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A Lindley Exponentiated-Exponential Distribution with Application To Waiting Time Data

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Abstract

In this paper, we introduced a new distribution called Lindley Exponentiated-Exponential (LEE) distribution. The distribution contains the Lindley and Quasi-Lindley distributions as special cases. The survival function, hazard function, moments, quantile function and Renyi entropy are derived. The maximum likelihood method of estimation was used in estimating the parameters of the LEE distribution. Finally, a real lifetime dataset was used to illustrate the usefulness of LEE distribution and compared with the fit of the Lindley and Quasi-Lindley distributions. Our finding was that the LEE distribution outperformed the other two distributions in terms of the Kolmogorov-Smirnov test statistic (K-S), the density plot, P-P plot and the Q-Q plot.

Keyword: Lindley; Quasi Lindley; Exponentiated-Exponential; T-X Family; Hazard Rate.

1. INTRODUTION

The Lindley distribution is a one parameter probability distribution introduced in [1] as a means of replacing the Fudicial statistics with the Bayesian statistics. The usefulness of the Lindley distribution was not seen until Ghitany et al [2] studied the properties and found the flexibility of Lindley distribution over exponential distribution, where they also showed its application in waiting time data analysis and Stress-Strength reliability.

The probability density function (pdf), $f(x)$ and cumulative distribution function (cdf), $F(x)$ of Lindley distribution are given respectively as follows:

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x>0,\ \theta>0 \qquad (1)$$

and

$$F(x) = 1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}, \ x>0,\ \theta>0 \qquad (2)$$

To improve on the flexibility of Lindley distribution, Ghitany et al [3] proposed a two-parameter weighted Lindley distribution and pointed out that it is used for modelling data from mortality studies. Shanker and Mishra [4] proposed the Quasi-Lindley distribution showing its application in social sciences, the Lindley-Exponential distribution was introduced in [5], Mavis, Gayan and Oluyede [6] introduced the Exponentiated Power Lindley distribution model. Recently, many methods of generating family of probability distribution have been introduced. Some of these includes the well-known MO-G generated family of distribution proposed in [7]. Eugene et al. [8] developed the beta class distribution, new family of generalized distributions [9], Oluyede et al [10] presented a new class of generalized Power Lindley distribution among others.

Alzaatreh et al. [11] proposed the T-X distribution family with cumulative distribution function given as

$$F(x) = \int_0^{W(G(x))} r(t)dt \qquad (3)$$

where $W(G(x)) = -\log[1-G(x)]$, provided that $WG(x)\in[0,\infty)$ is a monotonic non-decreasing and differentiable function. Using the idea in [11], we introduce and develop the statistical properties of LEE distribution as a means of improving the flexibility of Lindley distribution in modelling real lifetime data.

The rest of this paper is organized as follows: In Section 2, we present the formulation of the density function and the cumulative density function of LEE distribution. Section 3, the survival and the hazard rate functions are obtained, while the moments and the related measures are presented in Section 4. In Sections 5-7, we present the quantile function, the Renyi entropy and the maximum likelihood estimation. Section 8 is on the application of the LEE distribution and Section 9 concludes the paper.

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2. The Lindley Exponentiated-Exponential Distribution

Consider $G^{\alpha}(x)$ to be the exponentiated cumulative distribution function (cdf) of the random variable X and $r(t)$ be the density function of a random variable T from Lindley distribution. From equation (3), we define the cdf of Lindley exponentiated-G family as:

$$F(x) = \frac{\theta^2}{\theta+1}\int_0^{-\log[1-G(x)]^{\alpha}} (1+t)e^{-\theta t}\,dt,$$

$$= 1-\left\{1-\frac{\alpha\theta\log[1-G(x)]}{1+\theta}\right\}[1-G(x)]^{\alpha\theta},\ x>0;\ (\alpha,\ \theta)>0 \qquad (4)$$

and differentiating equation (4), the pdf is obtained:

$$f(x) = \frac{\alpha\theta^2 g(x)}{1+\theta}(1-\alpha\log[1-G(x)])[1-G(x)]^{\alpha\theta-1},\ x>0;\ (\alpha,\ \theta)>0 \qquad (5)$$

When $G(x) = 1-e^{-\lambda x}$ and $g(x) = \lambda e^{-\lambda x}$, the cdf and pdf of the LEE distribution are obtained from (4) and (5) respectively as:

$$F(x) = 1-\left(1+\frac{\alpha\lambda\theta x}{\theta+1}\right)e^{-\alpha\lambda\theta x},\ \ x>0;\ (\alpha,\lambda,\theta)>0 \qquad (6)$$

and

$$f(x) = \frac{\alpha\lambda\theta^2}{1+\theta}(1+\alpha\lambda x)e^{-\alpha\lambda\theta x},\ \ x>0;\ \ (\alpha,\ \lambda,\ \theta)>0 \qquad (7)$$

We reparametize the cdf and pdf in (6) and (7) respectively by setting $\beta = \alpha\lambda$ to get the cdf of LEE distribution as

$$F(x;) = 1-\left(1+\frac{\beta\theta x}{1+\theta}\right)e^{-\beta\theta x},\ \ x>0;\ (\theta,\beta)>0 \qquad (8)$$

the corresponding pdf becomes

$$f(x) = \frac{\beta\theta^2}{1+\theta}(1+\beta x)e^{-\beta\theta x},\ \ x>0;\ \ (\beta,\ \theta)>0 \qquad (9)$$

<u>Sub-Model</u>

I. When $\beta = 1$, we have the Lindley distribution with the pdf as:

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x},\ \ x>0;\quad \theta>0$$

II. When $\beta\theta = \phi$, we have the Quasi-Lindley distribution with pdf as:

$$f(x) = \frac{\phi}{1+\theta}(\theta+\phi x)e^{-\phi x},\ \ x>0;\ \ (\phi,\ \theta)>0$$

3. Survival and Hazard Rate Function

The survival (reliability) function of the LEE-distribution is obtained from (6) as

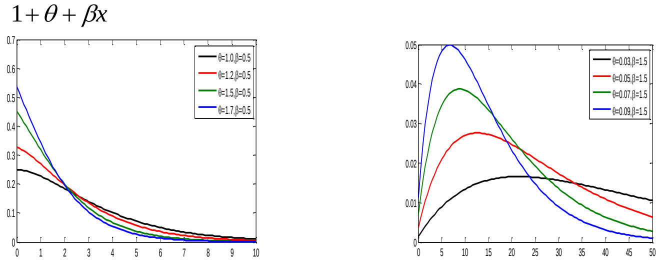
$$S(x) = 1-F(x) = \left(1+\frac{\beta\theta x}{1+\theta}\right)e^{-\beta\theta x}$$

$$= \left(1+\frac{\beta\theta x}{1+\theta}\right)e^{-\beta\theta x} \qquad (10)$$

and the corresponding Hazard Rate Function, $h(x)$ which of great interest in lifetime (Reliability) study becomes

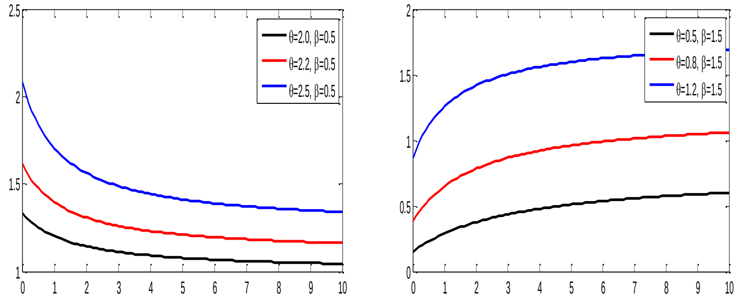
$$h(x) = \frac{f(x)}{1-F(x)}$$

$$= \frac{\beta\theta^2(1+\beta x)}{1+\theta+\beta x} \qquad (11)$$



(a) **(b)**

Figure 1: Plot of the probability density function (pdf) of LEE distribution



(a) **(b)**

Figure 2: Plot of the hazard rate function of LEE distribution

From Fig.1 (a), we observed that for fixed value of β and some fixed values of $\theta \geq 1$ the pdf is decreasing. In Fig.1 (b) for fixed value of β and some fixed values of $\theta < 1$ the pdf is right skewed unimodal. Similarly, Fig.2 (a) displays that the hazard rate function of LEE distribution is increasing for the values of $\beta > 1$ *and* $\theta < 2$ and Fig.2 (b) decreasing for some fixed values of $\beta < 1$ *and* $\theta \geq 2$.

4. Moment and Related Measure

The rth raw moment of a continuous random variable X denoted by μ_r' is defined as

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

Hence, rth raw moment of the LEE distribution is defined as:

$$\mu_r' = E\left(X^r\right) = \frac{\beta\theta^2}{1+\theta}\int_0^\infty x^r (1+\beta x)e^{-\beta\theta x}dx$$

$$= \frac{\beta\theta^2}{1+\theta}\left\{\int_0^\infty x^r e^{-\beta\theta x}dx + \beta\int_0^\infty x^{r+1}e^{-\beta\theta x}dx\right\}$$

Substituting $u = \beta\theta x$, then we have

$$\mu_r' = \frac{\beta\theta^2}{1+\theta}\left\{\int_0^\infty \left(\frac{u}{\beta\theta}\right)^r e^{-u}\frac{du}{\beta\theta} + \beta\int_0^\infty \left(\frac{u}{\beta\theta}\right)^{r+1}\frac{du}{\beta\theta}\right\}$$

$$= \frac{\beta\theta^2}{1+\theta}\left\{\frac{1}{(\beta)^{r+1}}\int_0^\infty u^r e^{-u}du + \frac{\beta}{(\beta\theta)^{r+2}}\int_0^\infty u^{r+1}du\right\}$$

Since $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, we have

$$\mu_r' = \frac{\beta\theta^2}{1+\theta}\left\{\frac{\Gamma(r+1)}{(\beta\theta)^r} + \frac{\beta\Gamma(r+2)}{(\beta\theta)^{r+2}}\right\}$$

$$= \frac{1}{1+\theta}\left\{\frac{\theta\Gamma(r+1)+\Gamma(r+2)}{(\beta\theta)^r}\right\},\ r = 1,\ 2,\ 3,\ \ldots \tag{12}$$

We obtain the first four raw moment using equation (12) as:

$$\mu_1' = \frac{\theta+2}{\beta\theta(1+\theta)},\quad \mu_2' = \frac{2\theta+6}{(\beta\theta)^2(1+\theta)},\quad \mu_3' = \frac{6\theta+24}{(\beta\theta)^3(1+\theta)},\quad \mu_4' = \frac{24\theta+120}{(\beta\theta)^4(1+\theta)}$$

Then the central moment of the LEE distribution is derived as follows

$$\mu_r = E(X-\mu)^r = E\left\{\sum_{i=0}^{r}\binom{r}{i}X^{r-i}(-\mu)^i\right\}$$

$$\mu_r = \sum_{i=0}^{r}(-1)^i\binom{r}{i}E\left(X^{r-i}\right)(\mu)^i$$

Hence

$$\mu_r = \sum_{i=1}^{r}(-1)^i\binom{r}{i}\mu_{r-i}'\mu^i \tag{13}$$

Thus the second, third and fourth central moments of LEE distribution are obtained:

$$\mu_2 = \frac{\theta^2+4\theta+2}{(\beta\theta)^2(1+\theta)^2},\quad \mu_3 = \frac{2\theta^3+12\theta^2+12\theta+4}{(\beta\theta)^3(1+\theta)^3},\quad \mu_4 = \frac{9\theta^4+72\theta^3+132\theta^2+96\theta+24}{(\beta\theta)^4(1+\theta)^4}$$

It follows that the mean (μ), variance (σ^2), coefficient of variation (C_V), coefficient of skewness (C_S) and coefficient of kurtosis (C_K) are obtained respectively

$$\mu = \mu_1' = \frac{\theta+2}{\beta\theta(1+\theta)}, \qquad \sigma^2 = \mu_2' - \mu^2 = \frac{\theta^2+4\theta+2}{(\beta\theta)^2(1+\theta)^2},$$

$$C_V = \frac{\sigma}{\mu} = \frac{\sqrt{\theta^2+4\theta+2}}{\theta+2}, \quad C_S = \frac{2\theta^3+12\theta^2+12\theta+4}{\left[\theta^2+4\theta+2\right]^{3/2}}$$

and $$C_K = \frac{9\theta^4+72\theta^3+132\theta^2+96\theta+24}{\left[\theta^2+4\theta+2\right]^2}$$

Table 1 presents the mean (μ), standard deviation (σ), coefficient of variation (C_V), coefficient of skewness (C_S) and coefficient of kurtosis (C_K) for some selected values of the parameters β and θ.

Table 1: The moments of the LEE distribution for some fixed parameter values of θ amd β

Parameters θ	β	μ	σ	C_V	C_S	C_K
0.5	0.5	6.6667	5.4975	0.8246	1.5123	6.3426
	1.2	2.7778	2.2906	0.8246	1.5123	6.3426
	1.5	2.2222	1.8325	0.8246	1.5123	6.3426
	1.5	1.6667	1.3744	0.8246	1.5123	6.3426
1.5	0.5	1.8667	1.7074	0.9147	1.6989	7.1725
	1.2	0.7778	0.7115	0.9147	1.6989	7.1725
	1.5	0.6222	0.5692	0.9147	1.6989	7.1725
	2.0	0.4667	0.4269	0.9147	1.6989	7.1725
2.0	0.5	1.3333	1.2472	0.9354	1.7563	7.4694
	1.2	0.5556	0.5197	0.9354	1.7563	7.4694
	1.5	0.4444	0.4157	0.9354	1.7563	7.4694
	2.0	0.3333	0.3118	0.9354	1.7563	7.4694
2.5	0.5	1.0286	0.9765	0.9493	1.7989	7.7029
	1.2	0.4286	0.4069	0.9493	1.7989	7.7029
	1.5	0.3429	0.3255	0.9493	1.7989	7.7029
	2.0	0.2571	0.2441	0.9493	1.7989	7.7029

5. Quantile Function of LEE Distribution

The quantile function of a probability distribution with cdf $F(x)$ is defined by $q = F^{-1}(x_q)$ where $0 < q < 1$, [12]. Thus, the quantile function of the LEE distribution is determined by solving for x_q at $q \in (0,1)$ from the equation.

$$q = 1 - \left(1 + \frac{\beta\theta x_q}{1+\theta}\right)e^{-\beta\theta x_q}$$

$$\left(\frac{1+\theta+\beta\theta x_q}{1+\theta}\right)e^{-\beta\theta x_q} = 1-q \qquad (14)$$

Substituting $W(x_q) = -1-\theta-\beta\theta x_q$ into (14) and simplifying, we have

$$W(x_q)e^{W(x_q)} = -(1+\theta)(1-q)e^{-(1+\theta)}$$

Then the solution becomes

$$W(x_q)e^{W(x_q)} = W_{-1}\left[-(1+\theta)(1\_q)e^{-(1+\theta)}\right]$$

$$x_q = -\frac{1}{\beta}\left\{1+\frac{1}{\theta}+\frac{1}{\theta}W_{-1}\left[-(1+\theta)(1-q)e^{-(1+\theta)}\right]\right\} \qquad (15)$$

For $0 < q < 1$, where $W(.)$ is the Lambert W f unction in [13]. Equation (15) can be used to generate random sample from the LEE distribution. Table 2 shows the quantile of LEE distribution for some selected parameter values of (β, θ).

Table 2: The quantile of the LEE distribution for some selected parameter values of (β,θ)

	(β,θ)						
Q	$(0.5,0.5)$	$(1.5,0.5)$	$(0.5,1.5)$	$(1.5,1.5)$	$(2.0,1.5)$	$(1.5,2.0)$	$(2.0, 2.0)$
0.1	1.088039	0.3626797	0.2291139	0.0763713	0.0572784	0.0520261	0.0390196
0.2	2.086144	0.6953825	0.4752306	0.1584102	0.1188076	0.1088104	0.0816078
0.3	3.087074	1.0290245	0.7442879	0.2480959	0.1860719	0.1717142	0.1287850
0.4	4.143660	1.3812201	1.0445609	0.3481869	0.2611402	0.2427054	0.1820291
0.5	5.307370	1.7691232	1.3884935	0.4628311	0.3471233	0.3248038	0.2436029
0.6	6.648178	2.2160592	1.7965290	0.5988430	0.4491322	0.4230252	0.3172689
0.7	8.285963	2.7619877	2.3064642	0.7688214	0.5766160	0.5466984	0.4100238
0.8	10.479081	3.4930269	3.0021816	1.0007272	0.7505454	0.7165931	0.5374448
0.9	14.032783	4.6775943	4.1480378	1.3826792	1.0370094	0.9982927	0.7487195

6. Renyi Entropy of LEE Distribution

Entropy is an important concept to measure the quantity of uncertainty in relation to a random variable say *X*. Renyi [14] defined the entropy of a random variable *X* with pdf $f(x)$ as

$$J_R(s)=\frac{1}{1-s}\log\left[\int_0^\infty f^s(x;\Phi)dx\right] \qquad s>0, s\neq 1$$

Then the Renyi entropy of LEE Distribution becomes

$$J_R(s)=\frac{1}{1-s}\log\left\{\left(\frac{\beta\theta^2}{1+\theta}\right)^s\int_0^\infty(1+\beta x)^s e^{-s\beta\theta x}dx\right\}$$

$$=\frac{1}{1-s}\log\left\{\left(\frac{\beta\theta^2}{1+\theta}\right)^s\left[\sum_{i=0}^{\infty}\binom{s}{i}\beta^i\int_0^\infty x^i e^{-s\beta\theta x}dx\right]\right\}$$

If $U=s\beta\theta x\Rightarrow x=\frac{U}{s\beta\theta}\Rightarrow dx=\frac{du}{s\beta\theta}$. Then we have

$$J_R(s)=\frac{1}{1-s}\log\left\{\left(\frac{\beta\theta^2}{1+\theta}\right)^s\sum_{i=0}^{\infty}\binom{s}{i}\beta^i\frac{1}{(s\beta\theta)^{i+1}}\int_0^\infty U^i e^{-U}dU\right\}$$

$$=\frac{1}{1-s}\log\left\{\left(\frac{\beta\theta^2}{1+\theta}\right)^s\left[\sum_{i=0}^{\infty}\binom{s}{i}\frac{\beta^i\Gamma(i+1)}{(s\beta\theta)^{i+1}}\right]\right\} \qquad (16)$$

7. Maximum Likelihood Estimation

Let $x_1, x_2, \cdots, x_n$ be a random variable from Lindley Exponentiated-Exponential (LEE) distribution, then the log-likelihood function denoted by $\ell_n(x_i,\Phi)$ is obtained as:

$$\ell_n(x_i,\Phi)=\sum_{i=1}^{n}\ln\left\{\frac{\beta\theta^2}{1+\theta}(1+\beta x_i)e^{-\beta\theta x}\right\}$$

$$=n\ln(\beta\theta^2)-n\ln(1+\theta)+\sum_{i=1}^{n}\ln(1+\beta x_i)-\beta\theta\sum_{i=1}^{n}x_i \qquad (17)$$

where $\Phi=(\beta,\theta)^T$ is a vector of parameter.

The partial derivatives of $\ell_n(x_i,\Phi)$ with respect to the parameters θ and β

are: $\frac{\partial \ell_n(\beta,\theta)}{\partial \theta} = \frac{n}{\theta} - \frac{n}{1+\theta} - \beta\sum_{i=1}^{n} x_i$ (18)

$$\frac{\partial \ell_n(\beta,\theta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} - \theta\sum_{i=1}^{n} x_i$$

Estimates $(\hat{\theta}, \hat{\beta})$ of (θ, β), can be obtained by solving the non-linear system of equations $\frac{\partial \ell_n}{\partial \beta} = 0$ and $\frac{\partial \ell_n}{\partial \phi} = 0$ simultaneously. We can achieve this solution by using a numerical method such as the Newton-Raphson iterative scheme. This was achieved using R software package. To do this in R environment, we supply the data set to be fitted, define the pdf, cdf and the quantile functions. Then loading the "fitdist" package and specifying the initial values of the distribution parameter(s) will enable R perform the iterative scheme and the desired output result is returned.

8. Application of the LEE Distribution

In this section we show the applicability of the LEE distribution by considering a real life data set consisting of the waiting times (in minutes) of 100 customers in a bank before they are being serviced as reported in [2]

Table 3: Waiting times (Minutes) of 100 bank customers

0.8,	0.8,	1.3,	1.5,	1.8,	1.9,	1.9,	2.1,	2.6,	2.7,	2.9,	3.1,	3.2,	3.3,
3.5,	3.6,	4.0,	4.1,	4.2,	4.2,	4.3,	4.3,	4.4,	4.4,	4.6,	4.7,	4.7,	4.8,
4.9,	4.9,	5.0,	5.3,	5.5,	5.7,	5.7,	6.1,	6.2,	6.2,	6.2,	6.3,	6.7,	6.9,
7.1,	7.1,	7.1,	7.1,	7.4,	7.6,	7.7,	8.0,	8.2,	8.6,	8.6,	8.6,	8.8,	8.8,
8.9,	8.9,	9.5,	9.6,	9.7,	9.8,	10.7,	10.9,	11.0,	11.0,	11.1,	11.2,	11.2,	11.5,
11.9,	12.4,	12.5,	12.9,	13.0,	13.1,	13.3,	13.6,	13.7,	13.9,	14.1,	15.4,	15.4,	17.3,
17.3,	18.1,	18.2,	18.4,	18.9,	19.0,	19.9,	20.6,	21.3,	21.4,	21.9,	23.0,	27.0,	31.6,
33.1,	38.5												

We fit this data set with the LEE distribution and compared with the fit of Quasi-Lindley (QL) and Lindley (L) distributions whose pdfs are given respectively by:

(I) Quasi Lindley Distribution:

$$f(x) = \frac{\phi}{1+\phi}(\theta + \phi x)e^{-\beta\theta x}, \; x > 0; \; (\phi, \theta) > 0$$

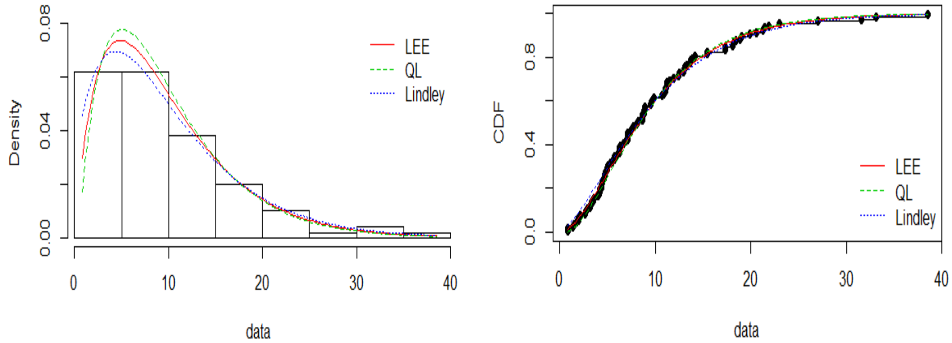
(II) Lindley Distribution:

$$f_L(x) = \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta}, \qquad x > 0, \theta > 0$$

For comparison of the goodness of fit of the distributions, we use the negative log likelihood $(-\ell)$, the Akaike Information Criterion (AIC), the Bayesian information criterion (BIC) and the Kolmogorov-Smirnov (K-S) test statistic for the dataset. The values of $-\ell$, AIC, BIC and K-S test Statistic are provided in Table 4.

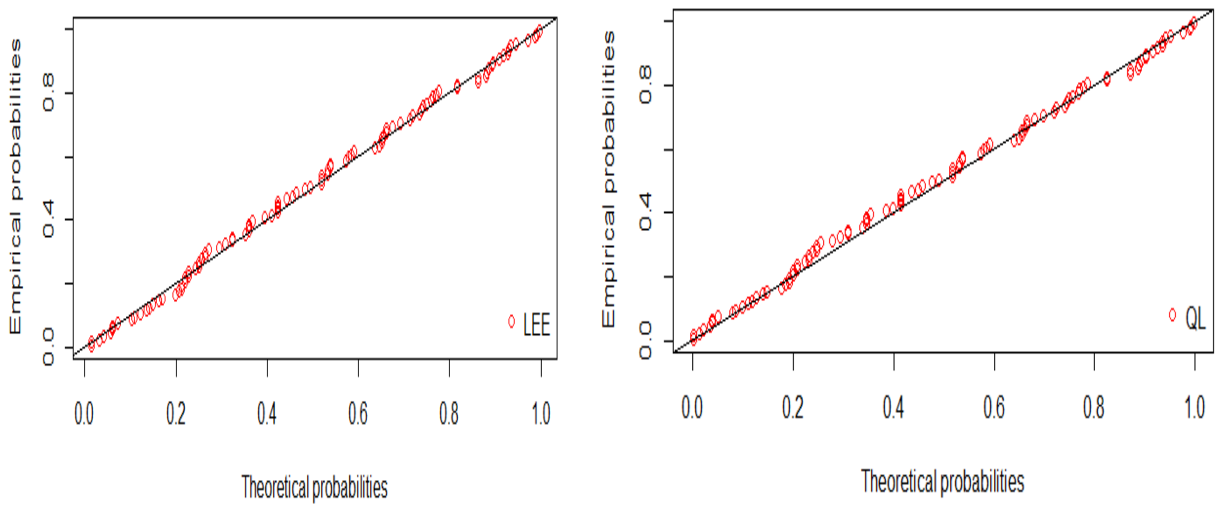
Table 4: Summary Statistics for the Data Set

Distribution	Estimates	$-\ell$	AIC	BIC	K-S
LEE	$\beta = 10.9904$ $\theta = 0.0182$	317.447	638.894	644.1043	0.0419
QL	$\phi = 0.2112$ $\theta = 0.0791$	316.9255	637.8511	643.0614	0.05666
LINDLEY	$\theta = 1865$	319.0374	640.0748	642.6800	0.06767



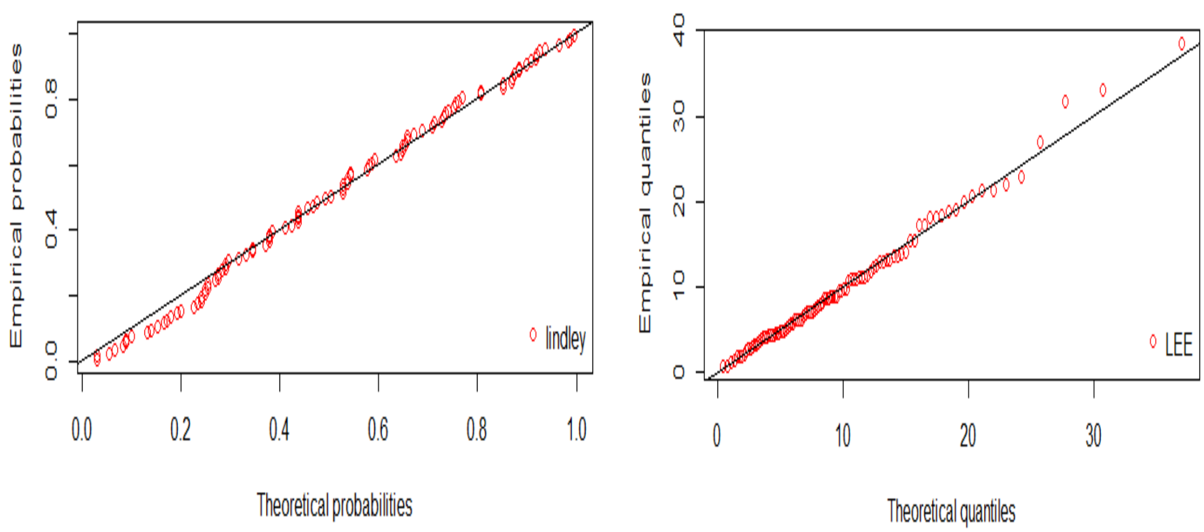
(a) **(b)**

Figure 3: PDF and CDF fits for the data set



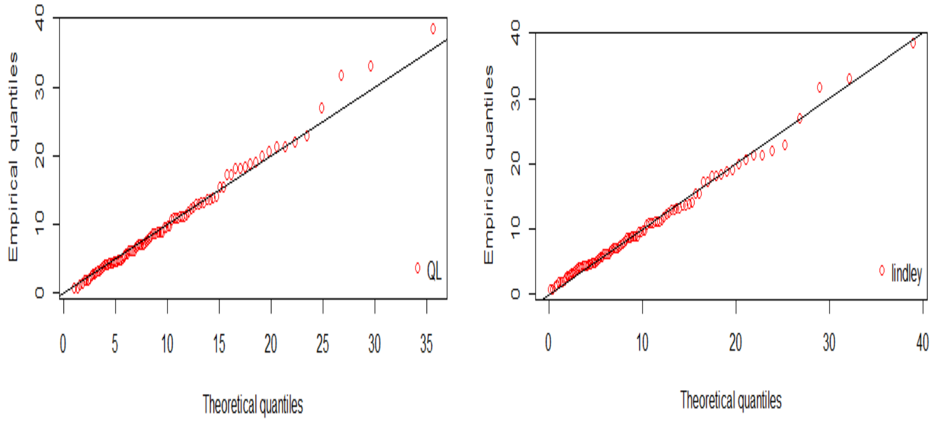
(a) (b)

Figure 4: The LEE and QL P-P fit for the data set



(a) **(b)**

Figure 5: The Lindley P-P fit and LEE Q-Q fit for the data set



(a) **(b)**

Figure 6: The QL and Lindley Q-Q fit for the Data Set

Figure 3(a) is the density fit plot, Figure 3(b) is the cumulative distribution fit plot, Figure 4(a), 4(b) and 5(a) are the P-P fit plot while Figure 5(b), 6(a) and 6(b) are the Q-Q fit plot. These plots show the superiority of the LEE distribution for the data set over the Lindley and Quasi-Lindley distributions.

9. Conclusion

In this paper, we propose a new family of Lindley distribution called the Lindley Exponentiated-Exponential (LEE) distribution using the Lindley distribution as the transformed random variable in the T-X family of distribution. We obtained some of the statistical properties such as the survival function, the hazard rate functions, the moments, quantile function and Renyi entropy. The maximum likelihood estimation (MLE) was used to estimate the parameters of the model and a real data set was used to compare the performance of the LEE distribution with the classical Lindley and Quasi Lindley distributions. The results show that the LEE distribution outperformed the others in terms of the K-S, density fit, P-P plot and Q-Q plot for the data set.

References

[1] Lindley, D., (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society 20(1): page 102-107.*

[2] Ghitany, M., Atieh, B. and Nadadrajah, S., (2008). Lindley distribution and its applications. *Mathematics and Computers in Simulation 78: page 493-506.*

[3] Ghitany, M.E., Alqallaf, F., Al-Mutairi, D.K., Husain, H.A. (2011). A two-parameter weighted Lindley distribution and its applications to survival data. *Mathematical Computations in Simulation. 81(6): page 1190–1201.*

[4] Shanker, R. and Mishra, A. (2013): A Quasi-Lindley distribution. *African Journal of mathematics and Computer Science Research,6(4): page 64-71.*

[5] Bhati, D., Malik, M. A. and Vaman, H. J. (2015). Lindley-Exponential distribution: Properties and applications. *METRON, 73(3), page 335-357.*

[6] Mavis P, Gayan W.L and Oluyede B, (2015). A New Class of Generalized Power Lindley with application to Lifetime Data. *Theoretical Mathematics and Application, 5(1), page 53-96.*

[7] Marshall A. N, Olkin I (1997). A New Method for Adding a parameter to a family of Distribution with Applications to the Exponential and Weibull Families. *Biometrika, 64: 641-552.*

[8] Eugene N, Lee C, Famoye F (2002). Bet-Normal Distribution and Its Applications. *Communications in Statistics Theory and Models. 31: 497-512*

[9] Cordeiro, G.M, de Castro M (2011). A New Family of Generalized Distribution. *Journal of Statistical Computation and Simulation,81: 883-893.*

[10] Oluyede B.O, Yang T and Makubate B, (2016). A New class of Generalized Power Lindley Distribution with Application to Lifetime Data. *Asian Journal of Mathematics and Application, ama0279 (34pages).*

[11] Alzaatreh A, Lee C, Famoye F (2013). A New Method for generating Families of Distributions. *Metron, 71: 63-79.*

[12] Jodra, P. (2010): Computer generation of random variables with Lindley or Poisson–Lindley distribution via the Lambert W function. *Mathematical Computations and Simulation, 81, page 851–859.*

[13] Corless R.M, Gonnet G.H, Hare D.E.G, Jeffery D. J and Knuth D.E, (1996). On Lambert W Function. *Advances in computational Mathematics, 5: 329-359.*

[14] Rényi, A, (1961). On measure of entropy and information. *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability 1, University of California Press, Berkeley, page 547-561.*