamma_6(\mathrm{tr}\mathbf{A_2})\mathbf{A_1^2} + [\gamma_7(\mathrm{tr}\mathbf{A_3}) + \gamma_8(\mathrm{tr}\mathbf{A_2}\mathbf{A_1})]\mathbf{A_1} \quad (5)$$

Where P is presusure , I is the identity matix, μ is the viscosity, $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$, and γ_8 are the material parameters . $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$ and $\mathbf{A_4}$ are Rilvlin-Ericksen tensors .The tensors are defined as

$$\mathbf{A_1} = (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T \quad (6)$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A_{n-1}}(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A_{n-1}}, \quad n > 1 \quad (7)$$

In which ∇ is the gradient operator, where T denotes the transpose of the resultant matrix. For the model (5) above, when $\alpha_i = 0, \beta_i = 0$, and $\gamma_i = 0$ the fluid is Newtonian , $\beta_i = 0$ and $\gamma_i = 0$ equivalent to second grade fluid, $\gamma_i = 0$ equivalent to the third grade fluid, $\alpha_i \neq 0, \beta_i \neq 0$, and $\gamma_i \neq 0$ equivalent to fourth grade fluid.

The component form of the Cauchy stress tensor (5) for a fourth grade fluid is a matrix given by

$$\mathbf{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \quad (8)$$

Where each τ_{ij} represents a fluid flow in the i-direction and a disturbance of the fluid flow in the j-direction such that $\tau_{ij} = \tau_{ji}$

The component form of the momentum equation(2 is found by using the velocity field (1) in conjunction with the component form of the Cauchy stress tensor as it is given in (8).This gives us the set of governing equation below

$$\rho\frac{du}{dt} = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{xz}}{\partial x} - \sigma B_0^2 u + R_x \quad (9)$$

$$0 = \frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\tau_{yy}}{\partial x} + \frac{\partial\tau_{yz}}{\partial x} + R_y \quad (10)$$

$$0 = \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial x} + \frac{\partial\tau_{zz}}{\partial x} + R_z \quad (11)$$

Where R_x, R_y, and R_z represents component of Darcy's resistance in the the x, y, z directions respectively.

Each component of the Cauchy stress tensor τ_{ij} illustrated in (8) is then computed using (3) along with the Rivlin-Ericksen tensors (6-7) in (5), then we have

$$\tau_{xx} = -P + \alpha_2\left(\frac{\partial u}{\partial y}\right)^2 + 2\beta_2\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t} - V_0\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial^2 y}\right) + 2\gamma_2\left(\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y\partial t^2} - 2V_0\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^2\partial t} + V_0^2\frac{\partial^3 u}{\partial y^3}\right) + \left[\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2 - 2V_0\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial y\partial t} + V_0^2\left(\frac{\partial^2 u}{\partial y^2}\right)^2\right] \quad (12)$$

$$\tau_{xy} = \tau_{yx} = \alpha_1\left(\frac{\partial^2 u}{\partial y\partial t} - V_0\frac{\partial^2 u}{\partial y^2}\right) + \beta_1\left(\frac{\partial^3 u}{\partial y\partial t^2} - 2V_0\frac{\partial^3 u}{\partial y^2\partial t} + V_0^2\frac{\partial^3 u}{\partial y^3}\right) + 2(\beta_2 + \beta_3)\left(\frac{\partial u}{\partial y}\right)^3 + \gamma_1\left(\frac{\partial^4 u}{\partial y\partial t^3} - 3V_0\frac{\partial^4 u}{\partial y^2\partial t^2} + 3V_0^2\frac{\partial^4 u}{\partial y^3\partial t} - V_0^3\frac{\partial^4 u}{\partial y^4}\right) + \mu\frac{\partial u}{\partial y} +$$
$$2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y\partial t} - V_0\left(\frac{\partial^2 u}{\partial y^2}\right)\right) \quad (13)$$

$$\tau_{xz} = \tau_{zx} = 0 \quad (14)$$

$$\tau_{yy} = -P + (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2 + 6\beta_1\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t} - V_0\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) + 2\beta_2\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t} - V_0\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) + 4\gamma_4\left(\frac{\partial u}{\partial y}\right)^4 + 4\gamma_5\left(\frac{\partial u}{\partial y}\right)^4 + \gamma_3\left[\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2 -\right.$$
$$\left.2V_0\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial y\partial t} + V_0^2\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^4\right] + 2\gamma_2\left(\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y\partial t^2} - 2V_0\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^2\partial t} + V_0^2\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^3}\right) + 2\gamma_6\left(\frac{\partial u}{\partial y}\right)^4 + \gamma_1\left[6\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2 + 8\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y\partial t^2} -\right.$$
$$\left.12V_0\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial y\partial t} + 6V_0^2\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 8V_0^2\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y^3} - 16V_0\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial t\partial y^2}\right] \quad (15)$$

$$\tau_{xz} = \tau_{zx} = 0 \quad (16)$$

Following the same procedure adopted by Hayat et al.[24, 32-33], Tan and Masuoka [34,35], then z-component of the Darcy's resistance R_z in the present case of unsteady fourth grade fluid flow is

$$R_z = -\frac{\emptyset}{\kappa}\left[\alpha_1\left(\frac{\partial u}{\partial y} - V_0\frac{\partial u}{\partial t}\right) + \beta_1\left(\frac{\partial^2 u}{\partial t^2} - 2V_0\frac{\partial^2 u}{\partial y\partial t} + V_0^2\frac{\partial^2 u}{\partial y^2}\right) + 2(\beta_2 + \beta_3)u\left(\frac{\partial u}{\partial y}\right)^2 + \gamma_1\left(\frac{\partial^3 u}{\partial t^3} - 3V_0\frac{\partial^4 u}{\partial y\partial t^2} + 3V_0^2\frac{\partial^3 u}{\partial y^2\partial t} - V_0^3\frac{\partial^3 u}{\partial y^3}\right) +\right.$$
$$\left.\mu u + 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)u\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t} - V_0\left(\frac{\partial^2 u}{\partial y^2}\right)\right)\right] \quad (17)$$

In the absent of pressure gradient, the following dimensional governing differential equation in u is obtained by implementing (12), (13), (14) and (17) in (9) give

$$\rho\left(\frac{\partial u}{\partial t} - V_0\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\left(\frac{\partial^3 u}{\partial y^2\partial t} - V_0\frac{\partial^3 u}{\partial y^3}\right) + \beta_1\left(\frac{\partial^4 u}{\partial y^2\partial t^2} - 2V_0\frac{\partial^4 u}{\partial y^3\partial t} + V_0^2\frac{\partial^4 u}{\partial y^4}\right) + 2(\beta_2 + \beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma_1\left(\frac{\partial^5 u}{\partial y^2\partial t^3} - 3V_0\frac{\partial^5 u}{\partial y^3\partial t^2} +\right.$$
$$\left.3V_0^2\frac{\partial^5 u}{\partial y^4\partial t} - V_0^3\frac{\partial^5 u}{\partial y^5}\right) + 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t} - V_0\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\right] - \sigma B_0^2 u - \frac{\emptyset}{\kappa}\left[\alpha_1\left(\frac{\partial u}{\partial y} - V_0\frac{\partial u}{\partial t}\right) + \beta_1\left(\frac{\partial^2 u}{\partial t^2} -\right.\right.$$
$$2V_0\frac{\partial^2 u}{\partial y\partial t} + V_0^2\frac{\partial^2 u}{\partial y^2}\Big) + 2(\beta_2 + \beta_3)u\left(\frac{\partial u}{\partial y}\right)^2 + \mu u + \gamma_1\left(\frac{\partial^3 u}{\partial t^3} - 3V_0\frac{\partial^4 u}{\partial y\partial t^2} + 3V_0^2\frac{\partial^3 u}{\partial y^2\partial t} - V_0^3\frac{\partial^3 u}{\partial y^3}\right) + 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 +$$
$$\left.\gamma_8)u\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t} - V_0\left(\frac{\partial^2 u}{\partial y^2}\right)\right)\right] \quad (18)$$

In order to solve (18), the relevant boundary and initial conditions are

$$u(0,t) = U_0V(t);\ u(\infty,t) = 0, t > 0,\ u(y,0) = I(y),\ \frac{\partial u(y,0)}{\partial t} = J(y), \frac{\partial^2 u(y,0)}{\partial t^2} = K(y),\ y > 0 \quad (19)$$

Where U_0 is the reference velocity and $V(t), I(y), J(y)$ and $K(y)$ are functions to be determined from further conditions. The first boundary condition indicates that the plate is moving with an impulsive velocity $V(t)$, also known as the no-slip condition and secondary boundary condition shows that the mainstream velocity is zero as the fluid to be considered is at rest far away from the plate. The initial condition indicates that initially, the fluid is moving with some non-uniform velocity $I(y)$. The remaining two initial conditions are the extra two conditions imposed to make the problem well posed.

We defined the following transformations and dimensionless quantities as

$$\bar{u}=\frac{u}{U_0},\bar{y}=\frac{U_0 y}{\nu},\ \bar{t}=\frac{U_0^2 t}{\nu},\bar{U}_0=\frac{V_0}{U_0} \quad (20)$$

Under these transformations the dimensional governing equation (18) and the corresponding initial and boundary conditions (19) take the non-dimensional form

$$\frac{\partial\bar{u}}{\partial\bar{t}}=\bar{V}_0\frac{\partial\bar{u}}{\partial\bar{y}}+\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+\bar{\alpha}\left(\frac{\partial^3\bar{u}}{\partial\bar{y}^2\partial\bar{t}}-\bar{V}_0\frac{\partial^3\bar{u}}{\partial\bar{y}^3}\right)+\bar{\beta}_1\left(\frac{\partial^4\bar{u}}{\partial\bar{y}^2\partial\bar{t}^2}-2\bar{V}_0\frac{\partial^4\bar{u}}{\partial\bar{y}^3\partial\bar{t}}+\bar{V}_0^2\frac{\partial^4\bar{u}}{\partial\bar{y}^4}\right)+3\bar{\beta}\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\frac{\partial^2\bar{u}}{\partial\bar{y}^2}+\bar{\gamma}\left(\frac{\partial^5\bar{u}}{\partial y^2\partial\bar{t}^3}-3\bar{V}_0\frac{\partial^5\bar{u}}{\partial y^3\partial\bar{t}^2}+3\bar{V}_0^2\frac{\partial^5\bar{u}}{\partial\bar{y}^4\partial\bar{t}}-\bar{V}_0^3\frac{\partial^5\bar{u}}{\partial\bar{y}^5}\right)+2\bar{\Gamma}\frac{\partial}{\partial\bar{y}}\left[\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\frac{\partial^2 u}{\partial\bar{y}\partial\bar{t}}-\bar{V}_0\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\right]-\sigma B_0^2\bar{u}-\bar{\phi}\left[\bar{\alpha}\left(\frac{\partial\bar{u}}{\partial\bar{y}}-\bar{V}_0\frac{\partial\bar{u}}{\partial\bar{t}}\right)+\bar{\beta}_1\left(\frac{\partial^2\bar{u}}{\partial\bar{t}^2}-2\bar{V}_0\frac{\partial^2\bar{u}}{\partial\bar{y}\partial\bar{t}}+\bar{V}_0^2\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\right)+\bar{\beta}\bar{u}\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)^2+\bar{\gamma}\left(\frac{\partial^3\bar{u}}{\partial\bar{t}^3}-3\bar{V}_0\frac{\partial^4\bar{u}}{\partial\bar{y}\partial\bar{t}^2}+3\bar{V}_0^2\frac{\partial^3\bar{u}}{\partial\bar{y}^2\partial\bar{t}}-\bar{V}_0^3\frac{\partial^3\bar{u}}{\partial\bar{y}^3}\right)+\bar{u}+2\bar{\Gamma}\bar{u}\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)\left(\frac{\partial^2\bar{u}}{\partial\bar{y}\partial\bar{t}}-\bar{V}_0\left(\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\right)\right)\right]-\bar{M}^2\bar{u} \quad (21)$$

Where $\bar{\alpha}=\frac{U_0^2\alpha_1}{\rho\nu^2},\ \bar{\beta}_1=\frac{U_0^2\beta_1}{\rho\nu^3},\bar{\beta}=2(\beta_2+\beta_3)\frac{U_0^4}{\rho\nu^3},\bar{\gamma}=\frac{U_0^6\gamma_1}{\rho\nu^4},\ \bar{\Gamma}=2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\frac{U_0^6\gamma_1}{\rho\nu^3}$

$$\bar{M}^2=\frac{\nu\sigma B_0^2}{\rho U_0^2},\bar{\phi}=\frac{\phi\nu^2}{\kappa U_0} \quad (22)$$

The dimensionless boundary and initial conditions are

$$\bar{u}(0,\bar{t})=U_0V(\bar{t});\ \bar{u}(\infty,\bar{t})=0,t>0,\ \bar{u}(\bar{y},0)=\bar{f}(\bar{y}),\ \frac{\partial\bar{u}(\bar{y},0)}{\partial\bar{t}}=\bar{g}(\bar{y}),\frac{\partial^2\bar{u}(\bar{y},0)}{\partial\bar{t}^2}=\bar{h}(\bar{y}),\ \bar{y}>0 \quad (23)$$

Where $\bar{f}(\bar{y})=\frac{I(y)}{V_0},\ \bar{g}(\bar{y})=\frac{J(y)}{V_0}$ and $\bar{h}(\bar{y})=\frac{K(y)}{V_0}$

where $\bar{f}(\bar{y}),\bar{g}(\bar{y})$ *and* $\bar{h}(\bar{y})$ are arbitrary functions

For simplicity we neglect the bars from all non-dimensional quantities in the further analysis

3. Invariant Solution of the problem

In this section, we present different types of group invariant solutions of the governing model partial differential equation (21) with the aid of the Lie group translation operator. By performing the classical Lie group analysis [25,26], the infinitesimals for the partial differential equation (21) are $\xi^1=c_1,\xi^2=c_2,\ \eta=0$,where $c_1,$ *and* c_2 are constants. The corresponding Lie pont symmetry generators are

$$X_1=\frac{\partial}{\partial t},X_2=\frac{\partial}{\partial y} \quad (24)$$

Where X_1 is the time-translational symmetry generator and X_2 is the space –translational symmetry generator

3.1 Travelling wave solution

Travelling wave solutions are one of a special kind of group invariant solutions which remain invariant under a linear combination of time –translational and space translational symmetry generators. These solutions are of two types (Backward and forward). We know that from the principle of the Lie symmetry methods that if a differential equation is explicitly independent of any dependent or independent variable, then this particular differential equation remains invariant under the translation symmetry corresponding to that particular variable. Partial differential equation (21) admits the Lie point symmetry generators, $X_1=\frac{\partial}{\partial t}$(time translation in *t*) and $X_2=\frac{\partial}{\partial y}$(space translation in *y*).Therefore, we can construct travelling g wave solutions for the model non-linear PDE (21)

3.1.1 **Backward wave front- type travelling wave solutions method for solving (21)**

X_1 *and* X_2 be the time –translation and space translation symmetry generators, respectively. We search for the invariant solution under the operator

$$X=X_1-cX_2=\frac{\partial}{\partial t}-c\frac{\partial}{\partial y}\quad c>0 \quad (25)$$

which represents a backward wave –front type travelling wave solution. In this case, the waves are propagating towards the plate with a constant wave speed c. The Lagrangian system correspond to (25) is

$$\frac{dy}{-c}=\frac{dt}{0}=\frac{du}{0} \quad (26)$$

Solving (26), the invariant solution is given as

$$u(y,t)=f_1(\eta)\text{ with }\eta=y+ct \quad (27)$$

Where $f_1(\eta)$ is an arbitrary function of the characteristic variable $\eta=y+ct$

Making use of (27) in (21) result in a fifth –order ordinary differential equation of the form

$$(c-V_0)\frac{df_1}{d\eta}+\frac{d^2f_1}{d\eta^2}-\alpha(c-V_0)\frac{d^3f_1}{d\eta^3}+\beta_1(c-V_0)^2\frac{d^4f_1}{d\eta^4}+3\beta\left(\frac{df_1}{d\eta}\right)^2\frac{d^2f_1}{d\eta^2}+\gamma(c-V_0)^3\frac{d^5f_1}{d\eta^5}+2\Gamma(c-V_0)\left(\frac{df_1}{d\eta}\right)^2\frac{d^3f_1}{d\eta^3}+4\Gamma(c-V_0)\frac{df_1}{d\eta}\left(\frac{d^2f_1}{d\eta^2}\right)^2-M^2f_1-\phi\left[\alpha(c-V_0)\frac{df_1}{d\eta}+\beta_1(c-V_0)^2\frac{d^2f_1}{d\eta^2}+\beta f_1\left(\frac{df_1}{d\eta}\right)^2+\gamma(c-V_0)^3\frac{d^3f_1}{d\eta^3}+f_1+2\Gamma(c-V_0)f_1\frac{df_1}{d\eta}\frac{d^3f_1}{d\eta^3}\right]=0 \quad (28)$$

Thus, the originally fifth –order nonlinear PDE (21) reduced to a fifth-order (28) along certain curve in the x-t plane .These curves are called characteristic curves

In order to solve (28) for f_1, we assume a solution of the form

$$f_1(\eta)=A\exp(B\eta) \quad (29)$$

Where A and B are the constants to be determined. Substituting (29) into (28), we obtain

$$-(c-V_0)ABe^0+AB^2e^0+\alpha(c-V_0)AB^3e^0+\beta_1(c-V_0)^2AB^4e^0+\gamma(c-V_0)^3AB^5e^0+3\beta A^3B^4e^{2B\eta}+6\Gamma A^3B^5(c-V_0)e^{2B\eta}-\phi[\alpha(c-V_0)ABe^0+\beta_1(c-V_0)^2AB^2e^0+\beta A^3B^2e^{2B\eta}+\gamma(c-V_0)^3AB^3e^0+Ae^0+2\Gamma A^3B^3(c-V_0)e^{2B\eta}]-M^2Ae^0=0 \quad (30)$$

Separating (30) in the exponent power of e^0 *and* $e^{2B\eta}$

$$e^0:\ -(c-V_0)AB+AB^2+\alpha(c-V_0)AB^3+\beta_1(c-V_0)^2AB^4+\gamma(c-V_0)^3AB^5-\phi[\alpha(c-V_0)AB+\beta_1(c-V_0)^2AB^2+\gamma(c-V_0)^3AB^3+A]+AM^2=0 \quad (31)$$

$$e^{2B\eta}:\ 3\beta A^3B^4+6\Gamma(c-V_0)A^3B^5-\phi[\beta A^3B^2+2\Gamma(c-V_0)A^3B^3]=0 \quad (32)$$

Solving (32), we deduce that

$$B=\frac{-\beta}{2\Gamma(c-V_0)} \quad (33)$$

Making use of the value of B in (31), we deduce that

$$\frac{\beta}{2\Gamma}+\frac{\beta^2}{(2\Gamma)^2(c-V_0)^2}-\frac{\alpha\beta^3}{(2\Gamma)^3(c-V_0)^2}+\frac{\beta_1\beta^4}{(2\Gamma)^4(c-V_0)^2}-\frac{\gamma\beta^5}{(2\Gamma)^5(c-V_0)^2}-\phi\left[1-\alpha\left(\frac{\beta}{2\Gamma}\right)+\beta_1\left(\frac{\beta}{2\Gamma}\right)^2-\gamma\left(\frac{\beta}{2\Gamma}\right)^3\right]-M^2=0 \qquad (34)$$

The exact solution $f_1(\eta)$ for (28) subject to condition (34) is

$$f_1(\eta)=Aexp\left(\frac{-\beta\eta}{2\Gamma(c-V_0)}\right) \qquad (35)$$

The exact solution $u(y,t)$ for (21) subject to condition (34) is given as

$$u(y,t)=exp\left(\frac{-\beta(y+ct)}{2\Gamma(c-V_0)}\right) \qquad (36)$$

We observe that the solution (36) does satisfy the physically relevant boundary and initial conditions (23) as follows

$$u(0,t)=V(t)=exp\left(\frac{-\beta ct}{2\Gamma(c-V_0)}\right) \qquad (37)$$

$$u(y,0)=f(y)=exp\left(\frac{-\beta y}{2\Gamma(c-V_0)}\right) \qquad (38)$$

$$\frac{\partial u}{\partial t}(y,0)=g(y)=exp\left(\frac{-\beta c}{2\Gamma(c-V_0)}\right)f(y) \qquad (38)$$

$$\frac{\partial^2 u}{\partial t^2}(y,0)=h(y)=exp\left(\frac{-(\beta m)^2}{(2\Gamma)^2(c-V_0)^2}\right)f(y) \qquad (40)$$

With

$A=V(0)=f(0)=1$

The function $V(t), f(y), g(y), h(y)$ *depend* on the physical parameters of the flow model.We note that by making use of the imposing condition (34) , we can express the exact solution (36) in the form

$$u(y,t)=exp\left[-\left(\frac{\gamma\beta^5}{(2\Gamma)^5(c-V_0)^3}-\frac{\beta_1\beta^4}{(2\Gamma)^4(c-V_0)^3}+\frac{\alpha\beta^3}{(2\Gamma)^3(c-V_0)^3}-\frac{\beta^2}{(2\Gamma)^2(c-V_0)^3}-\phi\left[1-\alpha\left(\frac{\beta}{2\Gamma}\right)+\beta_1\left(\frac{\beta}{2\Gamma}\right)^2-\gamma\left(\frac{\beta}{2\Gamma}\right)^3\right]+\frac{M^2}{(c-V_0)}\right)\right] \qquad (41)$$

The solution (41) is plotted for varying values of the emerging flow parameters of the flow problem.
Further, the imposing condition (34) can also be written as

$$(c-V_0)^2=\frac{\left(\frac{\beta}{2\Gamma}\right)^2\left[1-\alpha\left(\frac{\beta}{2\Gamma}\right)+\beta_1\left(\frac{\beta}{2\Gamma}\right)^2-\gamma\left(\frac{\beta}{2\Gamma}\right)^3\right]}{\left(M^2-\frac{\beta}{2\Gamma}\right)+\phi\left[1-\alpha\left(\frac{\beta}{2\Gamma}\right)+\beta_1\left(\frac{\beta}{2\Gamma}\right)^2-\gamma\left(\frac{\beta}{2\Gamma}\right)^3\right]} \qquad (42)$$

Equation (42) determines the wave speed c provided denominator of (42) is not equal to zero

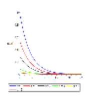

Fig **2.** Effect of t on the velocity profile (40)
With $\beta_1=2, \beta=2, \Gamma=6, c=5, V_0=0.5,$
$M=1, \phi=0.1, \alpha=0.5, \gamma=1$

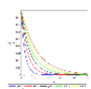

Fig 3**.** Effect of c on the velocity profile (40)
With $\beta_1=1, \beta=1, \Gamma=2, t=0.1, V_0=0.5,$
$M=1, \phi=0.1, \alpha=0.1, \gamma=0.5$

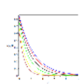

Fig 4**.** Effect of M on the velocity profile (40)
With $\beta_1=2, \beta=2, \Gamma=6, c=5, V_0=0.5,$
$\phi=0.1, t=0.1, \alpha=0.5, \gamma=1$

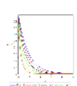

Fig 5**.** Effect of $V>0$ on the velocity profile (40)
With $\beta_1=1, \beta=1, \Gamma=2, t=0.1, c=2,$
$M=1, \phi=0.1, \alpha=0.1, \gamma=0.5$

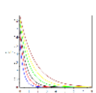

Fig **6.** Effect of $V<0$ on the velocity profile (40)
With $\beta_1=1, \beta=1, \Gamma=2, c=0.5,$
$\gamma=1, M=1, \beta=2, \alpha=1, t=0.5$

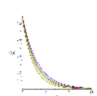

Fig 7. Effect of ϕ on the velocity profile (40)
with $\beta_1=1, \Gamma=1, c=5, V_0=0.5, t=0.2,$
$\gamma=1, M=1, \beta=2, \alpha=1,$

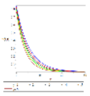

Fig 8**.** Effect of γ on the velocity profile (40)
With $\beta_1 = 1, \beta = 2, c = 5, V_0 = 0.5,$
$\phi = 0.1, t = 0.2, \alpha = 1, \Gamma = 1, M = 1$

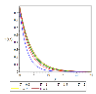

Fig 9. Effect of Γ on the velocity profile (40)
with $\beta = 2, \beta_1 = 1, c = 2, V_0 = 0.5, t = 0.3,$
$\gamma = 1, M = 0.5, \phi = 0.1, \alpha = 1$

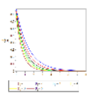

Fig 10. Effect of β_1 on the velocity profile (40)
With, $\beta = 1, c = 2, \Gamma = 2, V_0 = 0.1,$
$\phi = 1, t = 0.3, \alpha = 1, M = 1, \gamma = 1$

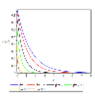

Fig 11. Effect of β on the velocity profile (40)
with $\phi = 1, \Gamma = 2, c = 2, V_0 = 0.1, t = 0.2,$
$\gamma = 1, M = 1, \alpha = 1, \beta_1 = 1$

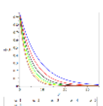

Fig 12**.** Effect of α on the velocity profile (40)
With $\beta_1 = 1, \beta = 1, c = 2, V_0 = 0.5, \Gamma = 1$
$\phi = 0.1, t = 0.2, M = 1, \Gamma = 1$

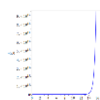

Fig 13. Shock wave behaviour osf solution (47)
with $\beta = 1, \Gamma = 2, c = 2, V_0 = 0.1, t = 3,$

3.1.2 **Forward wave front- type travelling wave solutions method for solving (21)**

We look for invariant solutions under $X = X_1 + cX_2$ with $c > 0$ which represent a forward wave –front type travelling wave solutions. In this case, the waves are propagating away from the plate with a constant wave speed c. These are solution of the form

$$u(y,t) = f_2(\eta) \text{ with } \eta = y - ct \quad (43)$$

using (43) in (21) result in a fifth –order ordinary differential equation of the form

$$(c+V_0)\frac{df_2}{d\eta} + \frac{d^2f_2}{d\eta^2} - \alpha(c+V_0)\frac{d^3f_2}{d\eta^3} + \beta_1(c+V_0)^2\frac{d^4f_2}{d\eta^4} + 3\beta\left(\frac{df_1}{d\eta}\right)^2\frac{d^2f_2}{d\eta^2} - \gamma(c+V_0)^3\frac{d^5f_2}{d\eta^5} - 2\Gamma(c+V_0)\left(\frac{df_2}{d\eta}\right)^2\frac{d^3f_2}{d\eta^3} - 4\Gamma(c+V_0)\frac{df_2}{d\eta}\left(\frac{d^2f_2}{d\eta^2}\right)^2 - M^2f_2 - \phi\left[-\alpha(c+V_0)\frac{df_2}{d\eta} + \beta_1(c+V_0)^2\frac{d^2f_2}{d\eta^2} + \beta f_1\left(\frac{df_1}{d\eta}\right)^2 - \gamma(c+V_0)^3\frac{d^3f_2}{d\eta^3} + f_2 - 2\Gamma(c+V_0)f_2\frac{df_2}{d\eta}\frac{d^3f_2}{d\eta^3}\right] = 0 \quad (44)$$

Following the same methodology adopted for obtaining the backward wave-front type travelling wave solutions above, equation (44) admits exact solutions of the form

$$f_2(\eta) = Aexp\left(\frac{\beta\eta}{2\Gamma(c-V_0)}\right) \quad (45)$$

Subject to condition

$$\frac{\beta}{2\Gamma} + \frac{\beta^2}{(2\Gamma)^2(c+V_0)^2} - \frac{\alpha\beta^3}{(2\Gamma)^3(c+V_0)^2} + \frac{\beta_1\beta^4}{(2\Gamma)^4(c-V_0)^2} - \frac{\gamma\beta^5}{(2\Gamma)^5(c-V_0)^2} - \phi\left[1 - \alpha\left(\frac{\beta}{2\Gamma}\right) + \beta_1\left(\frac{\beta}{2\Gamma}\right)^2 - \gamma\left(\frac{\beta}{2\Gamma}\right)^3\right] - M^2 = 0 \quad (46)$$

The exact solution $u(y,t)$ for (21) subject to condition (46) is given as

$$u(y,t) = exp\left(\frac{\beta(y-ct)}{2\Gamma(c+V_0)}\right) \quad (47)$$

The solution (47) is a shock wave solution to the governing partial differential equation (21) .The above solution is valid under a particular condition on the physical parameters of the flow problem given in (46).This solution does show the hidden shock wave behaviour of the flow problem with the slope of the velocity field or the velocity gradient approaching infinity such that

$$\frac{\partial u}{\partial y} \to \infty \; as \;\; y > 0$$

The solution (45) does not satisfy the second boundary condition at infinity but does satisfy the rest of the boundary conditions in (23)

The physical significance of the imposing condition (46) is that it gives the speed of the travelling shock wave. From (46), we deduce that

$$(c+V_0)^2 = \frac{\left(\frac{\beta}{2\Gamma}\right)^2\left[1-\alpha\left(\frac{\beta}{2\Gamma}\right)+\beta_1\left(\frac{\beta}{2\Gamma}\right)^2-\gamma\left(\frac{\beta}{2\Gamma}\right)^3\right]}{\left(M^2-\frac{\beta}{2\Gamma}\right)+\phi\left[1-\alpha\left(\frac{\beta}{2\Gamma}\right)+\beta_1\left(\frac{\beta}{2\Gamma}\right)^2-\gamma\left(\frac{\beta}{2\Gamma}\right)^3\right]} \quad (48)$$

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In order to get the physical insight of the flow model under investigation, the behaviour of various pertinent parameters on the structure of the velocity field is observed.

The effect of time t on the velocity field (40) is shown in Fig. 2. This figure shows that the velocity decreases for large value of time. The variation of velocity is observed for $0 \leq t \leq 5$. For $t > 5$, the velocity field remains the same. In other words, we can say that the steady state behaviours of velocity is achieved for $t > 5$

The influence of the wave speed c on the velocity field (40) has been presented in Fig 3. It is observed that the velocity increases by increasing c. In this way, we can remark that both time and wave speed have the opposite effects on the closed form solutions (40) of the model.

The dependence of the distribution of the velocity field (40) on the magnetic field is show in Fig. 4. From the figure, it is observed that the applied magnetic field tends to restrict the shearing to a thinner boundary layer near the porous plate. The reason for this thinning is that the magnetic field provides a resistance to the flow and hence decreases the velocity. Thus, the magnetic field provides some mechanism to control the boundary layer thickness.

Figures 5 and 6 describe the behaviour of the velocity field (40) with suction and injection parameters. It is depicted from Fig. 5. that the velocity field, as well as boundary layer thickness, decrease with an increase in the suction parameter. This structure of the velocity profile is also expected physically because always causes a reduction in the boundary layer thickness. Figure 6 is display to show the effect of the variation of the injection velocity. For injection through a porous plate, the fluid behaves qualitatively opposite to suction velocity. So, from these figure, it has been observed that our closed-form solution (40) remains stable both for the case of suction and injection provided that $c > V_0$

In figure 7, the influence of the porosity of the porous medium ϕ on the velocity profile (40) is illustrated .As anticipated, with an increase of the porosity of the porous medium causes an increase in drag force and hence the flow velocity decreases

In order to describe the influence of the fourth - grade fluid parameters γ and Γon the flow model, the velocity field (40) has been plotted in Figs 8 and 9 .These figures reveal that both γ and Γ have opposite roles on the structure of the velocity, i.e., with an increase in parameter γ ,the velocity field is decreasing, which shows the shear thickening behaviours of the fluid. However, the velocity profile increases for increasing values of Γ showing the shear thinning property of the fluid.

Also, figure 10 and 11 have been plotted to see the influence of the third grade parameters β_1 and β on velocity profile (40) of the flow problem. These figures reveal that both β_1 and β have similar behaviour on the structure of the velocity, that is, with an increase in the third grade parameters β_1 and β, the velocity profile decreases showing the shear thickening behaviour of the fluid. Effects of second grade fluid parameter have similar effect in figure 12.

In Fig 13, the close-form solution (47) is plotted .This figure predicts the shock wave behaviour of the flow with the slope approaching infinity along the characteristics .This solution does not show the physics of the model, but does predict the hidden shock wave phenomena in the flow. Some examples of shock waves are moving shock, detonation wave, detached shock, attached shock, decompression shock, shock in a pipe flow, shock waves in rapid granular flows, shock waves in astrophysics and so on.

5. Conclusion

In this study, we have focused on the modelling and solutions of a nonlinear time-dependent flow model of an electrically conducting fourth-grade fluid with plate suction and injection. Lie group theoretic analysis has been employed to perform reductions of the governing nonlinear partial differential equations to differential ordinary differential equations. Both forward and backward wave –front type travelling wave solutions (of the model equations) have been developed in the form of a closed- form exponential function.

The backward wave-front type travelling wave solution best represent the physics of the model under investigation in the sense that this solution satisfies all the boundary and initial conditions. It also shows the effects of the magnetic field and the plate injection/suction directly on the physics system. On the other hand, the forward wave-front type travelling wave solution does not satisfy the boundary condition at infinity. As a consequence, it does not show the physical behaviour of the flow model. But this solution, however, does describe the hidden shock wave behaviour of the flow. To emphasize, we say that this solution is going to be very useful in carrying out further analysis of the shock wave behaviour associated with non-Newtonian fluid flow problems. Moreover, the model considered herein is theoretical and prototype in nature, but the methodology used is quite useful to handle to handle a wide range of nonlinear problems. This is not only restricted to the field of non-Newtonian fluid mechanics, but also in other fields of science and engineering.

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