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Stream Function Modelling of Compressible Axisymmetric Flow of Air through the Riser of Casting Mould

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Abstract

In this work a simplified model that can be used to analyze compressible axisymmetric flow of air through the risers of casting mould was developed. Two fluids are involved in the casting process but casting engineers and researchers has focused on the flow of molten metal developing general casting design rules and empirical equations for the gating ratio, pouring time, and gating system dimensions, and consideration has not been given prior to now to the air flow behaviour in the casting mould. The model was developed from the basic equations governing fluid flow, the irrotationality flow conditions and the stream function. The model developed can be used to analyze fluid flow using any numerical method.

Keyword: Stream function, compressible axisymmetric flow, riser, air, casting mould

1. INTRODUTION

Casting is a manufacturing process for making complex shapes of metal materials. In the casting process two fluids are involved, the liquid molten metal poured from the ladle to fill the mould cavity and the air displaced from the mould cavity.

The most basic definition of a *fluid* is to state that a fluid is a material that conforms to the shape of its container. Thus, both liquids and gases are fluids. Alternately, it can be stated that a material which, in itself, cannot support shear stresses is a fluid. Thus, a fluid readily distorts, since the resistance to shear is very low, and such distortion results in flow [1]. Liquids exhibit constant density and the study of fluid mechanics of liquids is generally referred to as *incompressible flow*. On the other hand, gases are highly compressible and temperature dependent. Therefore, fluid mechanics problems involving gases are classified as *compressible flow* [1]. Fluids that exhibit very little viscosity are termed *inviscid* and shearing stresses are ignored. On the other hand, fluids with significant viscosity must be considered to have associated significant shear effects. In general, liquids are most often treated as incompressible but the viscosity effects depend specifically on the fluid. Gases, on the other hand, are generally treated as compressible but inviscid [1].

For manufacturing engineers there are many situations where compressible flow understanding is essential for adequate design. These processes include casting, injection moulding [2, 3]. Casting is a process in which liquid metal is injected into a mould to obtain a near final shape. The air in the mould is displaced by the liquid metal in a very rapid manner, in a matter of milliseconds through the riser; therefore its compressibility has to be taken into account [2, 3].

A riser (which is usually cylindrical in cross-section) is a reservoir built into a metal casting mould to prevent cavities due to shrinkage by providing molten metal at the point of likely shrinkage, so that the cavity forms in the riser, not the casting [4]. Apart from serving as a reservoir to compensate for shrinkage during solidification, the riser also serve as a channel through which the air displaced as a result of filling the mould cavity with the molten metal goes out of the mould cavity.

Numerous efforts have been made by castings engineers and researchers on gating system design over the past few decades [5 - 10]. Although there are general casting design rules and empirical equations for the gating ratio, pouring time, gating and riser system dimensions and optimization [6, 8], consideration has not been given prior to now to the air flow behaviour in the casting mould. Hence this work focuses on the development of a simplified model for compressible axisymmetric flow of air through the risers of casting mould.

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2.0 Basic Characteristics of Fluid

An *ideal* (or *perfect*) *fluid* is one that has zero viscosity and is incompressible. A *real fluid* or *viscous fluid* is one with finite viscosity, and may or may not be incompressible. *Non-viscous fluids (inviscid)* are those with zero viscosity, and again may or may not be incompressible. A *Newtonian fluid* is one with constant viscosity. Common examples of Newtonian fluids are air and water. A *non-Newtonian fluid* is one with variable viscosity. Common examples of non-Newtonian fluids are some plastics, colloidal suspensions, and emulsions [11, 12].

If the flow velocity is the same in magnitude and direction at every point in the fluid, the flow is said to be *uniform*. If at a given instant, the velocity is not the same at every point the flow is *non-uniform*. A *steady flow* is one in which the conditions (velocity, pressure and cross-section) may differ from point to point but DO NOT change with time. If at any point in the fluid, the conditions change with time, the flow is described as *unsteady*. (In practise there are always slight variations in velocity and pressure, but if the average values are constant, the flow is considered *steady* [1].

3.0 Basic Equations of Compressible Flows

The basic equations governing compressible flows are derived from the global laws of conversation of mass, momentum, and energy. Conservation of mass gives the continuity equation, while the conservation of momentum results in the equation of motion [2, 3, 11]. These equations are known as the Navier-Stocks equations.

3.1 Conservation of Mass

The continuity equation is obtained by applying the Conservation of Mass Law to an infinitesimal, fixed control volume [12, 13, 14]:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho u)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \qquad (1)$$

where ρ is density, *u, v, w* represent the *r, θ, z* components of the velocity vector.

Equation (1) represents the three-dimensional continuity equation for a fluid in unsteady flow.

3.2 Conservation of Momentum

The Conservation of Momentum Law is nothing but Newton's second law. Applying this law to a fluid passing through an infinitesimal, fixed control volume yields the momentum equations [12, 13, 14]

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (ru)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_r \qquad (2a)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+w\frac{\partial v}{\partial z}-\frac{vu}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_\theta \qquad (2b)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z \qquad (2c)$$

For steady flow all derivatives with respect to time will be zero in equations (1) and (2).

4.0 Flow Characteristics of Air in the Riser of Casting Mould

4.1 Axisymmetric Flow of Air in the Riser of Casting Mould

There is a group of three-dimensional field problems that can be solved using two-dimensional elements. These problems possess symmetry about an axis of rotation and are known as axisymmetric problems. The boundary conditions as well as the region geometry must be independent of the circumferential (*θ*) direction [12]. The equations governing physical processes in a cylindrical geometry are described analytically in terms of cylindrical coordinates (*r. θ, z*). When the geometry, loading (or source), and boundary conditions of the process are independent of the circumferential (angular) direction (i.e., *θ*-coordinate), the problem solution will also be independent of *θ.* Consequently, the flow is said to be Axisymmetric and the three-dimensional governing equations become a two-dimensional one in (*r, z*) coordinates [15, 16].

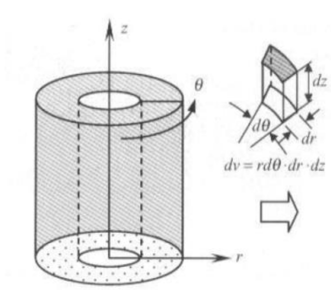

Figure 1: Axisymmetric flow [16]

The gas laws which explain the physical behaviour of gases can be explained by the Kinetic Theory of gases. It states that the gas molecules move randomly in straight lines, until they collide with one another or with the walls of their container. Upon collision, the particle will change its direction and continue moving in a straight line until another collision occurs [17].
Since gas molecules move randomly in straight lines, the motion of a gas in cylindrical coordinates ($r.\ \theta,\ z$) is independent of the circumferential (angular) direction (i.e., θ-coordinate), the problem solution will also be independent of θ. Consequently, gas flows in cylindrical coordinates is axisymmetric and thus have the coordinates (r, z).
Risers design are usually cylindrical, therefore the axisymmetric cylindrical coordinate (r, z) applies.
For axisymmetric flow equation (1) becomes:

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial(r\rho u)}{\partial r}+\frac{\partial(\rho w)}{\partial z}=0 \qquad (3)$$

and equation (2) becomes:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru)}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_r \qquad (3a)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z \qquad (3b)$$

4.2 Irrotationality Flow Condition of Air Flow in the Riser of Casting Mould

When the particles of a fluid are not rotating, the rotation is zero and the fluid is called irrotational [12]. A flow with negligible angular velocity is called irrotational $\nabla\times v=0$ [15]. It is convenient to introduce a function ϕ called the velocity potential in integrating equations (3). This function is defined in such a way that its partial derivative in any direction gives the velocity in that direction, [12, 18]: For axisymmetric flow we have

$$\frac{\partial\phi}{\partial r}=u \qquad \frac{\partial\phi}{\partial z}=w \qquad (4)$$

Differentiating u and w with respect to z and r, respectively, in equation (4)

$$\frac{\partial u}{\partial z}=\frac{\partial^2\phi}{\partial z\partial r}, \qquad \frac{\partial u}{\partial r}=\frac{\partial^2\phi}{\partial r^2} \qquad (5)$$

$$\frac{\partial w}{\partial z}=\frac{\partial^2\phi}{\partial z^2}, \qquad \frac{\partial w}{\partial r}=\frac{\partial^2\phi}{\partial r\partial z} \qquad (6)$$

Equating the first expression of (5) and the second expression of (6) gives the irrotationality condition

$$\frac{\partial u}{\partial z}=\frac{\partial w}{\partial r} \qquad \therefore \frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}=0 \qquad (7)$$

5.0 Stream Function Modelling of Air flow in the Riser of Casting Mould

For a two-dimensional compressible, steady state inviscid flow, equation (3) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho u)+\frac{\partial}{\partial z}(\rho w)=0 \qquad (8)$$

The problem of determining u, and w is simplified by introducing a stream function $\psi(r, z)$ such that the continuity equation is identically satisfied [11]. In cylindrical coordinates, the velocity components are related to the stream function as follows [19]:

$$u=\frac{1}{r}\frac{\partial\psi}{\partial z} \qquad w=-\frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (9)$$

In a steady, plane compressible flow, the stream function can be defined by including the density of the fluid so that the continuity equation is identically satisfied [18].

$$u=\frac{1}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial z} \qquad w=-\frac{1}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (10)$$

Then the irrotational flow condition in terms of ψ takes the form (i.e. by substituting equations (10) into equation (7))

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial z}\right)=-\frac{\partial}{\partial r}\left(\frac{1}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial r}\right) \qquad \therefore \frac{\partial^2\psi}{\partial z^2}=-\frac{\partial^2\psi}{\partial r^2} \qquad (11)$$

Equation (11) is used to determine ψ, and then the velocities of flow u and w are determined from (10) [11].

Conclusion

In this paper a simplified model that can be used to analyze compressible axisymmetric flow of air through the risers of casting mould has been developed. This model can be used to analyze compressible axisymmetric fluid flow using either the finite element or the finite difference method. Before now all researches do was to develop empirical equations and optimized molten metal flow in casting mould. This work has gone further to develop a simplified model that can be used to analyze the behaviour of air in casting mould.

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