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Simple Elastic Model For Thermal Stresses Induced By Surface Heating

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Abstract

Rapid surface heating can induce large stresses in solids. A relatively simple model, assuming full constraint in two dimensions and no constraint in the third dimension, can adequately model stresses in a wide variety of situations. This paper derives this simple model, and supports it with criteria for its validity. Phenomena that are considered include non-zero penetration depths for the heat deposition, spatial non-uniformity in the surface heating, and elastic waves. Models for each of these cases, using simplified geometries, are used to develop quantitative limits for their applicability.

Keyword:

1. INTRODUTION

Rapid surface heating can induce large stresses in solids, possibly leading to surface roughening, yielding, or fracture. The determination of the stresses for a given marital and set of loads can be quite difficult, requiring a time-dependent, three dimensional analysis [1] For many cases, though a relatively simple model, assuming full constraint in two dimensions and no constraint in the third dimension, can adequately model the peak stress [2]. This paper derives such a model, and supports it with criteria for its validity. Phenomena that are considered include non-zero penetration depths for the heat deposition, spatial non-uniformity in the surface heating, and elastic waves. Models for each of these cases,using simplified geometries, is used to develop quantitative limits for their applicability. Thermal waves are additional phenomenon that can be of concern for very short pulses [3], but the effect is left for future paper.

With this volumetric enlargement, the elements of a solid undergo greater levels of stress. Thermal stresses can have a significant effect on a structure's strength and stability, potentially causing cracks or breaks within certain components [4-5] Such failures compromise the overall design of the structure, which can lead to possible weakening and deformation.

Residual stress in welding is just one example [6]. In welding, a bond is formed between metal parts by melting their surfaces and placing them together so they are joined when the materials solidify again [7-9]. As the assembled structure cools down, some areas of the welding tend to contract more than other areas due to differing thermal expansion coefficients [10] . This causes residual stresses within the area of the weld [11].

2. BASELINE CASE

The base case considers a solid restrained from deformation in two dimensions and without constraint in the third dimension. The lack of constraint in the third dimension is a result of the free surface. The full constraint in the other dimensions assumes that the thermal field is shallow relative to the depth of the structure, so that cold material below the surface restrains motion of the heated material in that shallow layer. To develop a model for stresses in this situation, we begin with the stress-strain relations:

$$\begin{aligned}\sigma_{xx} &= \lambda(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{xx}-(3\lambda+2\mu)\alpha T\\ \sigma_{yy} &= \lambda(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{yy}-(3\lambda+2\mu)\alpha T\\ \sigma_{zz} &= \lambda(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{zz}-(3\lambda+2\mu)\alpha T\end{aligned} \tag{1}$$

where λ is the Lame constant, μ is the shear modulus, and α is the thermal expansion coefficient. This equation assumes that T is the temperature difference from a stress-free reference temperature. Under the assumptions discussed above, we impose the condition that

$$\varepsilon_{zz} = \varepsilon_{yy} = \sigma_{xx} = 0 \tag{2}$$

From the last of these conditions, combined with Eq. (1), we find the strain in the x direction to be

$$\varepsilon_{xx} = \left(\frac{3\lambda+2\lambda}{\lambda+2\mu}\right)\alpha T \tag{3}$$

This can then be substituted into Eq. (1) to find the transverse stresses, which can be written as:

$$\sigma_{yy} = \sigma_{zz} = \frac{-E\alpha T}{1-\upsilon} \tag{4}$$

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where E is the elastic modulus and υ is Poisson's ratio. To complete the model we need an estimate of the surface temperature induced by the surface heating. Assuming uniform heating applied on a half-space, the surface temperature is as in [1]

$$T_{surface} = \frac{2q}{\kappa}\sqrt{\frac{kt}{\pi}} \tag{5}$$

where q is the surface heat flux, k is the thermal conductivity, and κ is the thermal diffusivity. Combining Eqs. (4) and (5) provides the following model for stresses induced by rapid surface heating:

$$\sigma_{yy} = \sigma_{zz} = \frac{-2qE\alpha}{(1-v)k}\sqrt{\frac{kt}{\pi}}$$

$$\sigma_{xx} = 0 \tag{6}$$

This result provides a baseline estimation of the surface stresses induced by rapid surface heating. It assumes spatial uniformity of the applied heat, no volumetric heating below the surface, and it ignores both elastic and thermal waves.

3. DEPOSITION BELOW THE SURFACE

Most surface heating actually deposits heat as volumetric heating within a thin layer near the surface. A typical model for volumetric heating resulting from energy impinging on a surface is

$$Q''' = Ae^{-\gamma x} \tag{7}$$

where A is a constant and γ is the attenuation coefficient. To provide the same total heat input as a true surface heating flux q, we must enforce A= qγ. The temperature distribution resulting from volumetric heating of this type is :

$$T = \frac{2q}{k\gamma}\left[\zeta \; ierfc\left(\frac{\eta}{2\zeta}\right) - e^{-\eta} + e^{\zeta^2-\eta}\; erfc\left(\zeta - \frac{\eta}{2\zeta}\right) + e^{\zeta^2+\eta}\; erfc\left(\zeta + \frac{\eta}{2\zeta}\right)\right] \tag{8}$$

where $\zeta = \gamma\sqrt{kt}$, representing the ratio of the diffusion length in time t to the characteristic deposition length, and η=xγ. The surface temperature resulting from this solution is

$$T_{surface} = \frac{q}{k\gamma}\left[\frac{2\zeta}{\sqrt{\pi}} - 1 + e^{\zeta^2}\; erfc(\zeta)\right] \tag{9}$$

The ratio of the surface temperature from Eq. (9) to the surface temperature due to surface heating (Eq. 5) is

$$R = 1 - \frac{\sqrt{\pi}}{2\zeta}\left[1 - e^{\zeta^2}\; erfc(\zeta)\right] \tag{10}$$

The result of Eq. (10) was determined using the asymptotic result:[6]

$$R \sim 1 - \frac{\sqrt{\pi}}{2\zeta} + \frac{1}{2\zeta^2} \tag{11}$$

4. Spatial Non-Uniformity in Surface Heating

Quite often the heating distribution over the surface is non-uniform. For example, many lasers produce a Gaussian heating distribution when the laser is normally incident on a flat surface. To explore this effect, we consider the solution by Hector and Hetnarski [2]. This gives the peak stress due to laser heating with a Gaussian shape on a half-space as (at r < z < 0):

$$\sigma^*_{rr} = \int_0^\infty h(\beta^*)\int_0^{t^*} \left[\left(\frac{G^*}{2} - 1\right)\frac{\beta^{*2}}{2} + \left(\beta^{*2} - \frac{1}{2}\frac{\partial^2 G^*}{\partial z^{*2}}\right)\right]$$

$$+\left(\frac{1}{2} + v\right)\beta^{*2}\; erfc\left(\frac{\beta^*\sqrt{t^* - \tau^*}}{2}\right)d\tau^* d\beta^* \tag{12}$$

where

$$G^* = 2\; erf\left(\frac{\beta^*}{2}\sqrt{t^* - \tau^*}\right) \tag{13}$$

$$\frac{\partial^2 G^*}{\partial z^{*2}} = \beta^*\left[\frac{4}{\sqrt{\pi(t^* - \tau^*)}}\exp\left\{\frac{-\beta^{*2}}{4}(t^* - \tau^*)\right\} + 2\beta^*\; erf\left(\frac{\beta^*}{}\sqrt{t^* - \tau^*}\right)\right] \tag{14}$$

and

$$h(\beta^*) = \exp\left(\frac{-\beta^{*2}}{4}\right). \tag{15}$$

In these equations, the starred quantities are all dimensionless, according to [3]

$$\beta^* = \frac{\beta}{\sqrt{K_c}} \quad \sigma i^*_j = \frac{4(1-v)k\sqrt{K_c}}{(1+v)\alpha\mu q_0}\alpha_{ij}$$

$$t^* = 4kK_c t \tag{16}$$

$$\tau^* = 4kK_c\tau \quad T^* = \frac{k\sqrt{K_c}}{q_0}T$$

Here τ and β are integration variables, K_c is a measure of the width of the Gaussian laser profile on the surface, and q_0 is the peak surface heating. Putting Eqs. (13 – 15) together gives

$$\sigma_{rr}^{*} = \int_0^{\infty} \beta^{*} \exp\left(\frac{-\beta^{*2}}{4}\right)^{*} \times$$
$$\int_0^{t^*} \left[\beta^{*}(1+v) erfc\left(\frac{\beta^{*}}{2}\sqrt{t^{*}-\tau^{*}}\right) - \frac{2}{\sqrt{\pi(t^{*}-\tau^{*})}} \exp\left\{\frac{-\beta^{*2}}{4}(t^{*}-\tau^{*})\right\}\right] d\tau^{*} d\beta^{*} \qquad (17)$$

Carrying out this integration gives:

$$\sigma_{rr}^{*} = 2(1+v)\sqrt{\pi t^{*}} - 4(1+v)\sqrt{\frac{t^{*}}{\pi}} - \frac{4}{\sqrt{\pi}}\left[1+t^{*}-(1-t^{*})v\right]\tan^{-1}\left(\sqrt{t^{*}}\right) \qquad (18)$$

To compare this to our simple analytical solution, we can write the stress from Eq. (6) using the dimensionless variables in Eq. (16), giving

$$\sigma_{base}^{*} = -8\sqrt{\frac{t^{*}}{\pi}} \qquad (19)$$

Hence, the ratio of the stress due to a Gaussian heating profile to that of the uniform heating profile is

$$R_{\sigma} = \frac{-(1+v)}{4}\pi\sqrt{t^{*}} + \frac{(1+v)}{2} + \frac{1}{2}\left[1+t^{*}-(1-t^{*})v\right]\frac{\tan^{-1}\left(\sqrt{t^{*}}\right)}{\sqrt{t^{*}}} \qquad (20)$$

This ratio is a function of the dimensionless time for several values of the Poisson ratio.

A similar approach can be taken with the temperature. Hector and Hetnarski]2] give the temperature as

$$T^{*} = \frac{1}{\sqrt{4\pi}}\int_0^{t^*} \frac{d\tau^{*}}{(1+t^{*}-\tau^{*})\sqrt{t^{*}-\tau^{*}}} \qquad (21)$$

Carrying out this integral gives

$$T^{*} = \frac{Tan^{-1}\left(\sqrt{t^{*}}\right)}{\sqrt{\pi}} \qquad (22)$$

In the dimensionless units given in Eq. 16, the one-dimensional surface temperature from Eq.{ 5) becomes

$$T_{base}^{*} = \sqrt{\frac{t^{*}}{\pi}} \qquad (23)$$

Hence, the ratio of the temperature due to the Gaussian heating profile to that of the uniform profile is

$$R_{T} = \frac{\tan^{-1}\left(\sqrt{t^{*}}\right)}{\sqrt{t^{*}}} \qquad (24)$$

V. Elastic Waves

To model elastic waves, we must include inertial terms in the stress equations. To estimate their effects, consider thermoelastic deformation of a half-space, with x denoting the perpendicular distance from the surface. Following Sternberg and Chakravor [3], one can define the following dimensionless variables

$$\xi = \frac{x}{a} \qquad \phi = \frac{kT}{qa}$$
$$\tau_{w} = \frac{kt}{a^{2}} \qquad \hat{\sigma}_{x} = \frac{(1-2v)k\sigma_{x}}{2(1+v)\alpha qa\mu}$$
$$a = \frac{k}{c} \qquad c^{2} = \frac{2(1-v)\mu}{(1-2v)\rho} \qquad (25)$$

Here c is the wave speed and ρ is the density of the solid. With these definitions, the governing equations then become:

$$\frac{\partial^{2}\phi}{\partial\xi^{2}} = \frac{\partial\phi}{\partial\tau_{w}}$$
$$\frac{\partial^{2}\hat{\sigma}_{x}}{\partial\xi^{2}} = \frac{\partial^{2}\hat{\sigma}_{x}}{\partial t_{w}^{2}} + \frac{\partial^{2}\phi}{\partial\tau_{w}^{2}} \qquad (26)$$

It is assumed here that the only non-zero displacement is perpendicular to the surface of the half-space. that is, stresses, and time derivatives are nonexistent. The boundary conditions are that the temperatures and stresses vanish at x equals infinity, while at the surface

$$\frac{\partial\phi}{\partial\xi} = -1$$
$$\hat{\sigma}_{x} = 0 \qquad (27)$$

The solution for the dimensionless temperature is given by [1, 4]

$$\phi = 2\left(\sqrt{\frac{\tau_{w}}{\pi}}\exp\left[\frac{-\xi^{2}}{4\tau_{w}}\right] - \frac{\xi}{2} erfc \exp\left[\frac{\xi}{2\sqrt{\tau_{w}}}\right]\right) \qquad (28)$$

and the stresses can be found to be

$$\hat{\sigma}_{x} = \frac{1}{2}\exp(\tau_{w}-\xi)\left\{1-\exp(2\xi)\ erfc\left(\frac{2\tau+\xi}{2\sqrt{\tau_{w}}}\right)\right.$$

$$+ erfc\left(\frac{2\tau_w - \xi}{2\sqrt{\tau_w}}\right) - 2erf\left(\sqrt{\tau_w - \xi}\right)H(\tau_w - \xi)\Bigg\} \quad (29)$$

and

$$\hat{\sigma}_y = \hat{\sigma}_z = v\hat{\sigma}_x - (1-2v)\phi \quad (30)$$

This completes our solution for the stresses induced by surface heating on a half-space.

Since the longitudinal stress $(\hat{\sigma}_x)$ is zero at the surface, the transverse stress $(\hat{\sigma}_y)$ at the surface is given by

$$\hat{\sigma}_y = \hat{\sigma}_z = -(1-2v)\phi \quad (31)$$

or

$$\hat{\sigma}_y = \hat{\sigma}_z = -2(1-2v)\left(\sqrt{\frac{\tau_w}{\pi}}\right) \quad (32)$$

The peak stress in the wave occurs at $\xi = \tau_w$. Substituting this into Eqs. (29) and (30) gives

$$\hat{\sigma}_x = -\frac{1}{2}\left\{1 - \exp(2\tau_w)\ erfc\left(\frac{3\sqrt{\tau_w}}{2}\right) + erf\left(\frac{\sqrt{\tau_w}}{2}\right)\right\}$$

$$\hat{\sigma}_y = -\frac{v}{2}\left\{1 - \exp(2\tau_w)\ erfc\left(\frac{3\sqrt{\tau_w}}{2}\right) + erf\left(\frac{\sqrt{\tau_w}}{2}\right)\right\}$$

$$-2(1-2v)\left(\sqrt{\frac{\tau_w}{\pi}}\exp\left[\frac{-\tau_w}{4}\right] - \frac{\tau_w}{2}\ erf\left[\frac{\sqrt{\tau_w}}{2}\right]\right) \quad (33)$$

For long times the longitudinal stress approaches -1, while the transverse stress approaches (-v).

Typical wave shapes are shown in Fig. 4, which plots the two dimensionless stresses as a function of distance from the surface. These results are given for dimensionless times of 0.5, 1, and 10. As one would expect, the stresses are all compressive, and the peak stress in the wave occurs at $\xi = \tau_w$. Except at early times, the transverse stress peaks at the surface because that's where the temperature peaks. At early times, there is a local peak in the transverse stress where the wave front lies, and at this point the transverse stress is less than the longitudinal stress.

The peak in the longitudinal stress occurs at $\xi = \tau_w$, while the peak transverse stress occurs at the surface (except at short times). These peaks are plotted in Fig. 5, which gives both stresses at $\xi = \tau_w$ along with the surface stress at the same dimensionless time. It can be seen that beyond a dimensionless time of approximately 4, the surface stress exceeds the stress at the wave peak.

The ratio of the longitudinal stress at the wave peak to the surface stress is given by:

$$R_w = \frac{1 - \exp(2\tau_w)\ erfc\left(\frac{3\sqrt{\tau_w}}{2}\right) + erf\left(\frac{\sqrt{\tau_w}}{2}\right)}{4(1-2v)\left(\frac{\tau_w}{2}\right)} \quad (34)$$

This ratio is plotted in Fig. 6. For large times, this ratio is 0. As τ_w approaches 0, this ratio approaches 1.10/(1-2v).

5. Conclusions

For most situations, a simple formula provides an adequate representation of the thermal stress induced in a rapidly heated solid. When this formula is not valid, there are often simple analytical representations of these stresses. This paper provides these formulas, along with their regions of validity.

References

[1] H. Carslaw, and J. Jaeger, 1959, Conduction of Heat in Solids, Oxford Clarendon Press, p. 75.

[2] L. Hector and R. Hetnarski, 1996, "thermal Stresses Due to a Laser Pulse: Elastic Solution," J. Appl. Mech., 63, 38.

[3] E. Sternberg, and J. Chakravorty, 1959, "On Inertia Effects in a Transient Thermoelastic Problem," J. Appl. Mech., 26, 503,.

[4] Danilovskaya VI (1950). Thermal stresses in an elastic half-space arising after a sudden heating of its boundary. *Prikl Mat Mekh* 14: 316–318 MathSciNetGoogle Scholar

[5] Boley BA and Weiner JH (1997). Theory of thermal stresses. Dover, New York, 54 Google Scholar

[6] Noda N, Hetnarski RB and Tanigawa Y (2000). Thermal stresses. Lastran Co., New York, 333–337 Google Scholar

[7] Jadeja ND and Loo TC (1974). Heat induced vibration of a rectangular plate. *Trans ASME J Engi Ind* 96: 1015–1021

[8] Vel SS and Batra RC (2004). Three-dimensional exact solution for the vibration of functionally graded rectangular plates. *J Sound Vib* 272: 703–730 CrossRefADSGoogle Scholar 18.

[9] Tauchert TR (1987). Thermal stresses in plates – dynamical problems. In: Hetnarski, RB (eds) Thermal stresses, vol II., pp 1–56. North-Holland, Amsterdam

[10] Tauchert TR (1987). Thermal stresses in plates – dynamical problems. In: Hetnarski, RB (eds) Thermal stresses, vol II., pp 1–56.

[11] Udoh, P. J. (2016). Thermal Elastic Stresses in an Infinite Thin Plate with a Circular Cylindrical Pore. Transactions of the Nigerian Association of mathematical Physics. Vol.3 Pp71-78