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Decomposition Variation Iteration Method For Solving Delay Differential Equations

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Abstract

In this paper, we propose a new numerical algorithm called the decomposition variation iteration method (DVIM) for solving delay differential equations (DDEs). The proposed method is an elegant mixture of Adomian decomposition method with variation iteration method. The method is highly sufficient and efficient in solution of the DDEs as it has excellent rate of convergence, by-passing all linearization, perturbation or discretization procedures in the solution process. Two numerical examples (linear and nonlinear DDEs) were considered for experimentation of the method. The results obtained show the new method is effective, efficient and reliable for resolving DDEs. All computations were implemented with maple 18 software.

Keyword: Adomian decomposition method, Delay differential equations, Lagrange multiplier, Variation iteration method.

1. INTRODUTION

The delay differential equations (DDEs) have become relevant in recent times due to its explicit evaluation in mathematical modelling in a variety of fields, such as, biology, chemistry, physics, engineering, etc. The solutions to these problems are what give the precise interpretation of the model under consideration. To this effect, researchers in recent times have devoted their time and energy seeking for methods to evaluate mathematical problems explicitly. Conventional analytic methods are insufficient to handle these problems as most of these methods would require linearization, perturbation or even discretization. Numerical methods are becoming more explicit in resolving these problems as an approximation of the analytic solution. Such iterative methods include Adomian decomposition method (ADM) [1-2], Variation iteration method (VIM) [3], Differential transform method (DTM) [4], Runge-Kutta Method [5], Spline function method [6], etc. The stability of the solutions to DDEs was studied by Zennaro [7].

As knowledge is dynamic, we present a contribution to this dynamic nature of knowledge, a new numerical algorithm called the Decomposition Variation Iteration Method (DVIM) for resolving DDEs. This proposed method takes an elegant mixture of Adomian decomposition method and variation iteration method. In this new method, the Adomian polynomials are formulated for nonlinear cases, which we then employ to obtain the components u_n, $n \geq 0$ recursively. Thereafter, the correction functional is constructed for the given problem. The Lagrangean multiplier is then obtained optimally via variation theory [8]. We then substitute the value of the Lagrangean multiplier and iterate the given scheme for $n \geq 0$. For a linear case, Adomian polynomials formulation is by-passed.

The rate of convergence of the DVIM is excellent as compared with results available in the literature [1]. The method explicitly avoids linearization, perturbation or discretization. Also, computational and round-off errors are minimized in this method.

2. The Theory of Adomian Decomposition Method

In this section, we consider the Adomian decomposition method as presented in [1].

Consider the generalized ODE of the form;

$$Lu + Ru + Nu = f \qquad (1)$$

with prescribed auxiliary conditions, u is an unknown function, L is the highest power derivative which can be easily invertible, Nu is the nonlinear term, R is a linear operation whose order less than L, and f is the source term.

Applying the inverse operator L^{-1} to both sides of equation (1), we have

$$L^{-1}[Lu] = L^{-1}[G - Ru - Nu],$$

this implies that

$$u = L^{-1}(G) - L^{-1}(Ru) - L^{-1}(Nu), \qquad (2)$$

The standard Adomian defines the solution as

$$u = \sum_{n=0}^{\infty} u_n, \qquad (3)$$

And the nonlinear term series Nu is defined as

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$$Nu = \sum_{n=0}^{\infty} A_n \,, \tag{4}$$

where A_n are the Adomian polynomials (ADP) which can be determined recursively using the relation [9-12].

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} .$$

If we let $Nu = F(u_0)$, then the ADP are arranged into the form used in [9-12] as

$$\begin{aligned}
A_0 &= F(u_0)\\
A_1 &= u_1 F'(u_0)\\
A_2 &= u_2 F'(u_0) + \frac{u^2{}_1}{2!}F''(u_0)\\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{u_1^3}{3!}F'''(u_0)\\
&\vdots
\end{aligned} \tag{5}$$

The components of $u_0, u_1, u_2, u_4, \ldots,$ are determined recursively using the relation

$$u_0 = L^{-1}(G) - L^{-1}(Ru),$$

satisfying the prescribed conditions, and

$$u_{n+1} = L^{-1}(G) - L^{-1}(Ru) - L^{-1}(Nu) \,, \tag{6}$$

Where u_0 is the zero component. Hence, an n-component truncated series solution is obtained as

$$u_n = \sum_{i=0}^{n} u_i \tag{7}$$

3 Decomposition Variation Iteration Method (DVIM)

We can construct a correction functional for *Equation (1)* as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[Lu_n(s) + Ru_n(s) + N\tilde{u}_n(s) - f(s)\right]ds, \quad n \ge 0 \tag{8}$$

where $\lambda(s)$ is a general Lagrangean multiplier and $\tilde{u}_n$ is a restricted variable. Abbasbandy and Sivanian [13] obtained the generalized value of the Lagrangean multiplier via variation theory as

$$\lambda(s) = \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)} \,, \tag{9}$$

where m is the order of the derivatives appearing in the problem.

Two cases of the DDE are considered below.

Case 1: If the DDE has a linear term, then the approximate solution can be obtained as

$$u_{n+1}(x) = u_0(x) + \int_0^x \lambda(s)\left[Lu_n(s) + \sum_{n=0}^{\infty} u_n - f(s)\right]ds, \quad n \ge 0$$

Thus

$$u_{n+1}(x) = u_0(x) + \int_0^x \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)}\left[Lu_n(s) + \sum_{n=0}^{\infty} u_n - f(s)\right]ds, \quad n \ge 0 \tag{10}$$

Case 2: If the DDE has a nonlinear term, then the approximate solution can be obtained as

$$u_{n+1}(x) = u_0(x) + \int_0^x \lambda(s)\left[Lu_n(s) + Ru_n(s) + \sum_{i=0}^{\infty} A_n - f(s)\right]ds, \quad n \ge 0,$$

Thus

$$u_{n+1}(x) = u_0(x) + \int_0^x \frac{(-1)^m}{(m-1)!}(s-x)^{(m-1)}\left[Lu_n(s) + Ru_n(s) + \sum_{i=0}^{\infty} A_n - f(s)\right]ds, \quad n \ge 0 \tag{11}$$

where m is the order of the highest occurring derivative, $u_n,\ n \ge 0$, are the components derived recursively from the relation in *Equation (6)* and $A_n,\ n \ge 0$, are the Adomian polynomials (ADP).

4 Numerical Applications

In this section, we implement the method on linear and nonlinear delay differential equations. To enable us the sake do comparison, we use the same examples in [1].

The error formulation for this problem is

$$\left|u(x) - u_n(x)\right|,$$

where $u(x)$ is the exact and $u_n(x)$ is the approximate solution.

Example 4.1 [1]: Consider the following nonlinear delay differential equation of first order:

$$\frac{du}{dx} = 1 - 2u^2\left(\frac{x}{2}\right), \qquad 0 \le x \le 1, \tag{12}$$

with the initial condition $u(0) = 0.$

The exact solution of the problem is $u(x) = \sin(x)$

We first have to find the components $u_n,\ n \ge 0$, recursively.

Here,

$Lu = u'$, $Nu = 2u^2\left(\frac{x}{2}\right)$ and $f = 1$.

Thus, by *Equation (2)*, we have

$$u = L^{-1}(1) - L^{-1}\left(2u^2\left(\frac{x}{2}\right)\right),$$

where

$$L^{-1}(.) = \int_0^x (.)dx$$

and

$$u = x - L^{-1}\left(2u^2\left(\frac{x}{2}\right)\right).$$

Hence,

$$u_0(x) = x,$$

$$u_{n+1} = -2\int_0^x A_n. \qquad (13)$$

Using the structure of the Adomian polynomials in *Equation (5)* where the nonlinear term is

$$Nu = u^2\left(\frac{x}{2}\right),$$

we have the following ADP:

$$A_0 = \frac{x^2}{4},\ A_1 = -\frac{x^4}{48},\ A_2 = \frac{x^6}{1440} \text{ and } A_3 = -\frac{x^8}{80640}.$$

Thus, using *Equation (13)* we obtain the components $u_n,\ n \ge 1$ as follows

$$u_1 = -\frac{x^3}{6},\ u_2 = -\frac{x^5}{6} \text{ and } u_3 = -\frac{x^7}{645120}.$$

By the decomposition variation iteration method (DVIM), we construct a correction functional for *Equation (12)* as follows:

$$u_{n+1}(x) = u_0(x) + \int_0^x \lambda(s)\left[\frac{du_n}{ds} + 2u^2\left(\frac{s}{2}\right) - 1\right]ds,\quad n \ge 0, \qquad (14)$$

$\lambda(s) = -1$ obtained using *Equation (9)*. Since, a nonlinear term is involved in the given equation we adopt Case 2.

Thus,

$$u_{n+1}(x) = u_0(x) - \int_0^x \left[\frac{du_n}{ds} + 2\left(\sum_{n=0}^{\infty} A_n\right) - 1\right]ds,\quad n \ge 0 \qquad (15)$$

For $n \ge 0$, we have

$$u_1 = u_2 = u_3 = u_4 = \cdots = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 \qquad (16)$$

Using *Equation (16)*, we obtain the approximate solution $u(x)$ as shown in **Table 1**.

Table 1: The results obtained with DVIM compared with the exact solution and the ADM [1] using the partial sum of the first four approximations

X	Exact	DVIM	ADM	DVIM Error	ADM Error
0.2	0.1986693308	0.1986693308	0.1986693309	0.0000E+00	-1E-10
0.4	0.3894183423	0.3894183423	0.3894183422	0.0000E+00	1E-10
0.6	0.5646424734	0.5646424735	0.5645424735	1.0000E-10	1E-09
0.8	0.717356090	0.7173560930	0.717356093	2.1000E-09	-3E-09

Example 4.2 [1]: Consider the following linear delay differential equation of second order:

$$\frac{d^2u}{dx^2} = \frac{3}{4}u(x) + u\left(\frac{x}{2}\right) - x^2 + 2,\quad 0 \le x \le 1 \qquad (17)$$

with the initial condition

$$u(0) = 0,\quad u'(0) = 0.$$

The exact solution of the problem is

$$u(x) = x^2.$$

We first have to find the components $u_n,\ n \ge 0$, recursively.

Here,

$Lu = u''$, $Ru = \frac{3}{4}u(x) + u\left(\frac{x}{2}\right) - x^2$ and $f = 2$.

Thus, by *Equation (2)*, we have

$$u = L^{-1}\left(\frac{3}{4}u(x) + u\left(\frac{x}{2}\right) - x^2 + 2\right),\quad L^{-1}(.) = \int_0^x\int_0^x (.)dxdx,$$

this implies that

$$u_n(x) = L^{-1}\left(\frac{3}{4}u(x) + u\left(\frac{x}{2}\right) - x^2 + 2\right). \qquad (18)$$

Evaluating *Equation (18)* at $x=0$, we have our initial approximation as

$$u_0(x) = x^2 - \frac{x^4}{12}.$$

Hence,

$$u_{n+1}(x) = L^{-1}\left(\frac{3}{4}u(x) + u\left(\frac{x}{2}\right) - x^2 + 2\right). \qquad (19)$$

Thus, using *Equation (19)* we obtain the components u_n, $n \geq 1$ as follows

$$u_1(x) = \frac{191}{2304}x^4 - \frac{1}{480}x^6 ,$$

$$u_2(x) = \frac{2483}{1105920}x^6 - \frac{7}{245760}x^8 ,$$

and

$$u_3(x) = \frac{17381}{566231040}x^8 - \frac{1351}{5662310400}x^{10}.$$

By the decomposition variation iteration method (DVIM), we construct a correction functional for *Equation (17)* as follows:

$$u_{n+1}(x) = u_0(x) + \int_0^x \lambda(s)\left[\frac{d^2u_n}{ds^2} - \frac{3}{4}u(x) - u\left(\frac{s}{2}\right) + s^2 - 2\right]ds, \quad n \geq 0, \qquad (20)$$

with

$$\lambda(s) = (s - x)$$

obtained using *Equation (9)*.

Since, a linear is involved in the given equation we adopt the case 1.

Thus,

$$u_{n+1}(x) = u_0(x) + \int_0^x (s-x)\left[\frac{d^2u_n}{ds^2} - \frac{3}{4}\sum_{n=0}^{\infty}u_n(s) - \sum_{n=0}^{\infty}u_n\left(\frac{s}{2}\right) + s^2 - 2\right]ds, \quad n \geq 0 \qquad (21)$$

For $n = 0$, we have

$$u_1 = x^2 - \frac{13}{1105920}x^6 + \frac{1253}{566231040}x^8 + \frac{241829}{1304596316\ 1600}x^{10} - \frac{1038919}{7653631721\ 47200}x^{12} \qquad (22)$$

Using *Equation (22)*, we obtain the approximate solution $u(x)$ as shown in **Table 2**.

Table 2: The results obtained with DVIM compared with the exact solution and the ADM [1] using the partial sum of the first four approximations.

x	Exact	DVIM	ADM	DVIM Error	ADM Error
0.0	0.0000000	0.000000	0.000000	0.0000E+00	0.0000E+00
0.2	0.0400000	0.0400000	0.0399993	7.5000E-10	6.8408E-07
0.4	0.1600000	0.1600000	0.1599896	4.6700E-08	1.0447E-05
0.6	0.3600000	0.3599995	0.3599513	5.1120E-07	4.8663E-05
0.8	0.6400000	0.6399973	0.6398650	2.7083E-06	1.3500E-04
1.0	1.0000000	0.9999905	0.9997298	9.5249E-06	2.7020E-04

5 Conclusion

We have successively implemented the new numerical algorithm for linear and nonlinear delay differential equations (DDEs). It is evident from the results obtained as shown in the **Tables 1** and **2**, that the DVIM has an excellent rate of convergence. Hence, the method is efficient, effective and accurate for the numerical solution of the delay differential equations. Also, the method can be further explored for application in other areas of science and technology, such as in dynamic flow line analysis.

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